INTENSITY AND POLARIZATION OF RADIATION MULTIPLY SCATTERED BY FREELY ORIENTED PARTICLES OR MEDIUM FLUCTUATIONS

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Submitted April 14, 1969

Zh. Eksp. Teor. Fiz. 57, 988–1001 (September, 1969)

Multiple scattering of light by freely-oriented particles (electrons, atoms, molecules, macroscopic particles) and medium fluctuations are considered. The intensity and polarization of radiation from a point isotropic or anisotropic source in an infinite medium, and the polarization of radiation scattered by a layer of large optical thickness are considered.

1. INTRODUCTION

THE problem of the propagation of light in a medium consisting of individual scatterers (electrons, atoms, molecules, macroscopic particles) whose dimensions are much less than the wavelength of the incident radiation has great value in physics and astrophysics. The problem of the passage of light through such a medium with small optical thickness has been considered frequently.[1,2] The problem of the light intensity passing through an optically thick medium was also considered.[3,4] The problem of the polarization of light in such a medium, as the result of multiple scattering, has been investigated in much less detail. The problem of the polarization of an unbounded light beam in multiple scattering by free electrons in a plane layer of material has been studied most fully by Chandrasekhar.[1] He obtained equations which give a practical possibility of carrying out numerical calculations only for the case of not very large optical thickness, $\tau \lesssim 3$. For optical thicknesses $\tau \sim 1$, a similar problem has also been solved numerically by Germogenova.[5] It may seem that the problem of light polarization in the passage through a thick layer of material is not of interest, since the polarization becomes very small as the result of diffusion of the radiation, even in the case when it is large for a single scattering act. Actually, inside the medium and far from its boundaries, at points where radiation enters from all directions, the polarization is averaged and become small. Such an averaging does not take place close to the boundaries of the medium or near inhomogeneities. At the boundary of a homogeneous medium, the polarization is essentially determined by the last scattering acts. Therefore the polarization of the outgoing radiation can achieve an appreciable value even in the case of great optical thickness of the target. For example, polarization of light scattered by electrons and molecules in optically thick atmospheres of certain stars has been observed. This made it possible to obtain important information on the physical conditions in these atmospheres.[6-8]

The problem of light scattering on fluctuations of density and anisotropy in liquids and amorphous solids is also very important. Investigation of the polarization of scattered radiation gives necessary information on many properties of solid and liquid bodies.[9,10]

In the present research, the problem of multiple scattering of light on freely-oriented systems (in correspondence with the terminology of[11]) i.e., on electrons, atoms, molecules or macroscopic particles not present in the external field, has been considered in detail. Analytic expressions will be found for the polarization and the intensity of radiation in multiple scattering. The resultant formulas are suitable for the description of scattering by free electrons, atoms and molecules of gases, and in a number of cases, also by fluctuations in liquids and amorphous solids. We shall also consider scattering of collimated beam of light passing through a thick layer of material. We shall limit ourselves here to the Rayleigh scattering law. The case of scattering by particles with dimensions of the order of and greater than the wavelength of the incident radiation (dust particles, fog particles, colloidal particles) will be considered in a separate work.

2. THE TRANSPORT EQUATION FOR THE PHOTON DENSITY MATRIX

Let us consider the passage through a medium of a plane electromagnetic wave with length λ much greater than the dimensions of the scattering particles. We shall not assume that the particles possess special symmetry properties (including in our consideration, for example, extended dust particles, molecules with spin, etc.), but shall limit ourselves to the case in which they are randomly oriented in space, so that the medium is in the mean homogeneous and isotropic. If the average concentration of particles N_0 is such that there are many particles in a region with dimensions $\sim\lambda$, then the scattering is determined by the fluctuations of the dielectric susceptibility $\epsilon_{ik} = \epsilon\delta_{ik} + \Delta\epsilon_{ik}$, which are caused by the fluctuations of density and an isotropy in the volume element. For rarefied media ($N_0\lambda^3 \ll 1$) the scattering takes place on the individual particles which, as also the fluctuations of the medium, can be characterized by the polarizibility tensor α_{ik}. We shall neglect the change of frequency of the photons in the scattering process. This is admissible if the total width of the excited layer of particles (including the Doppler shift) is much less than the energy of the photon $\hbar\omega$ and less than the resolving width of the detector. For a dense medium, the analogous condition is $\omega t \gg 1$, where t is the mean lifetime of the fluctuations in the medium.

We represent the fluctuation of the tensor of the dielectric susceptibility $\Delta\epsilon_{ik}$ in the form of a sum of scalar, symmetric and antisymmetric components:

$$\Delta\varepsilon_{ik} = \Delta\varepsilon\delta_{ik} + \Delta\varepsilon_{ik}^{(s)} + \Delta\varepsilon_{ik}^{(a)}, \quad \mathrm{Sp}\,\Delta\varepsilon^{(s)} = 0.$$

The scalar $\Delta\epsilon$ determines the fluctuation of ϵ_{ik} associated with the fluctuations of density and temperature. The components $\Delta\epsilon_{ik}^{(n)} = \Delta\epsilon_{ki}^{(S)}$ and $\Delta\epsilon_{ik}^{(a)} = -\Delta\epsilon_{ki}^{(a)}$ describe anisotropy fluctuations which disappear for isotropic scattering particles $(\sum_i \Delta\epsilon_{ii}^{(S)} = 0)$. The anisotropy fluctuations in a volume V_0 containing N scattering particles are represented in terms of the symmetric and antisymmetric components of the polarizability tensor of the individual particle:

$$V_0\Delta\varepsilon_{ik}^{(s)} = 4\pi\sum_{n=1}^{N}\alpha_{ik}^{(s)}(n), \quad V_0\Delta\varepsilon_{ik}^{(a)} = 4\pi\sum_{n=1}^{N}\alpha_{ik}^{(a)}(n). \tag{1}$$

Summation in (1) is carried out over all the particles in the volume V_0. All discussions remain valid if there is only a single particle on the average in the volume V_0.

The transport equation for the radiation density matrix

$$\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = (c/8\pi)E_\alpha(\mathbf{n}_1\mathbf{r})E_\beta^*(\mathbf{n}_1\mathbf{r}),$$

for radiation propagating along the direction n_i has the form[12]

$$(\mathbf{n}_1\nabla)\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = -\varkappa_0\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) \times\frac{1}{V_0}\int d\mathbf{n}_x\,\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_x)\rho_{\gamma\nu}(\mathbf{n}_x\mathbf{r})t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n}_x)\rangle_{V_0} + R_{\alpha\beta}(\mathbf{n}_1\mathbf{r}). \tag{2}$$

Here κ_0 is the extinction coefficient of radiation associated with the imaginary part of the index of refraction n by the relation $\kappa_S = 2(\omega/c)\,\mathrm{Im}\,n$, $t_{\alpha\beta}(n_1n_0) = V_0\pi\lambda^{-2}M_{\alpha i}(n_0n_1)\Delta\epsilon_{i\beta}$ is the light scattering matrix due to the fluctuations in the volume V_0; the matrix $M_{ik}(n_0n_1) = e_i^{*(1)}e_k^{(0)}$ connects the components of the vector in the set of coordinates with the z axis along n_0 and unit vectors $e_k^{(0)}$ with its components in the set of coordinates with the z axis along n_1 and unit vectors $e_i^{(1)}$; the brackets $\langle\,\rangle_{V_0}$ denote statistical averaging of the quantities inside them over the volume V_0 and also averaging over the orientations of the particles; $R_{\alpha\beta}(n_1\mathbf{r})$ is a function describing the radiation sources. Greek indices can take on two values, α = x, y and Latin, three, i = x, y, z. Summation is understood over all repeated indices.

We shall henceforth denote the integral term in (2) by $B_{\alpha\beta}(n_1\mathbf{r})$. The sum of the diagonal elements of the matrix $B_{\alpha\beta}(n_1\mathbf{r})$ determines the radiation density $\mathcal{G}_0(n_1\mathbf{r})$ (erg-cm^{-3} sec^{-1}-sr^{-1} at the point $\mathbf{r}$. An integral equation can be written for this matrix if the Green's function of the transport equation is used:

$$[(\mathbf{n}_1\nabla) + \varkappa_0]G(\mathbf{r}-\mathbf{r}';\,\mathbf{n}_1) = \delta(\mathbf{r}-\mathbf{r}'). \tag{3}$$

For a homogeneous, unbounded medium, the Green's function has the form

$$G(\mathbf{r};\,\mathbf{n}) = \delta(\mathbf{n}-\mathbf{n}_r)r^{-2}\exp(-\varkappa_0|\mathbf{r}|).$$

With its help, we obtain the following integral transport equation:

$$B_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = B_{\alpha\beta}^{(1)}(\mathbf{n}_1\mathbf{r}) + \frac{1}{V_0}\int_{(V)} d\mathbf{r}'\,\frac{\exp\{-\varkappa_0|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|^2} \times\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_{\mathbf{r}-\mathbf{r}'})B_{\gamma\nu}(\mathbf{n}_{\mathbf{r}-\mathbf{r}'};\,\mathbf{r}')t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n}_{\mathbf{r}-\mathbf{r}'})\rangle_{V_0}, \tag{4}$$

$B_{\alpha\beta}^{(1)}(n_1\mathbf{r})$ is obtained from the integral term if we replace $R_{\gamma\nu}$ in it for $B_{\gamma\nu}$. Equation (4) is applicable for homogeneous media of volume V with convex boundaries.

It is easier mathematically to solve not Eq. (4) for $B_{\alpha\beta}(n_1\mathbf{r})$ but the equation for the Fourier transform of this matrix in the variable $\mathbf{r}$, for which we have

$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{u}) = \int d\mathbf{r}\exp(-i\mathbf{u}\mathbf{r})B_{\alpha\beta}(\mathbf{n}_1\mathbf{r}); \quad F(\mathbf{q}) = \int_{(V)} d\mathbf{r}\exp(-i\mathbf{q}\mathbf{r}),$$
$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{u}) = K_{\alpha\beta}^{(1)}(\mathbf{n}_1\mathbf{u}) + \frac{1}{V_0(2\pi)^3}\int d\mathbf{w}\int d\mathbf{n}_x\frac{F(\mathbf{u}-\mathbf{w})}{\varkappa_0 + i\mathbf{n}_x\mathbf{w}} \times\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_x)K_{\gamma\nu}(\mathbf{n}_x\mathbf{w})t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n}_x)\rangle_{V_0}. \tag{5}$$

To date, we have not specified that system of orthogonal unit vectors in which the amplitude $E_\alpha(n_1\cdot\mathbf{r})$ is calculated, and consequently, the density matrices $\rho_{\alpha\beta}(n_1\cdot\mathbf{r})$ and $B_{\alpha\beta}(n_1\cdot\mathbf{r})$. However, the coupling of the elements of the density matrix $\rho_{\alpha\beta}(n_1\cdot\mathbf{r})$ or $B_{\alpha\beta}(n_1\cdot\mathbf{r})$ with the Stokes parameters $I_i(n_1\cdot\mathbf{r})$ or $\mathcal{G}_i(n_1\cdot\mathbf{r})$ (i = 0, 1, 2, 3) observed experimentally depends on the specific choice of the set of unit vectors. Only the intensity $I_0(n_1\cdot\mathbf{r})$ and the radiation density $\mathcal{G}_0(n_1\cdot\mathbf{r})$ are always sums of the diagonal elements of the matrices $\rho_{\alpha\beta}(n_1\cdot\mathbf{r})$ and $B_{\alpha\beta}(n_1\cdot\mathbf{r})$, regardless of the specific choice of unit vectors. We recall that $I_2(\mathcal{G}_2)$ describes circularly polarized light, and I_1 and $I_3(\mathcal{G}_1$ and $\mathcal{G}_3)$, linear polarization.[1,11,13] The parameters I_0 and $I_2(\mathcal{G}_0$ and $\mathcal{G}_2)$ are invariant under rotations of the unit vectors e_x and e_y, in contrast with I_1 and $I_3(\mathcal{G}_1$ and $\mathcal{G}_3)$, which in this case transform into one another (for more details, see[1,13]).

One can write down directly transport equations similar to Eqs. (2) and (4) for the Stokes parameters I_i and $\mathcal{G}_i$, which has been done, for example, in[1]. Here we obtain very complicated systems of interlocking equations, in which it is difficult to find any symmetry properties which permit us to simplify the solution. It is much more convenient to use the cyclic unit vectors $\sqrt{2}\kappa_{-1} = -e_x - ie_y$, $\kappa_0 = e_z$, $\sqrt{2}\kappa_{+1} = e_x - ie_y$, and to solve Eqs. (2), (4), (5) for the density matrices $\rho_{\alpha\beta}$, $B_{\alpha\beta}$, $K_{\alpha\beta}$, using here the powerful tool of rotational functions (Wigner functions) $D_{mn}^{(l)}(\alpha\beta\gamma)$.[11,14] The connection of the Stokes parameters with the elements $\rho_{\alpha\beta}$ and $B_{\alpha\beta}$ is the following:

$$\rho_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} I_0 + I_2, & -I_3 + iI_1 \\ -I_3 - iI_1, & I_0 - I_2\end{pmatrix}; \quad B_{\alpha\beta} = \frac{1}{2}\begin{pmatrix}\mathcal{G}_0 + \mathcal{G}_2, & -\mathcal{G}_3 + i\mathcal{G}_1 \\ -\mathcal{G}_3 - i\mathcal{G}_1, & \mathcal{G}_0 - \mathcal{G}_2\end{pmatrix}. \tag{6}$$

As is known,[11,14] the cyclic components of the vector, in the transition to a new (primed) set of coordinates, having relative initial Euler angles α, β, γ are transformed with the help of the matrix

$$D_{mn}^{(1)}(\alpha\beta\gamma) = \exp[i(m\alpha + n\gamma)]\,d_{mn}^{(1)}(\beta),$$

where

$$2d_{-1-1}^{(1)} = 1 + \cos\beta, \quad \sqrt{2}\,d_{-10}^{(1)} = -\sin\beta, \quad 2d_{-11}^{(1)} = 1 - \cos\beta:$$
$$A_m' = D_{nm}^{*(1)}(\alpha\beta\gamma)A_n \equiv D_{mn}^{(1)+}(\alpha\beta\gamma)A_n; \quad n, m = 0, \pm 1. \tag{7}$$

The elements of the tensor of second rank α_{ik} are transformed similar to the product of components of the vector $A_iA_k^*$. In what follows, our Latin indices take on the values 0, +1, and the Greek, only +1. Moreover, for brevity, we shall denote by Ω_{01} the set of Euler angles for the transition from the set of coordinates with z axis along $\mathbf{n}_0$ to the set of coordinates with the z axis along $\mathbf{n}_1$.

In cyclic coordinates, we have $t_{\alpha\beta}(\mathbf{n}_1\mathbf{n}_0) = \pi\lambda^{-2}D^{(1)+}_{\alpha i}(\Omega_{01})\Delta\epsilon_{i\beta}$, and for the value of $V_0^{-1}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_x)t_{\nu\beta}(\mathbf{n}_1\mathbf{n}_x)\rangle_{V_0}$ entering into (2) and (5), we have

$$A_{\alpha\gamma\nu\beta}(\mathbf{n}_1\mathbf{n}_x) \equiv \frac{1}{V_0}\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_x)\,t_{\nu\beta}^+(\mathbf{n}_1\mathbf{n}_x)\rangle_{V_0} \tag{8}$$
$$= \left(\frac{\omega}{c}\right)^4 D^{(1)+}_{\alpha i}(\Omega_{x1})\left[\frac{1}{16\pi^2}V_0\langle\Delta\varepsilon^2\rangle_{V_0}\delta_{i\gamma}\delta_{\nu k} + N_0\langle\alpha^{(s)}_{i\gamma}\,\alpha^{(s)+}_{\nu k}\rangle \times N_0\langle\alpha^{(a)}_{i\gamma}\,\alpha^{(a)+}_{\nu k}\rangle\right]D^{(1)}_{k\beta}(\Omega_{x1})$$
$$= D^{(1)+}_{\alpha i}(\Omega_{x1})[\delta_{i\gamma}\delta_{\nu k}(a^{(0)} - a^{(s)}/15) + \delta_{ik}\delta_{\gamma\nu}(a^{(s)}/10 + a^{(a)}/6) \times \varphi_{i\nu}\varphi_{\gamma k}(a^{(s)}/10 - a^{(a)}/6)]D^{(1)}_{k\beta}(\Omega_{x1}).$$

Here, we have used the notation

$$a^{(0)} = \frac{V_0\omega^4}{16\pi^2c^4}\langle\Delta\varepsilon^2\rangle_{V_0} = \frac{\omega^4}{16\pi^2c^4}\left[\left(\frac{\partial\varepsilon}{\partial\rho}\right)^2_T\left(\frac{\partial\rho}{\partial p}\right)_T\rho T + \frac{T^2}{\rho c_V}\left(\frac{\partial\varepsilon}{\partial T}\right)\right],$$
$$a^{(s)} = N_0\left(\frac{\omega}{c}\right)^4\sum_{i,k}|\alpha^{(s)}_{ik}|^2,\quad a^{(a)} = N_0\left(\frac{\omega}{c}\right)^4\sum_{i,k}|\alpha^{(a)}_{ik}|^2, \tag{9}$$

ρ, p, T are the density, pressure and temperature of the medium, c_V the specific heat per unit mass of the medium, $\varphi_{-11} = \varphi_{1-1} = -\varphi_{00} = -1$, while the remaining φ_{ik} are equal to zero. Equation (9) is obtained upon completion of the averaging $\langle\ \rangle_{V_0}$.

Using Eq. (8), it is not difficult to write Eqs. (2) and (5) in cyclic coordinates:

$$(\mathbf{n}_1\nabla)\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = -\varkappa_0\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) + \int d\mathbf{n}_x A_{\alpha\gamma\nu\beta}(\mathbf{n}_1\mathbf{n}_x)\rho_{\gamma\nu}(\mathbf{n}_x\mathbf{r}) + R_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) \tag{10}$$

$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) = K^{(1)}_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) + \frac{1}{(2\pi)^3\varkappa_0}\int d\mathbf{q}\int d\mathbf{n}_x F(\mathbf{s}-\mathbf{q})(1+i\mathbf{n}_x\mathbf{q})^{-1} \times A_{\alpha\gamma\nu\beta}(\mathbf{n}_1\mathbf{n}_x)K_{\gamma\nu}(\mathbf{n}_x\mathbf{q}). \tag{11}$$

In (11), we have introduced the dimensionless variables $\mathbf{s} = \mathbf{u}/\kappa_0$, $\mathbf{q} = \mathbf{w}/\kappa_0$.

For the quantity $\mathscr{I}_2(\mathbf{n}_1\cdot\mathbf{s}) = K_{-1-1}(\mathbf{n}_1\cdot\mathbf{s}) - K_{11}(\mathbf{n}_1\cdot\mathbf{s})$, a separate equation is obtained from (11):

$$\mathscr{I}_2(\mathbf{n}_1\mathbf{s}) = \mathscr{I}^{(1)}_2(\mathbf{n}_1\mathbf{s}) + \frac{b_0}{(2\pi)^3}\int d\mathbf{q}\int d\mathbf{n}_x F(\mathbf{s}-\mathbf{q})(1+i\mathbf{q}\mathbf{n}_x)^{-1}D^{(1)}_{00}(\Omega_{x1})\mathscr{I}_2(\mathbf{n}_x\mathbf{q}), \tag{12}$$

where $\kappa_0 b_0 = a^{(0)} + (1/6)(a^{(a)} - a^{(s)})$. Thus if sources of circular polarization are absent, then it does not arise even in the process of multiple scattering. We note that the first separate equation for $I_2(\mathbf{n}_1\cdot\mathbf{r})$ was evidently written by Chandrasekhar.[1]

Inasmuch as a separate simple equation (12) has been isolated for $\mathscr{I}_2(\mathbf{n}_1\mathbf{s})$, then we shall also solve just this equation for determination of the circular polarization. The quantities $\mathscr{I}_0$, $\mathscr{I}_1$ and $\mathscr{I}_2$ will be dedetermined from the solution of the matrix equation (11) in which we shall now formally assume $\mathscr{I}_2 \equiv K_{-1-1} - K_{11} = 0$, because the parameters $\mathscr{I}_0$, $\mathscr{I}_1$ and $\mathscr{I}_3$ are not interconnected with $\mathscr{I}_2$ in this equation. Here Eq. (11) is greatly simplified, since for $K_{-1-1} = K_{11}$ the term with $\varphi_{i\nu}\varphi_{\gamma k}$ is joined with the first term in Eq. (8):

$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) = K^{(1)}_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) + (2\pi)^{-3}\int d\mathbf{q}\int d\mathbf{n}_x F(\mathbf{s}-\mathbf{q})(1+i\mathbf{n}_x\mathbf{q})^{-1} \times[b_2\delta_{\alpha\beta}K_{\gamma\gamma}(\mathbf{n}_x\mathbf{q}) + b_1 D^{(1)+}_{\alpha\gamma}(\Omega_{x1})K_{\gamma\nu}(\mathbf{n}_x\mathbf{q})D^{(1)}_{\nu\beta}(\Omega_{x1})], \tag{13}$$

where

$$\varkappa_0 b_1 = a^{(0)} + {}^1/_{30}(a^{(s)} - 5a^{(a)}),\quad \varkappa_0 b_2 = {}^1/_{10}a^{(s)} + {}^1/_6 a^{(a)}.$$

Thus, to find $\mathscr{I}_2(\mathbf{n}_1\cdot\mathbf{s})$, it is necessary to solve Eq. (12), and to find $\mathscr{I}_0$, $\mathscr{I}_1$ and $\mathscr{I}_3$, Eq. (13), in the free term of which there is no $\mathscr{I}^{(1)}_2(\mathbf{n}_1\cdot\mathbf{s})$.

When $b_2 = 0$, Eq. (13) describes the scattering by a spherical particle. The term with b_2 leads to depolarization of the radiation, associated with averaging over the orientations of the scattering particles. Thus, it is seen from Eq. (13) that the set of freely oriented particles does not scatter light in the same way as a system of spherical particles, no matter what their radius might be: Attention is called to the fact that in the solution of the problem of light scattering by dust particles, formulas describing the scattering by spherical particles (corresponding to $b_2 = 0$) are often used invalidly. At the same time, for particles of greatly elongated shape and particles of flattened shape, the ratio b_1/b_2 is equal to two and seven, respectively.

Equations (12) and (13) describe scattering by free, anisotropic molecules which form a strongly rarefied gaseous medium. In this case we denote by $a^{(0)}$ the quantity $N_0(\omega/c)^4|3^{-1}\sum_i\alpha_{ii}|^2$ ($(1/3)\sum\alpha_{ii}$ is the mean polarizability of the molecule).

3. POINT ANISOTROPIC SOURCE OF POLARIZED RADIATION IN AN UNBOUNDED MEDIUM

The advantages of Eq. (13) for the matrix $K_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{s})$ over the similar equation for the quantities $\mathscr{I}_i(\mathbf{n}_1\cdot\mathbf{s})$ are clearly evident already in the solution of a very simple problem in the theory of scattering—the problem of the scattering of radiation emitted by a point source which is located in an unbounded homogeneous medium. To solve the equations for $\mathscr{I}_i(\mathbf{n}_1\cdot\mathbf{s})$ is practically impossible because of their complexity (one must compute the determinants of high order). At the same time, as we shall see, Eqs. (12) and (13) are very simply solved in this case. In the case of scattering of radiation by a flat layer, Eqs. (12) and (13) are also comparatively simple.

In the case of a point anisotropic source, the matrix $R_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{r})$ in (2) has the form

$$R_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) \equiv B^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = \delta(\mathbf{r})\rho^{(0)}_{\alpha\beta}(\mathbf{n}_1).$$

As a free term of Eq. (4) we then have the density matrix $B^{(1)}_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{r})$ of the radiation, which reaches a depth r and is first scattered there:

$$B^{(1)}_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = \varkappa_0 r^{-2}\exp(-\varkappa_0 r)[b_1 D^{(1)+}_{\alpha\gamma}(\Omega_{r1})\rho^{(0)}_{\gamma\nu}(\mathbf{n}_r)D^{(1)}_{\nu\beta}(\Omega_{r1}) + b_2\delta_{\alpha\beta}\rho^{(0)}_{\gamma\gamma}(\mathbf{n}_r)]. \tag{14}$$

In order to obtain the free term of Eq. (13), it is necessary to calculate the Fourier transform in r of (14). Taking also into account the fact that for an unbounded medium, $F(\mathbf{u}-\mathbf{w}) = (2\pi)^3\delta(\mathbf{u}-\mathbf{w})$, we obtain the following expression for $K_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{s})$:

$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) = K^{(1)}_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) + \int d\mathbf{n}_x[b_1 D^{(1)+}_{\alpha\gamma}(\Omega_{x1})K_{\gamma\nu}(\mathbf{n}_x\mathbf{s})D^{(1)}_{\nu\beta}(\Omega_{x1}) \times b_2\delta_{\alpha\beta}K_{\gamma\gamma}(\mathbf{n}_x\mathbf{s})](1+i\mathbf{s}\mathbf{n}_x)^{-1}, \tag{15}$$

where $K^{(1)}_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{s})$ is determined by an expression similar to the integral term of the right hand side of (15), if we replace $K_{\gamma\nu}$ by $\rho^{(0)}_{\gamma\nu}$ in it.

Using the group property of D functions, we separate the angular dependence in $K_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{s})$ on the vector $\mathbf{n}_1$:

$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) = b_1 D^{(1)+}_{\alpha n}(\Omega_{s1}) D^{(1)}_{m\beta}(\Omega_{s1}) H_{mn}(\mathbf{s}) + b_2\delta_{\alpha\beta}H(\mathbf{s}). \tag{16}$$

Here

$$H_{mn}(\mathbf{s}) = \Gamma^{(0)}_{mn}(\mathbf{s}) + \Gamma_{mn}(\mathbf{s});\quad H(\mathbf{s}) = \Gamma^{(0)}(\mathbf{s}) + \Gamma(\mathbf{s}),$$

where

$$\Gamma^{(0)}_{mn}(\mathbf{s}) = \int d\mathbf{n}_x D^{(1)}_{n\gamma}(\Omega_{sx})\rho^{(0)}_{\gamma\nu}(\mathbf{n}_x) D^{(1)+}_{\nu m}(\Omega_{sx})(1+i\mathbf{s}\mathbf{n}_x)^{-1},$$
$$\Gamma^{(0)}(\mathbf{s}) = \int d\mathbf{n}_x \rho^{(0)}_{\gamma\gamma}(\mathbf{n}_x)(1+i\mathbf{s}\mathbf{n}_x)^{-1}; \tag{17}$$

$\Gamma_{mn}(\mathbf{s})$ and $\Gamma(\mathbf{s})$ are expressed by the formulas (17), in which we must substitute $K_{\gamma\nu}(\mathbf{n}_x\cdot\mathbf{s})$ and $K_{\gamma\gamma}(\mathbf{n}_x\cdot\mathbf{s})$ in place of $\rho^{(0)}_{\gamma\nu}(\mathbf{n}_x)$ and $\rho^{(0)}_{\gamma\gamma}(\mathbf{n}_x)$. According to Eqs. (15) and (16), we obtain in the following algebraic equations for the matrices $H_{mn}(\mathbf{s})$ and $H(\mathbf{s})$:

$$H_{kl}(\mathbf{s}) = \Gamma^{(0)}_{kl}(\mathbf{s}) + b_1 B_{nlkm}(s^2) H_{mn}(\mathbf{s}) + b_2 N_l \delta_{kl} H(\mathbf{s}), \tag{18}$$
$$H(\mathbf{s}) = \Gamma^{(0)}(\mathbf{s}) + b_1 N_n(s) H_{nn}(\mathbf{s}) + 8\pi D_0(s^2) b_2 H(\mathbf{s}),$$
$$N_0(s) = 4\pi[D_0(s^2) - D_2(s^2)];$$
$$N_1(s) = N_{-1}(s) = 2\pi[D_0(s^2) + D_2(s^2)].$$

The functions $D_n(s^2)$ have the following explicit form:

$$D_n(s^2) = \int_0^1 dx\, x^n (1+s^2x^2)^{-1},\quad D_0(s^2) = s^{-1}\operatorname{arc\,tg} s, \tag{19}$$

$$D_2(s^2) = s^{-2} - s^{-3}\operatorname{arctg} s,\quad D_4(s^2) = \frac{1}{s^2}\left[\frac{1}{3} - \frac{1}{s^2} - \frac{1}{s^3}\operatorname{arctg} s\right].$$

The matrix $B_{mlkm}(s^2)$ possesses a high degree of symmetry and is equal to

$$B_{nlkm}(s^2) = \int d\mathbf{n}_x D^{(1)}_{l\alpha}(\Omega_{sx}) D^{(1)+}_{\alpha n}(\Omega_{sx}) D^{(1)}_{m\beta}(\Omega_{sx}) D^{(1)+}_{\beta k}(\Omega_{sx})(1+i\mathbf{s}\mathbf{n}_x)^{-1},$$
$$B_{nlkm}(s^2) = \delta_{ln}\delta_{km}B^{(1)}_{lk}(s^2) + \delta_{kl}\delta_{nm}B^{(2)}_{ln}(s^2). \tag{20}$$

The symmetric matrices $B^{(1)}_{lk}(s^2)$ and $B^{(2)}_{lk}(s^2)$ have the following form:

$$B^{(1)}_{mn} = \pi[D_0 + 2D_2 + D_4],\quad m, n = -1, 1; \tag{21}$$

$$\left.\begin{aligned} B^{(1)}_{mn} &= 2\pi[D_0 - D_4] \\ B^{(2)}_{mn} &= 2\pi[D_2 - D_4] \end{aligned}\right\} m = -1, 1,\ n = 0 \text{ or } m = 0,\ n = -1, 1;$$

$$B^{(1)}_{00} = 4\pi[D_0 - 2D_2 + D_4],\quad B^{(2)}_{nn} = 0,\quad B^{(2)}_{mn} = {}^1/_4 B^{(1)}_{00},\quad m, n = -1, 1.$$

Substituting (20) in (18), we immediately obtain the solution for the nondiagonal elements of the matrix H_{mn}:

$$H_{mn}(\mathbf{s}) = \Gamma^{(0)}_{mn}(\mathbf{s})[1 - b_1 B^{(1)}_{mn}(s^2)]^{-1},\quad m \neq n. \tag{22}$$

According to what was pointed out at the end of Sec. 2, one can assume that $\rho^{(0)}_{-1-1} = \rho^{(0)}_{11}$ in the solution of Eq. (15). This leads to the equality $H_{11} = H_{-1-1}$. For the elements H_{00}, H_{11} and H, a set of three equations is obtained whose solution is of the form

$$H_{11} = H_{-1-1} = \Delta_1\Delta^{-1},\quad H_{00} = \Delta_2\Delta^{-1},\quad H = H_{00} + 2H_{11}, \tag{23}$$

where the corresponding determinants of the system are equal to

$$\Delta(s^2) = 1 - 2\pi b_1[3D_0 - 4D_2 + 3D_4] - 8\pi b_2 D_0$$
$$3\pi(1-p) b_1[D_0^2 - D_2^2 - 2D_0D_2 + 2D_0D_4],$$
$$\Delta_1(\mathbf{s}) = \Gamma^{(0)}_{11}(\mathbf{s})[1 - 4\pi b_1(D_0 - 2D_2 + D_4) - 4\pi b_2(D_0 - D_2)]$$
$$\times 2\pi\Gamma^{(0)}_{00}(\mathbf{s})[b_1(D_2 - D_4) + b_2(D_0 + D_2)],$$

$$\Delta_2(\mathbf{s}) = 4\pi\Gamma^{(0)}_{11}(\mathbf{s})[b_1(D_2 - D_4) + 2(D_0 - D_2)]$$
$$\times\Gamma^{(0)}_{00}(\mathbf{s})[1 - 2\pi b_1(D_0 + D_4) - 4\pi b_2(D_0 + D_2)].$$

Here we have introduced the new quantity p, which is the degree of actual quantum absorption in the scattering; it is determined by the ratio of the scattering extinction coefficient κ_0:

$$1 - p = \varkappa_s\varkappa_0^{-1} = {}^8/_3\pi(b_1 + 3b_2). \tag{25}$$

The equation $H(\mathbf{s}) = H_{00}(\mathbf{s}) + 2H_{11}(\mathbf{s})$ is obtained from Eq. (18) under assumption of the equality $\Gamma^{(0)}(\mathbf{s}) = \Gamma^{(0)}_{00}(\mathbf{s}) + 2\Gamma^{(0)}_{11}(\mathbf{s})$, which is not difficult to obtain by starting out from (17). The function $\Delta(s^2)$ behaves in the following manner for small s and p:

$$\Delta(s^2) \approx (s^2 + 3p)(b_1 + 10b_2)4\pi/15,\quad s^2 \ll 1,\ p \ll 1. \tag{26}$$

Equations (16), (22) and (23) give an exact solution of the problem for the functions $\mathscr{G}_0(\mathbf{n}_1\cdot\mathbf{s})$, $\mathscr{G}_1(\mathbf{n}_1\cdot\mathbf{s})$ and $\mathscr{G}_3(\mathbf{n}_1\cdot\mathbf{s})$. To determine $\mathscr{G}_2(\mathbf{n}_1\cdot\mathbf{s})$ we use Eq. (12). The solution of this equation by a similar method leads to the following result:

$$\mathscr{G}_2(\mathbf{n}_1\mathbf{s}) = b_0 D^{(1)}_{m0}(\Omega_{s1})\Gamma^{(0)}_m(\mathbf{s})[1 - b_0 C_m(s^2)]^{-1},$$
$$C_0 = 4\pi D_2,\ C_1 = C_{-1} = 2\pi[D_0 - D_2], \tag{27}$$
$$\Gamma^{(0)}_m(\mathbf{s}) = \int d\mathbf{n}_x D^{(1)+}_{0m}(\Omega_{sx})\mathscr{G}^{(0)}_2(\mathbf{n}_x)(1 + i\mathbf{s}\mathbf{n}_x)^{-1},$$
$$\mathscr{G}^{(0)}_2(\mathbf{n}) = \rho^{(0)}_{-1-1}(\mathbf{n}) - \rho^{(0)}_{11}(\mathbf{n}).$$

The desired density matrix $B_{\alpha\beta}(\mathbf{n}_1\mathbf{r})$ is obtained after carrying out the inverse Fourier transformation of $K_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{s})$:

$$B_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = \frac{\varkappa_0^3}{(2\pi)^3}\int d\mathbf{s}\exp\{i\varkappa_0\mathbf{r}\mathbf{s}\}K_{\alpha\beta}(\mathbf{n}_1\mathbf{s}). \tag{28}$$

In taking the integrals (28), the behavior of the denominators in the formulas for the coefficients $H_{mn}(\mathbf{s})$, $H(\mathbf{s})$ and also the behavior of the function $[1 - b_0C_m(s^2)]$ in (27) are important. It is easy to show that for small s only function $\Delta(s^2)$ has roots. Their approximate values, in accord with (26), are equal to $s_{1,2}\kappa \pm i\sqrt{3p}$, where we assume $p \ll 1$. These are the so-called diffusion roots. Radiation becomes evidently diffuse far away from the source $\kappa_0 r \gg 1$, but in this case important contributions to the integral (28) are made only by terms containing $\Delta(s^2)$ (small s are essential). Therefore, in the diffusion approximation, we can assume the matrix $H_{mn}(\mathbf{s})$ to be diagonal. Furthermore, in this approximation the functions $H_{11}(\mathbf{s})$ and $H_{00}(\mathbf{s})$ are almost equal to one another. Actually, we get from (24) for the difference $\Delta_1 - \Delta_2$ the expression

$$\Delta_1 - \Delta_2 = \Gamma^{(0)}_{11}[1 - {}^3/_2(1-p)(D_0 - D_2)] - \Gamma^{(0)}_{00}[1 - {}^3/_4(1-p)(D_0 + D_2)],$$

which, for small s, is of the order $(p + s^2) \sim p + (\kappa_0 r)^{-2} \ll 1$. To this accuracy, the density matrix $B_{\alpha\beta}(\mathbf{n}_1\mathbf{r})$ is diagonal in the diffusion approximation. The polarization of the radiation, which is determined by the nondiagonal elements of $B_{\alpha\beta}(\mathbf{n}_1\mathbf{r})$, is therefore small. The degree of linear polarization $\xi_3 = \mathscr{G}_3(\mathbf{n}_1\mathbf{r})\mathscr{G}_0^{-1}(\mathbf{n}_1\mathbf{r})$ is of the order of $p + (\kappa_0 r)^{-2}$. In the diffusion approximation, the circular polarization is practically absent. Thus, for purely Rayleigh (scalar) scattering, in the case $\mathscr{G}^{(0)}_2(\mathbf{n}_x) = \text{const}$ only a single term remains in (27) with $m = 0$, and its denominator $[1 - (^3/_2)(1-p)D_2(s^2)]$ has roots $s_{1,2} \approx \pm i0.85(1 - (^{14}/_{35})p)$. The contribution from these roots is of the order $r^{-1}\exp(-0.85\kappa_0 r)$ and is much smaller than the contribution from the diffusion roots in the intensity $\mathcal{G}_0(\mathbf{n}_1\mathbf{r})$, which is of the order $r^{-1}\exp(-\kappa_0 r\sqrt{3p})$.

All these qualitative conclusions will be confirmed in the subsequent sections by specific quantitative calculations.

4. POINT ISOTROPIC SOURCE IN AN UNBOUNDED MEDIUM

We apply the formulas obtained in the previous section to the analysis of a series of specific cases. As a first example, we calculate the intensity and polarization of radiation from a point isotropic source in an infinite medium. In this case, the density matrix of the initial radiation has the form $\rho^{(0)}_{\alpha\beta}(\mathbf{n}_1) = \frac{1}{2}\delta_{\alpha\beta}\mathcal{G}_0^{(0)}$. The diagonality of the matrix $\Gamma^{(0)}_{mn}(\mathbf{s})$ then follows from equation (17), and therefore, according to (22), the matrix $H_{mn}(\mathbf{s})$ will also be diagonal: $H_{mn}(\mathbf{s}) = \delta_{mn}H_n(\mathbf{s})$. Here the expression (16) for $K_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{s})$ takes the form

$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{s}) = \delta_{\alpha\beta}[(b_1+2b_2)H_1(s^2)+b_2H_0(s^2)] \times b_1[H_0(s^2)-H_1(s^2)]D^{(1)+}_{\alpha 0}(\Omega_{s1})D^{(1)}_{0\beta}(\Omega_{s1}). \tag{29}$$

The desired density matrix $B_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{r})$ is obtained from (29) after carrying out the inverse Fourier transformation (28). Substituting (29) in (28), and carrying out integration over the angles, we get the following expression for $B_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{r})$:

$$B_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = \frac{\varkappa_0^3}{2\pi^2}\left\{\delta_{\alpha\beta}\int_0^\infty ds\,s^2 j_0(\varkappa_0 rs)[(b_1+2b_2)H_1(s^2)+b_2H_0(s^2)] + \delta_{\alpha\beta}\frac{b_1}{3}\int_0^\infty ds\,s^2[j_0(\varkappa_0 rs)+j_2(\varkappa_0 rs)][H_0(s^2)-H_1(s^2)] - b_1D^{(1)+}_{\alpha 0}(\Omega_{r1})D^{(1)}_{0\beta}(\Omega_{r1})\int_0^\infty ds\,s^2 j_2(\varkappa_0 rs)[H_0(s^2)-H_1(s^2)]\right\}, \tag{30}$$

where j_0 and j_2 are spherical Bessel functions: $j_n(x) = \sqrt{\pi/2x}\,J_{n+1/2}(x)$, and the explicit form of $H_1(s^2)$ and $H(s^2)$ is the following

$$H_1(s^2) = \pi\mathcal{G}_0^{(0)}\Delta_1\Delta^{-1}, \quad H_0(s^2)-H_1(s^2) = \pi\mathcal{G}_0^{(0)}[D_0-3D_2]\Delta^{-1},$$
$$\Delta_1(s^2) = D_0+D_2-4\pi b_1[D_0^2-D_2^2-2D_0D_2+2D_0D_4]. \tag{31}$$

The dependence of the density matrix $B_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{r})$ on the angles is included in the matrix

$$D^{(1)+}_{\alpha 0}(\Omega_{r1})D^{(1)}_{0\beta}(\Omega_{r1}) = \frac{4}{3}\pi Y^*_{1\alpha}(\gamma_{r1};\beta_{r1})Y_{1\beta}(\gamma_{r1};\beta_{r1}).$$

Comparing it with (6), one can write out the expressions for $\mathcal{G}_1(\mathbf{n}_1\cdot\mathbf{r})$ and $\mathcal{G}_3(\mathbf{n}_1\cdot\mathbf{r})$. The dependence of this matrix on γ_{rl} assures the law of transformation of the Stokes parameters $\mathcal{G}_1$ and $\mathcal{G}_3$ into one another under rotation of the system of coordinates of the recording apparatus (polarimeter). Usually, one uses a system for which $\gamma_{rl} = 0$. In this system, according to (6), the parameter $\mathcal{G}_1(\mathbf{n}_1\cdot\mathbf{r})$ is equal to zero.

Under the condition $\kappa_0 r \gg 1$ (diffusion approximation), analytic expressions can be found for the density matrix $B_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{r})$. Here it is necessary to integrate over a contour in the upper half plane in (30); in this calculation, one must take into account only the residue from the diffusion pole $s = i\sqrt{3p}$, discarding the integrals around the cut from the branch point $s = i$ of the integrand. The discarded terms are $\sim(\kappa_0 r)^{-2}\exp(-\kappa_0 r)$ and can be neglected when $\kappa_0 r \gg 1$, since the diffusion terms are of the order $(\kappa_0 r)^{-1}$. We shall also neglect terms $\sim p\sqrt{p}$, p^2, which corresponds to the physically interesting case in which the actual absorption in the medium is small. As a result, we obtain the following expressions for $\mathcal{G}_0(\mathbf{n}_1\cdot\mathbf{r})$ and $\mathcal{G}_3(\mathbf{n}_1\cdot\mathbf{r})$ in the frame of reference with $\gamma_{rl} = 0$, where $\mathcal{G}_1(\mathbf{n}_1\cdot\mathbf{r}) = 0$:

$$\mathcal{G}_0(\mathbf{n}_1\mathbf{r}) = \mathcal{G}_0^{(0)}\frac{3\varkappa_0^2}{4\pi r}\exp(-\varkappa_0 r\sqrt{3p})\left\{1+\frac{b_1}{b_1+10b_2}\left[\frac{34}{21}p+\frac{2}{3}(\varkappa_0 r)^{-2} \times \frac{2}{3}\sqrt{3p}(\varkappa_0 r)^{-1} - \sin^2\beta_{r1}(p+(\varkappa_0 r)^{-2}+\sqrt{3p}(\varkappa_0 r)^{-1})\right]\right\},$$
$$\mathcal{G}_3(\mathbf{n}_1\mathbf{r}) = -\mathcal{G}_0^{(0)}\frac{3\varkappa_0^2}{4\pi r}\exp(-\varkappa_0 r\sqrt{3p})\sin^2\beta_{r1}[p+(\varkappa_0 r)^{-2} \times \sqrt{3p}(\varkappa_0 r)^{-1}]\,b_1(b_1+10b_2)^{-1}. \tag{32}$$

For the degree of linear polarization of the radiation ξ_3, we have, from (32)

$$\xi_3 = \mathcal{G}_3(\mathbf{n}_1\mathbf{r})\mathcal{G}_0^{-1}(\mathbf{n}_1\mathbf{r}) = -\frac{b_1}{b_1+10b_2}\left[p+\frac{1}{(\varkappa_0 r)^2}+\frac{\sqrt{3p}}{\varkappa_0 r}\right]\sin^2\beta_{r1}. \tag{33}$$

The integrals over the cut discarded in obtaining (32) are easily computed on an electronic computer. Such a calculation was carried out for the case of Rayleigh scattering ($b_2 = 0$, $b_1 = 3(1-p)/8\pi$) for $p = 0$, when actual absorption of photons is absent. In this case the exact formulas for $\mathcal{G}_0(\mathbf{n}_1\cdot\mathbf{r})$ and $\mathcal{G}_3(\mathbf{n}_1\cdot\mathbf{r})$ have the form

$$\mathcal{G}_0(\mathbf{n}_1\mathbf{r}) = \mathcal{G}_0^{(0)}\frac{3\varkappa_0^2}{4\pi r}\left[f_1(\varkappa_0 r)+f_2(\varkappa_0 r)\left(\frac{2}{3}-\sin^2\beta_{r1}\right)\right],$$
$$\mathcal{G}_3(\mathbf{n}_1\mathbf{r}) = -\mathcal{G}_0^{(0)}\frac{3\varkappa_0^2}{4\pi r}f_2(\varkappa_0 r)\sin^2\beta_{r1}. \tag{34}$$

The functions $f_1(\kappa_0 r)$ and $f_2(\kappa_0 r)$ are shown graphically in the range of $\kappa_0 r$ from 0.25 to 4 in Fig. 1. Figures 1 and 2 also show the graphs of $\mathcal{G}_0(\mathbf{n}_1\cdot\mathbf{r})$ and $\mathcal{G}_3(\mathbf{n}_1\cdot\mathbf{r})$ for the angles $\beta_{rl} = 0°$, 90°, and also the degree of linear polarization ξ_3 for $\beta_{rl} = 90°$. For comparison, the corresponding graphs are also given, computed from the analytic formulas (32) and (33). From these graphs may be seen that (32 are satisfac-

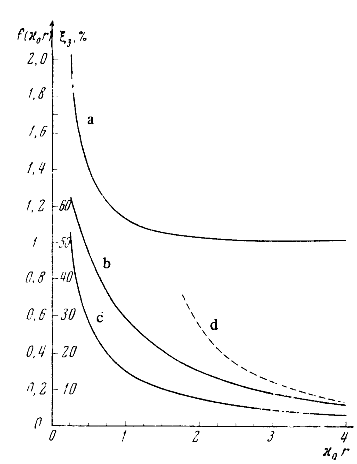

FIG. 1. Graphs of tabulated functions from 0.25 to $4\kappa_0 r$: a—$f_1(\kappa_0 r)$, b—$f_2(\kappa_0 r)$, c—degree of linear polarization $\xi_3(\mathbf{n}_1\cdot\mathbf{r})$ for $\beta_{rl} = 90°$, calculated on an electronic computer, d—degree of linear polarization of $\xi_3(\mathbf{n}_1\cdot\mathbf{r})$ for $\beta_{rl} = 90°$, calculated according to the asympotic formula (33).

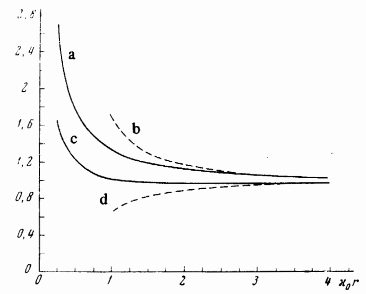

FIG. 2. Dependence of $\mathscr{G}_0(nr)\,(3\kappa_0{}^2/4\pi r\,\mathscr{G}_0{}^{(0)})^{-1}$ on $\kappa_0 r$: a and b—exact values of $\mathscr{G}_0(\mathbf{n}_1\cdot\mathbf{r})$, for $\beta_{rl} = 0^0$ and 90°, c and d—the corresponding values of $\mathscr{G}_0(\mathbf{n}_1\cdot\mathbf{r})$ calculated from the asymptotic formula (32).

torily accurate when $\kappa_0 r = 2$ for $\mathscr{G}_0(\mathbf{n}_1\cdot\mathbf{r})$ and when $\kappa_0 r = 3$ for $\mathscr{G}_3(\mathbf{n}_1\cdot\mathbf{r})$.

We now discuss the results. First of all, it is evident that in the diffusion approximation (i.e., at large distances from the sources and boundaries) the value of the polarization is insignificant. In the case of pure scattering $p = 0$, the polarization is a quantity of the order of $(\kappa_0 r)^{-2}$. Such a dependence is explained by the fact that the polarization is proportional to the gradient of the density (current), which in our case is directed from the source and which changes with distance as $(\kappa_0 r)^{-2}$. The current creates a distinct direction in the medium, which is necessary for the appearance of polarization.

In the expressions (32) and (33), it is easy to proceed to the case of Rayleigh scattering by fluctuations of the density and temperature. Here $b_2 = 0$ and $b_1 = 3(1-p)/8\pi$, as follows from Eq. (25). We note that in this case, the degree of linear polarization ξ_3 is maximum, since the factor $b_1(b_1 + 10b_2)^{-1} = 1$. This factor characterizes the decrease of the degree of polarization of the radiation under the action of the depolarizing effect of scattering by anisotropy fluctuations. Depending on the relative contribution of the different types of scattering (scalar, symmetric and antisymmetric) to the cross section, it can take on different values, but the modulus does not exceed unity. Thus, for scalar (Rayleigh) scattering ($a^{(s)} = a^{(a)} = 0$) this factor is equal to unity, for purely symmetric scattering ($a^{(0)} = a^{(a)} = 0$) it has the value $1/31$, and for antisymmetric scattering ($a^{(0)} = a^{(s)} = 0$) it becomes negative and is equal to $-1/g$. Thus, measurement of ξ_3 can serve as a means of obtaining information on the presence of one type of scattering or another in the medium.

5. POINT COLLIMATED SOURCE OF POLARIZED RADIATION IN AN UNBOUNDED MEDIUM

We now consider the problem of scattering in an unbounded medium of an infinitely thin beam of polarized photons radiated from a point source. The consideration is made on the basis of the general formulas of Sec. 3. A practical case is that of the scattering of a light ray with small optical diameter ($\kappa_0 d \ll 1$) where d is the geometric diameter of the ray. The source is given in this case in the following way:

$$B^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = \delta(\mathbf{r})\,\delta(\mathbf{n}_1 - \mathbf{n}_0)\,I^{(0)}_{\alpha\beta}(\mathbf{n}_0),$$

$I^{(0)}_{\alpha\beta}(\mathbf{n}_0)$ is the matrix of the source radiation density in the frame of reference connected with the unit vector $\mathbf{n}_0$.

We limit ourselves here to the case $\kappa_0 r \gg 1$ (diffusion approximation), when it is possible to obtain asymptotic formulas. For simplicity, we consider purely Rayleigh (scalar) scattering $b_2 = 0$, $b_1 = 3(1-p)/8\pi$. In the diffusion approximation,

$$H_{mn}(\mathbf{s}\mathbf{n}_0) = \delta_{mn} H_n(\mathbf{s}\mathbf{n}_0)\,(1 + i\mathbf{s}\mathbf{n}_0)^{-1}.$$

We obtain the expression $H_n(\mathbf{s}\cdot\mathbf{n}_0)$ from the general formulas (17) and (23). Just as in the previous section, we shall take into account in the calculation of the Fourier transform of $K_{\alpha\beta}(\mathbf{n}_1\cdot\mathbf{s})$ only the contribution of the diffusion root, neglecting the integrals over the cut. As a result, we obtain the following asymptotic formulas in the frame of reference with $\gamma_{\mathbf{r}l} = 0$, where $\mathscr{G}_1(\mathbf{n}_1\cdot\mathbf{r}\mathbf{n}_0) = 0$:

$$\mathscr{G}_0(\mathbf{n}_1\mathbf{r}\mathbf{n}_0) = \frac{3\varkappa_0{}^2}{16\pi^2 r}\exp(-\varkappa_0 r\sqrt{3p})\Big\{ I^{(0)}_{\gamma\gamma}\Big[1 + \frac{20}{21}p \times \Big(\frac{1}{\varkappa_0 r} + \sqrt{3p}\Big)\cos\beta_{r0}\Big] + \frac{2}{3}\Big(p + \frac{1}{(\varkappa_0 r)^2} + \sqrt{3p}\,(\varkappa_0 r)^{-1}\Big) \times [I^{(0)}_{\gamma\gamma}\,(4P_2(\cos\beta_{0r}) + P_2(\cos\beta_{r1})) + 3\sin^2\beta_{0r}\,\mathrm{Re}\,(I^{(0)}_{-11}e^{-2i\alpha_{r0}})]\Big\},$$

$$\mathscr{G}_3(\mathbf{n}_1\mathbf{r}\mathbf{n}_0) = -\frac{3\varkappa_0{}^2}{16\pi^2 r}\exp(-\varkappa_0 r\sqrt{3p})\,I^{(0)}_{\gamma\gamma}\,[p + (\varkappa_0 r)^{-2} + \sqrt{3p}\,(\varkappa_0 r)^{-1}]\sin^2\beta_{r1}.$$

If (35) is integrated over all directions of the vector $\mathbf{n}_0$, formula (32) is obtained (for the Rayleigh case), which describes the radiation from an isotropic point source. To obtain (35), we discarded terms $\sim p\sqrt{p}$, p^2; $p(\kappa_0 r)^{-1}$, $(\kappa_0 r)^{-4}$ and so forth.

As follows from (35), the dependence on the initial direction of the source of radiation decreases sharply with increase in the distance from the source if the absorption in the medium is small. Actually, the correlation coefficient between the directions of the radius vectors $\mathbf{r}$ and $\mathbf{n}_0$ is proportional to the quantity $(\kappa_0 r)^{-1} + \sqrt{3p}$ for the radiation density. The dependence of the radiation density on the linear polarization of the source is also weak and is characterized by the quantity $p + (\kappa_0 r)^{-2} + \sqrt{3p}\,(\kappa_0 r)^{-1}$. In this approximation, the polarization of $\mathscr{G}_3$ is completely independent of the direction $\mathbf{n}_0$ of the source of the radiation. This phenomenon is easily understood physically, since the diffusion of the radiation, on the one hand, leads to the isotropization of the flow of photons at great distances from the source, and on the other, to an equilibrium distribution of the scattered photons and to equalization of the gradient of the flow.

In conclusion, we shall write out the formula for the radiation density $\mathscr{G}_0(\mathbf{n}_1\cdot\mathbf{r}\,\mathbf{n}_0)$, obtained from the same assumptions as (35), from the ordinary non-matrix transport equation with Rayleigh scattering indicatrix $\kappa(\mathbf{n}_1\cdot\mathbf{n}_0) = 3[1 + (\mathbf{n}_1\cdot\mathbf{n}_0)^2]/16\pi$:

$$\mathscr{G}_0(\mathbf{n}_1\mathbf{r}\mathbf{n}_0) = \frac{3\varkappa_0{}^2}{16\pi^2 r}\exp(-\varkappa_0 r\sqrt{3p})\,I^{(0)}_{\gamma\gamma}\Big\{1 - \frac{4}{21}p + \cos\beta_{r1}\Big(\frac{1}{\varkappa_0 r} + \sqrt{3p}\Big) \times \frac{2}{3}\Big[p + \frac{1}{(\varkappa_0 r)^2} + \frac{\sqrt{3p}}{\varkappa_0 r}\Big][4P_2(\cos\beta_{0r}) + P_2(\cos\beta_{r1})]\Big\}. \quad (36)$$

It follows from a comparison of Eqs. (35) and (36) that for $p = 0$ the difference between them lies in the absence of a term with polarizations of the source $I^{(0)}_{-11}$

in (36), i.e., the non-matrix transport equation gives the correct result with accuracy up to terms of the order $(\kappa_0 r)^{-2}$.

6. PHOTON SCATTERING FROM A FLAT LAYER OF MATTER

The equations obtained in the previous sections are easily generalized to the case of scattering by a flat layer of matter. Let an unbounded beam of polarized photons be incident on a flat layer of thickness 2L in the direction $\mathbf{n}_0$. The state of this beam is described by the density matrix $I^{(0)}_{\alpha\nu}$. The density matrix of the photons which reach a depth z and are first scattered there in the direction $\mathbf{n}_1$, is equal to

$$B^{(1)}_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) \equiv B^{(1)}_{\alpha\beta}(\mathbf{n}_1 z) = \varkappa_0 [b_1 D^{(1)+}_{\alpha\gamma}(\Omega_{01}) I^{(0)}_{\gamma\nu}(\mathbf{n}_0) D^{(1)}_{\nu\beta}(\Omega_{01}) \times b_2 \delta_{\alpha\beta} I^{(0)}_{\gamma\gamma}(\mathbf{n}_0)] \exp\{-\varkappa_0 z \sec\vartheta_0\}. \quad (37)$$

Here $\cos\vartheta_0 = \mathbf{n}_0 \cdot \mathbf{n}_Z$. Since, for a flat layer,

$$\mathbf{F}(\mathbf{q}) = 8\pi^2 \delta(q_x)\delta(q_y)\exp\{-iq_z L\} f(q_z), \text{ where } f(x) = x^{-1}\sin xL,$$

then Eq. (13) in this case takes the following form:

$$K_{\alpha\beta}(\mathbf{n}_1\mathbf{u}) = 8\pi^2\delta(u_x)\delta(u_y)\exp[-i(u_z - i\varkappa_0\sec\vartheta_0)L] K_{\alpha\beta}(\mathbf{n}_1 u_z), \quad (38)$$

$$K_{\alpha\beta}(\mathbf{n}_1 s) = f(s - i\sec\vartheta_0)[b_1 D^{(1)+}_{\alpha\gamma}(\Omega_{01}) I^{(0)}_{\gamma\nu}(\mathbf{n}_0) D^{(1)}_{\nu\beta}(\Omega_{01}) + b_2\delta_{\alpha\beta} I^{(0)}_{\gamma\gamma}(\mathbf{n}_0)] + \frac{1}{\pi}\int_{-\infty}^{\infty} dr \int d\mathbf{n}_x f(s-r)(1 + ir\mathbf{n}_x\mathbf{n}_z)^{-1}[b_1 D^{(1)+}_{\alpha\gamma}(\Omega_{x1}) K_{\gamma\nu}(\mathbf{n}_x r) \times D^{(1)}_{\nu\beta}(\Omega_{x1}) + b_2\delta_{\alpha\beta} K_{\gamma\gamma}(\mathbf{n}_x r)]. \quad (39)$$

Equation (39) possesses the same symmetry properties as the equation for an unbounded medium (15). We note that Eq. (39) is not more complicated than the ordinary transport equation except for the intensity with indicatrix, containing the zeroth and second Legendre polynomials. The desired density matrix $B_{\alpha\beta}(\mathbf{n}_1 \cdot \mathbf{r})$ is found after satisfying the inverse Fourier transform $K_{\alpha\beta}(\mathbf{n}_1 \cdot \mathbf{u})$ and $\mathscr{I}_2(\mathbf{n}_1 \cdot \mathbf{u})$.

In the work of the authors of,[12] a set of integral equations is solved for the observed values of the Stokes parameters $\mathscr{I}_i$, similar to the equation for the case of Thomson and Compton scattering. Here the authors start out from the following physical picture of the process of multiple scattering. As is seen from the results of the present paper (see, for example, Eqs. (32), (33)), the Stokes parameters of the scattered radiation, which describe the polarization, are small in comparison with the scattering intensity $\mathscr{I}_0$ at great distances from the source. This effect is easily explained by the diffusion radiation which decreases the anisotropy of the radiation flux at large distances. Therefore, it should be expected that, inside the flat layer, at distances from the boundaries of the medium greater than the total free path, that is, where the photon motion is diffusive in character, the polarization of the radiation is small ($\mathscr{I}_3 \approx 0$). The fundamental polarization of the outgoing radiation arises in a single scattering process close to the boundaries of the layer.

The physical considerations set forth above allow us to suggest the following method for the solution of Eqs. (39) or (10). As the zeroth approximation, we choose $\mathscr{I}_3 = 0$, $\mathscr{I}_0 = I(\mathbf{n}_1 z)$ where $I(\mathbf{n}_1 z)$ is determined by the solution of (39), if we neglect the polarization terms in it. After substituting the zeroth approximation in (39), we determine the Stokes parameters in first approximation. Substituting these results again in (39), we obtain the values of the Stokes parameters in second approximation, and so on. Thus, the problem reduces to the solution of Eq. (39), if the polarization terms (the non-matrix transport equation for the scattering intensity) in the latter are neglected. Solution of such an equation is obtained in the previous work of the authors.[15] The substance of the method of solution is the following. As is known, photons can go from one point of the medium to another either directly, without scattering (direct "flight"), or as a result of a certain number of collisions, i.e., diffusion.

In[15] the authors transformed the usual transport equation to integral form, in which the processes of diffusion and direct "flight" without scattering are described by different kernels of the integral equation. In this case, it was shown that the process of diffusion can be taken into account exactly, and the process of direct "flight" can be taken into account by the method of successive approximations. The advantage of the resultant integral equation is that it automatically takes into account the correct boundary conditions of the diffusion equation and allows us to obtain a practical solution with the necessary accuracy. The intensity thus calculated is substituted in Eq. (39) and the polarization of scattered radiation in optically thick layers of matter is calculated (see[12] for details).

In[2,16] the linear polarization is calculated by a numerical method for Rayleigh scattering for different optical thicknesses of the scatterer. The results of the calculations of the polarization for this case by the method given by us lead to a simple analytical formula, and in first approximation, better agreement is obtained with the numerical calculation (see Fig. 3). We note that this method does not depend on the specific angular dependence of the cross section. Moreover, for thick layers, the polarization of the scattered radiation does not depend on the angle of incidence of the initial ray, since the initial direction is completely "forgotten," in the diffusion process, and the polarization is determined by the subsequent acts of photon collisions.

We express our gratitude to Yu. A. Shibanov for carrying out the numerical computations on an electronic computer.

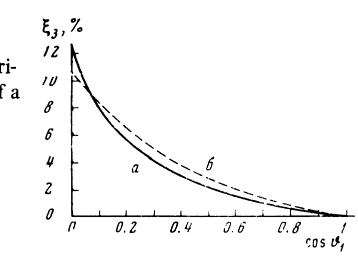

FIG. 3. Degree of linear polarization of radiation coming out of a thick flat layer of matter under Rayleigh scattering: a—result of numerical solution, b—result of analytic solution in first approximation.

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Translated by R. T. Beyer
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