NONLINEAR THEORY OF INTERACTION BETWEEN A "MONOCHROMATIC" BEAM OF RELATIVISTIC ELECTRONS AND A PLASMA

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Excitation of oscillations as a result of interaction between a relativistic "monochromatic" beam and a plasma is considered in the quasilinear approximation. The variation of the beam distribution function is found and the energy of the excited oscillations and the characteristic time for transfer of energy from the beam to the oscillations are determined. It is demonstrated that when the increments are sufficiently large the fraction of energy used to excite oscillations in the hydrodynamic stage increases considerably on going over to a relativistic beam.

1. RECENTLY, owing to the rapid development of accelerator technology (see, for example, [1]), it became possible to perform experiments on the interaction between relativistic beams and a plasma. In this connection, interest attaches to a theoretical analysis of the instabilities that arise in such an interaction. The linear theory of these instabilities was considered in [2–5], where the conditions for their occurrence, their frequencies and their growth increments were determined for different branches of the oscillations excited by the beam in the plasma.

However, to determine the effectiveness of the plasma-beam interaction at relativistic beam energies, a nonlinear theory of instability is necessary, which makes it possible to determine the fraction of the beam energy going to the excitation of the plasma oscillations, and the characteristic time of energy transferred to the plasma. Such a nonlinear theory was considered by the author[6] for a model of one-dimensional oscillations excited by a relativistic beam in a plasma.

According to [7, 8] the slowing down of a low-density beam in a plasma, due to the excitation of oscillations, can be considered in many cases within the framework of the quasilinear theory. We present in this paper such an analysis for the case of a relativistic beam. We consider the interaction between a beam and a plasma in the absence of an external longitudinal magnetic field, when the spectrum of the oscillations excited by the beam is not one-dimensional, and in a strong field, when the spectrum is close to one-dimensional.

The influence of the magnetic field produced by the beam current itself on the process of the quasilinear relaxation can be neglected, both in the presence of a sufficiently strong external magnetic field in the plasma and in the absence of such a field, provided the cyclotron frequency corresponding to the self-field is much smaller than the oscillation growth increment. If the current in the beam is sufficiently strong, the self-field is offset by the field of the inverse current that is produced in the plasma. When $n_b \ll n_p$ (n_b and n_p are respectively the beam and plasma densities), the energy of the inverse current is small compared with the energy of the beam, by virtue of the fact that the particle velocity in the inverse current is $v_{inv} \ll c$.

The excitation of the oscillations is considered under conditions when the beam is "monochromatic." The condition of beam monochromaticity, as is well known, is of the form $k|\Delta v| \ll \delta$ ($|\Delta v|$—thermal velocity scatter in the beam, δ—increment). Under this condition, the entire beam is at resonance with each harmonic of the packet of unstable oscillations, and the instability is the strongest. Excitation of oscillations by the beam is accompanied by the "smearing" of its velocity distribution function, and if the energy of the oscillations exceeds a definite level, then the beam ceases to be monochromatic. The growth increments of the oscillations then decrease appreciably. In the nonrelativistic case considered in [8], the hydrodynamic stage corresponding to the monochromatic beam terminates already under conditions when the oscillation energy is smaller than the beam energy by a factor $(n_b/n_p)^{1/3}$. The subsequent excitation of the oscillations is produced then by a beam having a smeared velocity distribution. The energy of the oscillations excited in this stage is comparable with the beam energy, but their excitation occurs quite slowly (when the thermal scatter Δv in the beam increases, the increment decrease from the value $\omega_p(n_b/n_p)^{1/3}$ during the hydrodynamic stage to $\omega_p n_b/n_p$ when $|\Delta v| \sim v_0$).

On going over to relativistic beam energies, there occurs, on the one hand, a certain decrease in the increments, owing to the increase of the particle mass. On the other hand, owing to the mass increase, an appreciable velocity scatter is obtained in the beam at much higher values of the oscillation energy. As a result, the fraction of the energy lost by the beam at the hydrodynamic stage, when the oscillations are excited most effectively, is increased. In the ultrarelativistic case $(n_b/n_p)^{1/3} \gtrsim 1$ ($\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$), the energy of the oscillations excited during the hydrodynamic stage is comparable with the beam energy.

2. In the absence of an external magnetic field, the dispersion equation for the oscillations excited by a monochromatic beam ($|\mathbf{k}\Delta\mathbf{v}| \ll |\omega_k - k_z v_0|$) in a plasma, neglecting the beam's own magnetic field, can be written in the form

$$\left[1-\frac{\omega_p^2}{\omega_k^2}-\frac{\omega_b^2}{\gamma_0^3(\omega_k-k_zv_0)^2}\right]\left[k^2c^2+\omega_p^2+\frac{\omega_b^2}{\gamma_0}-\omega_k^2\right]$$

$$-\frac{\omega_b^2\omega_p^2 k_x^2 v_0^2}{\gamma_0(\omega_k - k_z v_0)^2 \omega_k^2} = 0. \quad (1)$$

$\omega_b^2 = 4\pi e^2 n_b/m$, $\omega_p^2 = 4\pi e^2 n_p/m$; k_x and k_z are the transverse and longitudinal components of the wave vector relative to the beam.

At a low beam density, $n_b \ll n_p$, the oscillations building up with the largest increment are those whose frequency is close to the plasma frequency ω_p. The excited oscillations are electrostatic—the electric field component transverse to the wave vector is small compared with the longitudinal in a ratio $k_x k_z \delta/k^2\omega_p$ (δ is the increment).

The maximum of the growth increment with respect to the longitudinal wave number k_z is reached when $k_z = \omega_p/v_0$ and is equal to

$$\delta = \frac{\sqrt{3}}{2^{4/3}}\omega_p\left(\frac{n_b}{n_p\gamma_0}\right)^{1/3}\left(\frac{k_z^2}{k^2}\frac{1}{\gamma_0^2}+\frac{k_x^2}{k^2}\right)^{1/3}, \quad (2)$$

which coincides with the result obtained in [2]. At relativistic energies, the oscillations first to be excited are those propagating almost perpendicular to the beam ($k_x \gg k_z$). Such an angular dependence of the increment in (2) is connected with the fact that when $\gamma_0 \gg 1$ the transverse mass of the beam particles is much smaller than the longitudinal mass, and the oscillations that are transverse to the beam are easiest to build up. At large k_x/k_z, it becomes important to take into account the transverse thermal motion in the beam. In this case the growth increment of the oscillations is determined by the formula

$$\delta = \frac{\sqrt{3}}{2^{4/3}}\omega_p\left(\frac{n_b}{n_p\gamma_0}\right)^{1/3}\left[\frac{k_x^{2/3}}{k^{2/3}} - \frac{3k_x^2}{8k_z^2}\frac{\overline{v_\perp^2}}{c^2}\left(\frac{2n_p\gamma_0}{n_b}\right)^{2/3}\right], \quad (3)$$

where

$$\overline{v_\perp^2} = \frac{1}{n_b m^2\gamma_0^2}\int p_\perp^2 f_0 d\mathbf{p}$$

is the square of the transverse thermal velocity of the beam particles. The maximum increment, namely

$$\delta = \frac{\sqrt{3}}{2^{4/3}}\omega_p\left(\frac{n_b}{n_p\gamma_0}\right)^{1/3},$$

is reached when

$$\left(\frac{k_x^2}{k_z^2}\right)^{ext} = \frac{2^{7/6}}{3}\frac{c}{\sqrt{\overline{v_\perp^2}}}\left(\frac{n_b}{n_p\gamma_0}\right)^{1/3}, \quad (4)$$

and by virtue of the monochromaticity of the beam we have

$$\frac{\sqrt{\overline{v_\perp^2}}}{c}\left(\frac{n_p\gamma_0}{n_b}\right)^{1/3} \sim \frac{k_x^2\overline{v_\perp^2}}{\delta_k^2} \ll 1.$$

The conditions for neglecting the beam's own magnetic field, as shown by calculations, take the form

$$\omega_{Hy} \ll \delta\gamma_0^2, \quad \omega_{Hy} \ll \omega_p \quad (5)$$

($\omega_{H_y} = eH_y/mc$; we are considering a ribbon beam along the x axis, with a beam dimension much greater than the transverse wavelength $k_x a \gg 1$). If the plasma density is sufficiently high, then condition (5) is satisfied also at very appreciable beam currents.

If the plasma is in a sufficiently strong external magnetic field H_0, with $\omega_{H_0} \gg \delta\gamma_0^2$ and $\omega_{H_0} \gg \omega_p$, the oscillations of the beam particles and of the plasma occur following Cerenkov excitation ($\omega_k \approx k_z v_0$) only along the field H_0. The self-field H_y is negligible in this case. The frequency of the excited oscillations is equal to $\omega_p k_z/k$, and the growth increment corresponding to the Cerenkov resonance is determined by

$$\delta = \frac{\sqrt{3}}{2^{4/3}}\omega_p\left(\frac{n_b}{n_p}\right)^{1/3}\frac{1}{\gamma_0}\frac{k_z}{k}. \quad (6)$$

In this case the growth increment is much smaller when $\gamma_0 \gg 1$ than in the case when $H_0 = 0$, and decreases rapidly with increasing beam energy. The maximum of the increment corresponds to oscillations propagating along the beam ($k_z \approx k$), in such a way that in a strong magnetic field the spectrum of the excited oscillations is close to the one-dimensional spectrum.[1)] The monochromaticity condition leads to a limitation on the scatter with respect to the longitudinal velocities in the beam $|\Delta v_z| \ll c(n_b/n_p)^{1/2}\gamma_0^{-1}$

3. In this section we consider the quasilinear relaxation of the beam in a strong magnetic field, when the spectrum of the oscillations is one-dimensional. In the absence of collisions, the beam-particle distribution function $f_0(t, \mathbf{r}, \mathbf{p})$ is determined from the equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} - e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}_0]\right)\frac{\partial f}{\partial \mathbf{p}} = 0 \quad (7)$$

($\mathbf{E}$ is the electric field of the oscillations). In the case of a monochromatic beam $k|\Delta v| \ll |\omega_k - kv_0|$ and of a one-dimensional spectrum of oscillations, we obtain from this equation, by means of the usual procedure, the following quasilinear-approximation equation for the "background" distribution function of the beam f_0:

$$\frac{\partial f_0}{\partial t} = e^2\sum_{k>0}\frac{2\delta_k|E_k|^2}{(kv_0-\omega_k)^2+\delta_k^2}\frac{\partial^2 f_0}{\partial p^2}$$
$$- e^2\sum_{k>0}\frac{4\delta_k(kv_0-\omega_k)k|E_k|^2}{[(kv_0-\omega_k)^2+\delta_k^2]^2}\frac{\partial}{\partial p}\left[(v-v_0)\frac{\partial f_0}{\partial p}\right]. \quad (8)$$

Here $f_0(t, p) = \langle f(t, z, p)\rangle$, the angle brackets denote averaging over distances that are large compared with the oscillation wavelength; E_k is the amplitude of the Fourier harmonic of the electric field of the oscillations:

$$E(t,z) = \frac{1}{2}\sum_k E_k(t)\exp[i(kz-\omega_k t)] + \text{c.c.} \quad (9)$$

The time variation of E_k is determined from the equation

$$\frac{\partial|E_k|^2}{\partial t} = 2\delta_k|E_k|^2 = \sqrt{3}\left(\frac{n_b}{2n_p}\right)^{1/3}\frac{\omega_p}{\gamma(t)}|E_k|^2, \quad (10)$$

where

$$\gamma(t) = \left[\frac{1}{n_b}\int\frac{dp f_0(t,p)}{(p^2/m^2c^2+1)^{3/2}}\right]^{-1/3}.$$

1) Under this condition, $\omega_{H_0} \gg \omega_p$ is not a necessary condition for the spectrum to be one-dimensional, for when $\omega_{H_0} \ll \omega_p$ the growth increment of the oscillations is determined by formula (6), in which the substitution $k_z/k \to (k_z/k)^{2/3}$ is made. The strongest limitation on the magnetic field intensity H_0, which if satisfied makes it possible to regard the spectrum of the excited oscillations as one-dimensional, is obtained from the condition that the increment of the anomalous Doppler effect ($\omega_k \approx k_z v_0 - \omega_{H_0}/\gamma_0$) be small compared with the increment (6), and takes the form $\omega_{H_0} \gg \omega_p^{1/3}\omega_b^{2/3}\gamma_0^2$.

The second term in the equation for f_0 is small and becomes appreciable only in calculation of the mean values, which is not connected with the symmetrical smearing of the distribution function, for example for the calculation of the change of the beam energy. It is convenient to introduce in (8) and (10) dimensionless variables $\xi = p/mc$ and τ, the latter being connected with t by the relation

$$\frac{d\tau}{dt} = \sqrt{3}\left(\frac{n_b}{2n_p}\right)^{2/3} \omega_p \frac{\gamma(\tau)}{4\pi n_b mc^2} \sum_k |E_k|^2.$$

In terms of these variables, neglecting the small second term in the equation for f_0, we obtain

$$\frac{\partial f_0}{\partial \tau} = \frac{\partial^2 f_0}{\partial \xi^2} \quad \frac{d}{d\tau}\sum_k \frac{|E_k|^2}{8\pi} = \left(\frac{2n_p}{n_b}\right)^{1/3} \frac{n_b mc^2}{2\gamma^2(\tau)}. \qquad (11)$$

The solution of the equation for f_0, subject to the initial condition $f_0(\tau = 0) = (n_b/mc)\delta(\xi - \xi_0)$, where $\xi_0 = p_0/mc$, is

$$f_0 = \frac{n_b}{2mc\sqrt{\pi\tau}} \exp\left[-\frac{(\xi - \xi_0)^2}{4\tau}\right]. \qquad (12)$$

The oscillations thus lead to a smearing of the momentum distribution function of the beam, and for the distribution (12) the momentum scatter is $\Delta p \sim 2mc\sqrt{\tau}$. It follows from (8) that the width of the resonance between the beam particles and the wave, in terms of the velocity, is of the order of δ_k/k, and by the end of the hydrodynamic stage corresponding to the monochromatic beam the velocity spread in the beam is

$$|\Delta v| \sim \frac{\delta_k}{k} \sim \left(\frac{n_b}{n_p}\right)^{1/3} \frac{c}{\gamma}.$$

The corresponding momentum spread is

$$|\Delta p| \approx \frac{dp}{dv}|\Delta v| \sim mc\gamma^2\left(\frac{n_b}{n_p}\right)^{1/3}, \text{ т. е. } \tau \approx \frac{1}{4}\left(\frac{n_b}{n_p}\right)^{2/3} \gamma^4.$$

These formulas were obtained for the case $|\Delta p| \ll p_0$, corresponding to the condition

$$(n_b / n_p)^{1/3}\gamma_0 \ll 1. \qquad (13)$$

Under this condition, the change of the value of γ during the stage corresponding to the monochromatic beam is small, $\gamma(\tau) \approx \gamma_0$. Then, integrating the second equation of (11), we obtain the following relation for the energy of the oscillations excited by the beam during the hydrodynamic stage:[2)]

$$W = \sum_k \frac{|E_k|^2}{8\pi} \approx \left(\frac{2n_p}{n_b}\right)^{1/3} \frac{n_b mc^2\tau}{2\gamma_0^2} \approx \mathscr{E}_0\left(\frac{n_b}{n_p}\right)^{1/3} \frac{\gamma_0}{2^{8/3}}, \qquad (14)$$

where $\mathscr{E}_0 = n_b mc^2\gamma_0$ is the initial value of the beam energy. In comparison with the nonrelativistic case, the fraction of the beam energy going to excitation of oscillations during the hydrodynamic stage increases by a factor γ_0. Such an increase of W is connected with the fact that, owing to the increase of the particle mass at relativistic energies, the velocity scatter in the beam is small, and the beam remains monochromatic at a large momentum scatter.

In the ultrarelativistic case $(n_b/n_p)^{1/3}\gamma_0 > 1$, the relative momentum scattering in a monochromatic beam can be of the order of unity, $\Delta p/p_0 \sim 1$. In this case the value of $\gamma(\tau)$ changes appreciably when the beam becomes spread out. In calculating this quantity and other momentum-averaged quantities, it is possible with sufficient accuracy, to replace the distribution function (12) by a step function

$$f_0 = \frac{n_b}{4mc\sqrt{\tau}} \cdot \begin{cases} 1 & \text{if } |\xi - \xi_0| < 2\sqrt{\tau}, \\ 0 & \text{if } |\xi - \xi_0| > 2\sqrt{\tau}. \end{cases}$$

We then obtain for $\gamma(\tau)$ the relation

$$\frac{1}{\gamma^3} = \frac{1}{4\sqrt{\tau}} \int_{\xi_0 - 2\sqrt{\tau}}^{\xi_0 + 2\sqrt{\tau}} \frac{d\xi}{[\xi^2 + 1]^{3/2}} \approx \frac{1}{4\gamma_0^3\varepsilon^2}. \qquad (15)$$

Here $\epsilon = (\xi_0 - 2\sqrt{\tau})/\xi_0$. In calculating the integral in (15) it was assumed that $1/\xi_0 \ll \epsilon \ll 1$. The velocity scatter in the beam $|\Delta v|$ is determined in similar fashion:

$$|\Delta v| = \frac{1}{n_b}\left|\int (v - v_0) f_0 dp\right| = \frac{c}{4\sqrt{\tau}} \int_{\xi_0 - 2\sqrt{\tau}}^{\xi_0 + 2\sqrt{\tau}} \left(\frac{1}{2\xi^2} - \frac{1}{2\xi_0^2}\right) d\xi \approx \frac{c}{4\gamma_0^2\varepsilon}. \qquad (16)$$

At the end of the hydrodynamic stage the scatter is

$$|\Delta v| \sim \frac{\delta_k}{k} \approx \frac{c\sqrt{3}}{4}\left(\frac{n_b}{n_p}\right)^{1/3} \frac{1}{\gamma_0\varepsilon^{2/3}},$$

where $\gamma(\tau)$ is taken from (15). By determining $|\Delta v|$ from (16), we find that the value of ϵ corresponding to this scatter is

$$\varepsilon = \frac{1}{3^{3/2}} \frac{n_p}{n_b} \frac{1}{\gamma_0^3}. \qquad (17)$$

Integrating Eq. (11) for the oscillation energy with respect to τ, we find that in the ultrarelativistic case $(n_b/n_p)^{1/3}\gamma_0 \gtrsim 1)$ the energy of the oscillations excited during the hydrodynamic stage becomes comparable with the beam energy:

$$W = \frac{1}{4}\left(\frac{n_p}{n_b}\right)^{1/3} \frac{n_b mc^2}{\gamma_0^{2/3}} \int_0^\tau \frac{d\tau'}{(\xi_0 - 2\sqrt{\tau'})^{4/3}} \sim n_b mc^2\left(\frac{n_p}{\varepsilon n_b}\right)^{1/3} \sim \mathscr{E}_0. \qquad (18)$$

The energy lost by the beam in the case of interaction with the oscillations can be determined from (8). Multiplying this equation by $c\sqrt{p^2 + m^2c^2}$ and integrating with respect to the momenta, we get

$$\frac{d\mathscr{E}}{d\tau} = mc^2 \int \sqrt{\xi^2 + 1} \frac{\partial f_0}{\partial \tau} dp = -m^2c^3\left(\frac{2n_p}{n_b}\right)^{1/3} \gamma(\tau) \int \frac{f_0(\tau, \xi)\, d\xi}{(\xi^2 + 1)^{3/2}}$$
$$= -\left(\frac{2n_p}{n_b}\right)^{1/3} \frac{n_b mc^2}{\gamma^2(\tau)} = -2\frac{dW}{d\tau}. \qquad (19)$$

In obtaining the last relation, we used the second equation of (11).

The largest contribution to $d\mathscr{E}/d\tau$ is made by the second term in the right side of (8). The first term describes the symmetrical smearing of the momentum

2) We are considering a case when the unstable part of the spectrum $\Delta k/k \sim (n_b/n_p)^{1/3}\gamma^{-1}$ contains many harmonics of the oscillations, i.e., the condition $(\omega_p L/c)(n_b/n_p)^{1/3}\gamma^{-1} \gg 1$ is satisfied (L – longitudinal dimension of the region occupied by the plasma). In the opposite case, when the oscillation spectrum is essentially discrete, the decisive role in the development of the instability is played by beam particles captured in a potential well produced by the oscillations. In this case, the maximum amplitude of the oscillations excited by the monochromatic beam is determined from the condition $\sqrt{e\varphi_0/m\gamma_0^3} \sim \delta/k$ (φ_0 – amplitude of the potential), and is of the same order of magnitude as the amplitudes from (14).

distribution function of the beam and its contribution to $d\mathscr{E}/dt$ is small compared with (19) in a ratio $(n_b/n_p)^{1/3}\gamma_0^{-1} \ll 1$.

We now proceed to consider the question of the role of nonlinear wave interaction. The most appreciable in this problem is the induced scattering of the oscillations excited by the beam from the thermal ions of the plasma. This process influences strongly the dynamics of the beam relaxation in the case when the increment of the induced scattering $\delta^{(2)}$ exceeds the linear growth increment $\delta^{(1)}$ at the maximum amplitudes of the oscillations excited by the beam. When $T_i \ll T_p m_i v_{Te}^2/m_e c^2$, the scattering of the oscillations takes place in a sufficiently narrow interval of wave numbers.

$$\varkappa \approx |k+k'| \approx v_{Ti}\omega_p / v_{Te}^2 \ll k$$

and leads to transfer of the energy from the oscillations excited by the beam to waves propagating opposite to the beam ($v_{ph} = -v_0$). We shall assume that $\kappa \gg \Delta k$ ($\Delta k \sim k(n_b/n_p)^{1/3}\gamma^{-1}$ is the width of the spectrum of the oscillations excited by the beam), and the increment is $\delta < kv_{Ti}$. Then a simple estimate, based on the formulas of [9], shows that at maximum oscillation amplitudes, determined by relations (14) and (18), the ratio $\delta^{(2)}/\delta^{(1)}$ is equal to

$$\frac{\delta^{(2)}}{\delta^{(1)}} \approx \frac{\omega_p W^{\max}}{n_p T_e^2/T_i} \frac{1}{\omega_p (n_b/n_p)^{1/3}\gamma^{-1}} \approx \Lambda \frac{T_i}{T_e}\frac{c^2}{v_{Te}^2}, \tag{20}$$

where

$$\Lambda \approx \gamma_0^3 n_b/n_p \text{ if } (n_b/n_p)^{1/3}\gamma_0 \ll 1, \quad \Lambda \sim 1 \text{ if } (n_b/n_p)^{1/3}\gamma_0 \gtrsim 1.$$

It follows from (20) that the scattering induced during the hydrodynamic stage can become appreciable only in the ultrarelativistic case $(n_b/n_p)^{1/3}\gamma_0 \gtrsim 1$. If $\delta^{(2)} > \delta^{(1)}$, then the spectral transfer of oscillation energy due to the induced scattering leads in the case of a monochromatic beam to the occurrence of a new instability with increment $\sim \delta^{(1)}\sqrt{\delta^{(1)}/\delta^{(2)}}$, analogous to the dissipative instability of a beam in a plasma with frequent collisions.[3)] The instability results from the fact that the waves excited by the beam during the hydrodynamic stage have a negative energy.

Excitation of oscillations by a beam with a smeared velocity distribution $k|\Delta v| \gg \delta$ is described by the equations of the quasilinear theory, which were obtained for the relativistic case and were solved in [6]. We shall not consider in detail the relaxation of a beam with a smeared velocity distribution. We note only that, with the exception of the ultrarelativistic case $(n_b/n_p)^{1/3}\gamma_0 \gtrsim 1$, the increments of excitation of oscillations by a smeared beam are much smaller than for a monochromatic beam. As a result, the nonlinear stabilization of two-stream instability considered in [9] may take place during the stage corresponding to the smeared beam. Such a stabilization is connected with the transfer of the oscillation energy into the spectral region which is nonresonant for the beam, as a result of induced-scattering processes. Comparing the increment of the excitation of oscillations by a smeared beam at $|\Delta p|/p \ll 1$

$$\delta^{(1)} \approx \omega_p \frac{n_b\gamma_0^3}{n_p}\left(\frac{mc}{\Delta p}\right)^2$$

with the induced-scattering increment $\delta^{(2)}$ determined from (20) with $W \sim n_b c|\Delta p|$, we obtain the condition for the momentum spread in the beam, satisfaction of which leads to nonlinear stabilization:

$$\frac{|\Delta p|}{p} > \left(\frac{T_e^2}{T_i m_e c^2}\right)^{1/3} > \left(\frac{m_e}{m_i}\right)^{1/3}. \tag{21}$$

The question of nonlinear stabilization of relativistic beams is considered also in [10].

4. We consider now quasilinear relaxation of a monochromatic beam in the absence of a magnetic field. In this case, if the beam is monochromatic both with respect to the transverse and with respect to the longitudinal velocity, $k_\perp v_\perp \ll \delta_k$ and $k_z|\Delta v_z| \ll \delta_k$, then it excites "oblique" waves with $k_\perp \gg k_z$. The growth increment for these waves is determined by formula (3). For the beam distribution function $f_0(t, \mathbf{p})$ averaged over the spatial oscillations, we have in the quasilinear approximation the following equation:

$$\frac{\partial f_0}{\partial t} = e^2 \sum_{\mathbf{k}} |E_{\mathbf{k}}|^2 \frac{1}{k^2}\left[k_\perp \cos\varphi\left(\frac{\partial}{\partial p_\perp} + \frac{1}{p_\perp}\right) + k_z\frac{\partial}{\partial p_z}\right]$$
$$\times \frac{\delta_{\mathbf{k}}}{(k_\perp v_\perp \cos\varphi + k_z v_z - \omega_{\mathbf{k}})^2 + \delta_{\mathbf{k}}^2}\left[k_\perp \cos\varphi \frac{\partial f_0}{\partial p_\perp} + k_z\frac{\partial f_0}{\partial p_z}\right]. \tag{22}$$

We have introduced in this equation the momentum-space cylindrical coordinates $p_\perp$, p_z, and φ, and have used the fact that, by virtue of the axial symmetry of the spectrum $|E_k|^2$, the distribution function f_0 does not depend on the azimuthal angle φ between the vectors $\mathbf{p}_\perp$ and $\mathbf{k}_\perp$. In the case of a monochromatic beam, it is necessary to expand in (22) with respect to the parameter

$$\frac{k_\perp v_\perp \cos\varphi}{|\omega_{\mathbf{k}} - k_z v_0|} \sim \frac{k_\perp v_\perp}{\delta_{\mathbf{k}}} \ll 1,$$

and retain terms that are linear in small parameters. The quantities ω_k and δ_k can be replaced by their values for the most unstable harmonics of the spectrum,

$$\omega_{\mathbf{k}} + i\delta_{\mathbf{k}} = k_z v_0 + 2^{-4/3}(-1+i\sqrt{3})\,\omega_p\left(\frac{n_b}{n_p\gamma}\right)^{1/3}.$$

Then, going over to the dimensionless variables $\xi_\perp = p_\perp/mc$, $\xi_z = p_z/mc$, and τ, the latter connected with t by the relation

$$\frac{d\tau}{dt} = \sqrt{3}\,\omega_p\left(\frac{n_b}{2n_p}\right)^{2/3}\frac{\gamma^{1/3}(\tau)}{8\pi n_b mc^2}\sum_{\mathbf{k}} |E_{\mathbf{k}}|^2\frac{k_\perp^2}{k^2},$$

we rewrite the equation for f_0 in the form

$$\frac{\partial f_0}{\partial \tau} = \frac{1}{\xi_\perp}\frac{\partial}{\partial \xi_\perp}\left[\xi_\perp\frac{\partial f_0}{\partial \xi_\perp} - \alpha\frac{\xi_\perp^2}{(\xi_z^2+\xi_\perp^2+1)^{1/2}}\frac{\partial f_0}{\partial \xi_z}\right]$$
$$+\frac{\partial}{\partial \xi_z}\left[\beta\frac{\partial f_0}{\partial \xi_z} - \alpha\frac{\xi_\perp}{(\xi_z^2+\xi_\perp^2+1)^{1/2}}\frac{\partial f_0}{\partial \xi_\perp}\right]. \tag{23}$$

Here $\alpha = (2n_p\gamma/n_b)^{1/3}$, and for $\beta(\tau)$, using (4), we get the relation

$$\beta(\tau) = 2\left(\frac{k_z^2}{k_\perp^2}\right)^{ext} = 3\alpha\left(\frac{v_\perp^2}{2c^2}\right)^{1/2} = 3\alpha\left(\frac{1}{2n_b}\int\frac{\xi_\perp^2 f_0 d\mathbf{p}}{\xi_\perp^2+\xi_z^2+1}\right)^{1/2}. \tag{24}$$

It is easy to obtain from (23) a system of equations for the moments of the distribution function f_0. As noted in [11], this system coincides with that obtained in the

3) This circumstance was pointed out to the authors by L. I. Rudakov.

solution of the purely hydrodynamic equations in the quasilinear approximation.

The beam relaxation is described in this case by equations that determine the increase of the transverse thermal energy in the beam and its deceleration following excitation of the oscillations. These equations are of the form

$$\frac{dv_\perp^2}{d\tau}=\frac{c^2}{n_b}\int\frac{\xi_\perp^2}{\xi_z^2+\xi_\perp^2+1}\frac{\partial f_0}{\partial\tau}dp=c^2\left(\frac{4}{\xi_0^2}+3\alpha\frac{\overline{v_\perp^2}}{c^2\xi_0^2}\right)\approx\frac{4c^2}{\xi_0^2},\quad(25)$$

$$\frac{d\mathscr{E}}{d\tau}=mc^2\int(\xi_z^2+\xi_\perp^2+1)^{1/2}\frac{\partial f_0}{\partial\tau}dp=-2\alpha\frac{n_bmc^2}{\xi_0}.\quad(26)$$

Equations (25) and (26) were obtained from (24) under the conditions $\xi_0\gg1$ and $\Delta\xi_z$, $\xi_\perp\ll\xi_0$. With this, $\gamma(\tau)\approx\gamma_0\approx\xi_0$. The system (25) and (26) must be supplemented with an equation for the energy W of the oscillations excited by the beam:

$$\frac{dW}{d\tau}=\alpha\frac{n_bmc^2}{\xi_0},\quad W=\frac{1}{8\pi}\sum_{\mathbf{k}}|E_{\mathbf{k}}|^2.\quad(27)$$

An investigation of the obtained system of equations entails no difficulty. From (25) we get $v_\perp^2\approx4c^2\tau/\gamma_0^2$. At the end of the hydrodynamic stage, corresponding to a beam that is monochromatic with respect to the transverse scatter, the transverse thermal velocity is determined from the condition

$$\overline{v_\perp^2}\sim\frac{\delta_{\mathbf{k}}^2}{k_\perp^2}\approx\frac{3}{4}\frac{c^2}{\alpha^2}\left(\frac{k_z^2}{k_\perp^2}\right)^{ext}\approx\frac{c}{\alpha}\sqrt{\overline{v_\perp^2}},$$

i.e., $\overline{v_\perp^2}\approx c^2/\alpha^2$, which corresponds to $\tau_{max}\approx\gamma_0^2/4\alpha^2$. We then get from (27) that the energy of the oscillations excited by the beam during the hydrodynamic stage is small compared with the beam energy in a ratio $1/\alpha=(n_b/2n_p\gamma_0)^{1/3}$:

$$W=\alpha\frac{n_bmc^2\tau_{max}}{\gamma_0}\approx\frac{n_bmc^2\gamma_0}{4\alpha}.\quad(28)$$

At the same time, in the case considered above, that of excitation of oscillations propagating along the beam, the energy of the oscillations excited during the hydrodynamic stage was an appreciable fraction of the beam energy. This is connected with the fact that at relativistic energies of the beam particles, their transverse mass is much smaller than the longitudinal one ($m_\perp=m\gamma_0$, $m_\parallel=m\gamma_0^3$), and a noticeable velocity scattering is easier to obtain as a result of the "oblique" oscillations with large transverse components of the electric field.

From a comparison of (27) and (26) it follows that $\Delta\mathscr{E}=-2W$, i.e., the energy lost by the beam goes over mainly into potential energy of the oscillations and into the kinetic energy of the plasma particles. The excitation of the oscillations leads also to the appearance of a rather appreciable transverse momentum component $p_\perp\approx p_0/\alpha$ for the beam particles, but the corresponding transverse beam energy $n_bp^2/2m_\perp$ is small compared with W in a ratio $1/\alpha$.

According to (24), the ratio $(k_z^2/k^2)^{ext}$, corresponding to the maximum growth increment, increases with increasing $\overline{v_\perp}$, and by the end of the hydrodynamic stage there is produced in the beam also a sufficiently strong scatter with respect to the longitudinal momenta:

$$|\Delta p_z|\approx mc\left[\int_0^{\tau_{max}}\beta\,d\tau\right]^{1/2}\approx mc\left(\frac{\alpha}{\gamma_0}\right)^{1/2}(2\tau_{max})^{3/4}\approx\frac{mc\gamma_0}{2^{3/4}\alpha}.\quad(29)$$

However, owing to the large magnitude of the longitudinal mass, the corresponding velocity scatter is small:

$$|\Delta v_z|\sim c/\alpha\gamma_0^2\ll\delta/k_z,$$

i.e., at the oscillation energy W determined from (28) the beam becomes smeared out only with respect to the transverse velocity, and remains monochromatic with respect to the longitudinal velocity.

The subsequent relaxation is described by equations corresponding to a large transverse scatter $k_\perp v_\perp\gg\delta_{\mathbf{k}}$ and a small longitudinal scatter $k_z|\Delta v_z|\ll\delta_{\mathbf{k}}$. Under these conditions, the equation obtained from (22) for f_0 takes the form

$$\frac{\partial f_0}{\partial t}=\frac{e^2}{8\pi^2}\int_{-\infty}^{\infty}dk_z\left(\frac{1}{p_\perp}\frac{\partial}{\partial p_\perp}p_\perp\frac{\tilde\omega}{v_\perp}+\frac{\partial}{\partial p_z}k_z\right)\int_{|\tilde\omega|/v_\perp}^{\infty}dk_\perp k_\perp\frac{|E_{\mathbf{k}}|^2}{k^2}$$
$$\times\frac{1}{\sqrt{k_\perp^2v_\perp^2-\tilde\omega^2}}\left(\frac{\tilde\omega}{v_\perp}\frac{\partial f_0}{\partial p_\perp}+k_z\frac{\partial f_0}{\partial p_z}\right),\quad(30)$$

where $\tilde\omega=\omega_{\mathbf{k}}-k_zv_0$. We obtain similarly an equation for the spectral density of the oscillation energy $|E_{\mathbf{k}}|^2$:

$$\frac{\partial|E_{\mathbf{k}}|^2}{\partial t}=\frac{4\pi^2e^2}{k^2}\omega_p|E_{\mathbf{k}}|^2\int_{-\infty}^{\infty}dp_z\int_{v_\perp\geq|\tilde\omega|/k_\perp}dp_\perp$$
$$\times\frac{p_\perp}{\sqrt{k_\perp^2v_\perp^2-\tilde\omega^2}}\left(\frac{\tilde\omega}{v_\perp}\frac{\partial f_0}{\partial p_\perp}+k_z\frac{\partial f_0}{\partial p_z}\right).\quad(31)$$

We confine ourselves to a qualitative investigation of the system (30) and (31). The oscillation growth increment, determined from (31), can be represented in the form

$$\delta_{\mathbf{k}}\approx\frac{\pi\omega_p}{2}\frac{k_z^2n_b}{k^2n_p}\frac{c^2}{\bar v_\perp^2}\frac{1}{\gamma}\left[A\left(\frac{\tilde\omega}{k_\perp\bar v_\perp}\right)+\frac{k_z\bar v_\perp}{k_\perp c}\frac{\bar p_z}{|\Delta p_z|}B\left(\frac{\tilde\omega}{k_\perp\bar v_\perp}\right)\right],\quad(32)$$

where

$$A\left(\frac{\tilde\omega}{k_\perp\bar v_\perp}\right)=\frac{\tilde\omega}{k_\perp\bar v_\perp}\frac{1}{n_b}\int dp_z\int_{\zeta\geq|\tilde\omega|/k_\perp\bar v_\perp}\frac{dp_\perp p_\perp}{\sqrt{\zeta^2-\tilde\omega^2/k_\perp^2\bar v_\perp^2}}\frac{1}{\zeta}\bar p_\perp\frac{\partial f_0}{\partial p_\perp},$$

$$B\left(\frac{\tilde\omega}{k_\perp\bar v_\perp}\right)=\frac{1}{n_b}\int dp_z\int_{\zeta\geq|\tilde\omega|/k_\perp\bar v_\perp}\frac{dp_\perp p_\perp}{\sqrt{\zeta^2-\tilde\omega^2/k_\perp^2\bar v_\perp^2}}|\Delta p_z|\frac{\partial f_0}{\partial p_z};$$

$\bar p_\perp$—average transverse momentum, Δp_z—longitudinal-momentum scatter, $\zeta=v_\perp/\bar v_\perp$. In deriving (32) we have assumed that $\bar p_\perp\ll\bar p_z$, and then $\bar p_\perp=\bar v_\perp\bar p_z/c=m\bar v_\perp\gamma$.

The maximum of the growth increment determined by formula (32) corresponds to the condition $\bar k_\perp\bar v_\perp\sim|\tilde\omega|$, and under this condition the coefficients A and B are of the order of unity. At the end of the hydrodynamic stage considered above, i.e., when $\bar v_\perp\sim c/\alpha$ and $\Delta p_z\sim p_0/\alpha$, the growth increment (32) coincides in order of magnitude with the hydrodynamic increment $\delta_h\sim\omega_p(n_b/n_p\gamma_0)^{1/3}$. The diffusion of the beam particles leads to a decrease of the increment. It follows from (30) that the diffusion in the beam occurs along lines which

$$\frac{dp_z}{dp_\perp}=-\frac{k_zv_\perp}{\tilde\omega}=-\frac{k_z}{m\tilde\omega}p_\perp\Big/\sqrt{\frac{p_z^2}{m^2c^2}+\frac{p_\perp^2}{m^2c^2}+1}.\quad(33)$$

From this we get when $p_\perp\ll p_z$

$$p_z^2=p_0^2-\frac{\omega_p}{\tilde\omega}p_\perp^2\quad(34)$$

(we have taken into account the fact that in the beam $p_z\approx p_0$ when $p_\perp\to0$). From (34) we get that the transverse velocity component occurring in the case of beam-particle diffusion, is equal to

$$\frac{\overline{v_\perp^2}}{c^2} \approx 2\frac{\tilde{\omega}}{\omega_p}\frac{\Delta p_z}{p_z}. \qquad (34')$$

Substituting $\overline{v_\perp^2}$ into formula (32) for the increment and replacing, for the estimate of the maximum increment, the frequency deviation $\tilde{\omega}$ by its minimal value $\sim\delta$, we obtain in order of magnitude

$$\delta \approx \frac{\sqrt{\pi}}{2}\omega_p\frac{k_z}{k}\left(\frac{n_b}{n_p\gamma}\frac{\bar{p}_z}{|\Delta p_z|}\right)^{1/2} \qquad (35)$$

(when $\bar{k}_\perp\bar{v}_\perp \sim |\tilde{\omega}|$, $A \sim B \sim 1$, and, as can be readily shown by using (34′), both terms in (32) are of the same order of magnitude).

Equations (30) and (31) can be greatly simplified by changing over in the right side of these equations to the independent variables $\eta = p_z^2 + \tilde{\omega}p_\perp^2/\omega_p$ and p_z. As a result we obtain the following system:

$$\frac{\partial f_0}{\partial t} = \frac{e^2}{8\pi^2}\int_{-\infty}^{\infty} dk_z \frac{\partial}{\partial p_z}\left[\int_{|\tilde{\omega}|/v_\perp}^{\infty} \frac{dk_\perp k_\perp |E_k|^2}{\sqrt{k_\perp^2 v_\perp^2(\eta, p_z) - \tilde{\omega}^2}}\frac{k_z^2}{k^2}\frac{\partial f_0}{\partial p_z}\right], \qquad (36)$$

$$\frac{\partial |E_k|^2}{\partial t} = \frac{4\pi^2 e^2}{k^2}\omega_p\int_{-\infty}^{\infty} dp_z \int_{v_\perp \geqslant |\tilde{\omega}|/k_\perp} dp_\perp p_\perp \frac{k_z\,\partial f_0/\partial p_z}{\sqrt{k_\perp^2 v_\perp^2 - \tilde{\omega}^2}}|E_k|^2. \qquad (37)$$

Equation (36) describes the diffusion of the beam particles along the lines η = const.[4)] The diffusion leads to a smoothing of the distribution function along these lines. The diffusion continues until a scatter of the longitudinal velocity $|\Delta v_z| \sim \delta_k/k_z$ is produced in the beam, corresponding to a momentum scatter

$$|\Delta p_z| \approx \frac{dp_z}{dv_z}|\Delta v_z| \sim \bar{p}_z\gamma^2\frac{\delta_k}{\omega_p}.$$

Substituting $|\Delta p_z|$ from this condition in (35), we obtain the following estimate for the increment and for the maximum scatter $|\Delta p_z|$ under conditions when the beam still remains monochromatic with respect to the longitudinal velocity:

$$\delta \approx \left(\frac{n_b}{n_p}\right)^{1/3}\omega_p\frac{1}{\gamma_0}, \quad |\Delta p_z| \approx p_0\left(\frac{n_b}{n_p}\right)^{1/3}\gamma_0. \qquad (38)$$

These formulas were obtained under the condition $(n_b/n_p)^{1/3}\gamma_0 \ll 1$, when $|\Delta p_z| \ll p_0$ and $\gamma \approx \gamma_0$. When $(n_b/n_p)^{1/3}\gamma_0 \gtrsim 1$, we have $|\Delta p_z| \sim p_0$ and γ changes appreciably during the diffusion process.

From (36) with $v_\perp \sim \bar{v}_\perp$ we get, in order of magnitude, $(\Delta p_z)^2 \sim (mc)^2\tau'$, where

$$\tau' = \frac{e^2}{8\pi^2 m^2c^2}\int_0^t dt'\int_{-\infty}^{\infty} dk_z \int_{|\tilde{\omega}|/v_\perp}^{\infty} dk_\perp k_\perp \frac{|E_k|^2}{\sqrt{k_\perp^2\bar{v}_\perp^2 - \tilde{\omega}^2}} \approx \approx \frac{e^2}{16\pi^3 m^2c^2|\tilde{\omega}|}\int d\mathbf{k}\int dt\,|E_k|^2,$$

and integrating in (37) with respect to **k**, we obtain the following estimates for the oscillation energy W $= (1/64\pi^4)\int d\mathbf{k}|E_k|^2$:

$$W \approx \frac{n_b m^2c^3\tau'}{2|\Delta p_z|} \sim n_b c|\Delta p_z| \sim \begin{cases} n_b mc^2(n_b/n_p)^{1/3}\gamma_0^2 & \text{if} \quad (n_b/n_p)^{1/3}\gamma_0 \ll 1, \\ n_b mc^2\gamma_0 & \text{if} \quad (n_b/n_p)^{1/3}\gamma_0 \gtrsim 1. \end{cases} \qquad (39)$$

4) The transverse magnetic field produced by the beam current causes the diffusion lines to rotate in a plane perpendicular to the magnetic field, but if the first condition of (5) is satisfied this process is slower than the diffusion of the particles along the lines η = const.

The excited oscillations propagate almost parallel to the beam ($k_\perp \ll k_z$), and the transverse velocity arising upon diffusion of the beam particles is small. Determining $\bar{v}_\perp$ from (34′) and $k_\perp$ from the condition $k_\perp\bar{v}_\perp \sim |\tilde{\omega}|$, and replacing $|\tilde{\omega}|$ by δ for the oscillations with the maximum increment, we get in the case when $(n_b/n_p)^{1/3}\gamma_0 \ll 1$

$$\frac{k_\perp}{k_z} \approx \sqrt{\frac{\delta}{\omega_p}\frac{p_0}{|\Delta p_z|}} \approx \frac{1}{\gamma_0}, \quad \bar{v}_\perp \approx \left(\frac{n_b}{n_p}\right)^{1/3}c. \qquad (40)$$

Thus, with the exception of the fastest initial stage, in which the transverse diffusion plays a principal role and the oscillation energy remains much smaller than the beam energy, the beam relaxation occurs mainly in analogy with the relaxation by one-dimensional oscillations, which was considered in the preceding section. The principal role is played in this case by the longitudinal diffusion of the beam particles. Owing to the large longitudinal mass, the transverse-velocity scatter in the beam increases slowly, and the energy of the oscillations excited by a beam that is monochromatic with respect to the longitudinal velocity may amount, according to (39), to an appreciable fraction of its energy. Nonlinear stabilization of the instability is possible with further relaxation of a beam with a smeared longitudinal-velocity distribution ($k_z|\Delta v_z| > \delta$), just as in the preceding section.

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