NONLINEAR EVOLUTION OF DISTURBANCES IN A ONE-DIMENSIONAL UNIVERSE

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An exact solution is obtained for the problem of the development of arbitrary disturbances of the density and velocity in a one-dimensional universe. As an illustration, the nonlinear distortion of a sinusoidal perturbation of the initial density is calculated.

1. The problem of the origin of galaxies and their distributions with respect to masses and angular momenta is, first of all, a problem of the nonlinear evolution of disturbances of the density. The spherically symmetric solution of R. Tolman[1] and the first-order perturbations to the Friedmann solution, which were obtained by E. M. Lifshitz,[2] are well known. These results, and also the formulation of the problem about the origin of galaxies, are significantly simplified in the Newtonian approximation,[3,4] and also the Newtonian approximation turns out to be adequate for the problem under consideration, at least for the epoch after recombination of the plasma. But even in the Newtonian approximation the problem reduces to a rather complicated system of nonlinear equations with partial derivatives,[4] which must be investigated by approximate methods (using different orders of perturbation theory and numerical procedures). In this connection it is useful to have a simple model of the problem possessing an exact solution, which may serve as a test of the approximate methods.

2. Let us consider a one-dimensional universe filled with dust (we shall be interested in scales of distance which are larger than the Jeans wavelength). The initial equations (corresponding to a special case of the equations given in[4]) have the form

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{\partial\varphi}{\partial x}, \tag{1}$$

$$\frac{\partial\sigma}{\partial t}+\frac{\partial}{\partial x}(\sigma v)=0, \tag{2}$$

$$\frac{\partial^2\varphi}{\partial x^2}=\sigma, \tag{3}$$

where v is the velocity, φ is the gravitational potential, and σ is a quantity which is proportional to the product of the density times the gravitational constant.[1] Differentiating (1) with respect to x, with Eq. (3) taken into account we obtain

$$\frac{dh}{dt}+h^2=-\sigma,\quad \frac{d\sigma}{dt}+\sigma h=0, \tag{4}$$

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+v\frac{\partial}{\partial x},\quad h\equiv\frac{\partial v}{\partial x}.$$

The system (4) reduces to a single equation of the second order:

$$\frac{d^2}{dt^2}(\sigma^{-1})=-1. \tag{5}$$

[1]One can show that Eqs. (1) – (3) are the nonrelativistic limit of Einstein's equations for a one-dimensional universe.

In the Lagrangian description the solution has the form

$$\sigma(t,x_0)=\sigma_0(x_0)\left[1+h_0(x_0)t-{}^1/_2\sigma_0(x_0)t^2\right]^{-1}, \tag{6}$$

$$h(t,x_0)=(h_0(x_0)-\sigma_0(x_0)t)\left[1+h_0(x_0)t-{}^1/_2\sigma_0(x_0)t^2\right]^{-1}, \tag{7}$$

where x_0 is the initial distance of a fluid particle from the fixed particle to which the reference system is attached, and $\sigma_0(x_0)$ and $h_0(x_0)$ denote the initial distributions.

We see that in a one-dimensional universe, in the final analysis a contraction to a singularity always occurs after a time

$$t_K(x_0)=\sigma_0^{-1}(x_0)\left[h_0(x_0)+\sqrt{h_0^2(x_0)+2\sigma_0(x_0)}\right],$$

although for $h_0 > 0$ at the beginning, of course, there will be an expansion. One can show that the same result will also hold in a two-dimensional universe.[2]

To formulas (6) and (7) it is necessary to add the dependence of the moving coordinate $x(t, x_0)$ of the fluid particle on the time. It is easy to obtain this dependence from the equation of continuity

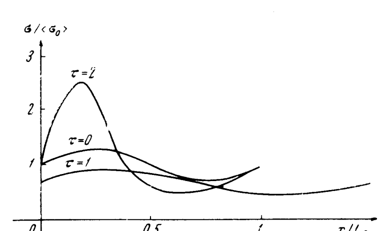
formula $$\partial x/\partial x_0=\sigma_0/\sigma,$$

which is another way of writing Eq. (2). We have

$$x(t,x_0)=x_0+u_0(x_0)t-{}^1/_2\mu_0(x_0)t^2, \tag{8}$$

$$u_0(x_0)=\int_0^{x_0}h_0(z)\,dz,\qquad \mu_0(x_0)=\int_0^{x_0}\sigma_0(z)\,dz. \tag{9}$$

Formulas (6), (7), (8), and (9) completely solve the problem in the Lagrangian description. The same formulas in parametric form (x_0 is the parameter) give



Evolution of a sinusoidal perturbation of the initial density.

[2]We present here one fantastic conjecture. Perhaps the universe was not always three-dimensional. The dimensionality might change during a transition through the singular state with zero dimensionality. Only starting with dimensionality equal to three did the universe gain the possibility "to survive".

the solution in the Eulerian description.

Let us emphasize the basic physical fact which enabled us to obtain a simple and exact solution: in a one-dimensional universe the relative acceleration of the fluid particles is constant and is proportional to the mass confined between the particles.

3. Let us consider the following example: a sinusoidal perturbation of the initial density

$$\sigma_0(x_0) = \langle\sigma_0\rangle\left(1 + a_0 \sin\frac{2\pi x_0}{L_0}\right), \qquad h_0 = \text{const},$$

where $\langle\sigma_0\rangle$ denotes the average density, a_0 is the relative initial amplitude, and L_0 is the initial spatial period. Formulas (6) and (8) take the form

$$s \equiv \frac{x}{L_0} = s_0 + s_0\theta_0\tau - \frac{1}{2}\tau^2\left[s_0 + \frac{1}{\pi} a_0 \sin^2 \pi s_0\right],$$

$$s_0 = \frac{x_0}{L_0}, \quad \theta_0 = h_0\langle\sigma_0\rangle^{-1/2}, \quad \tau = t\langle\sigma_0\rangle^{1/2}.$$

The graphs are constructed in the Eulerian description for

$$\theta_0 = 1, \quad a_0 = 1/4, \quad \tau = 0,\ 1,\ 2.$$

The value $\tau = 2$ corresponds to that moment when the spatial period, after a temporary increase, is again equal to the initial period, but the shape of the perturbation is now substantially distorted in accordance with gravitational condensation.

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[1] L. D. Landau and E. M. Lifshitz, Teoriya polya (Theory of Fields), Fizmatgiz, M. 1962.

[2] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. 16, 587 (1946).

[3] Ya. B. Zel'dovich and I. D. Novikov, Relyativistskaya astrofizika (Relativistic Astrophysics), Nauka, M. 1967.

[4] Ya. B. Zel'dovich, Usp. Mat. Nauk 23, 171 (1968).

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