MODEL OF A SHOCK WAVE IN SOLAR WIND PLASMA

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A model for a collisionless shock wave in a solar-wind plasma is considered. The energy dissipation of the shock wave front in the direction of motion of the plasma is due to the development of firehose instability. The development of instability in weak shock waves is described within the framework of the quasilinear theory. The shock wave velocity lies in the interval between the velocities of sound in the one-dimensional and three-dimensional hydrodynamic models; the width of the front is on the order of the ion Larmor radius.

1. INTRODUCTION

AT the present time, the existence of shock waves in solar-wind plasma has been firmly established on the basis of data from satellites and space probes[1]. The bulk of the data refers to stationary lateral waves caused by supersonic solar-wind flow around the earth's magnetosphere. According to the data, the upper limit of the thickness of the lateral shock wave front turns out to be on the order of a thousand kilometers, which is at least five orders of magnitude less than the length of the plasma-particle Coulomb path. Research on collisionless shock waves conducted with laboratory plasma[2] has shown that intensive electrostatic noise (ion-acoustic turbulence) arises in such wave fronts. The plasma-particle scattering length in this type of noise is the characteristic scale that determines the thickness of the shock wave front. Thus far, laboratory research has been concerned with the case of transverse or almost transverse shock wave propagation with respect to the direction of the initial magnetic field in the plasma. In this case the magnetic field does not permit the wave front to spread a distance greater than the ion Larmor radius. As a rule, the magnetic field was sufficiently strong ($\beta = 8\pi nT/H^2 < 1$).

Conditions in a solar-wind plasma differ greatly from those in laboratory experiments, namely, $\beta \gtrsim 1$ (the solar-wind magnetic field is not strong, the wave may even be directed along the magnetic field, etc. The hypothesis was put forth[3] that the basic mechanism for the formation of the shock wave front in these cases might be particle scattering by magnetic field fluctuations, similar to the interactions between particles and magnetic-field inhomogeneities in the Fermi mechanism. However, in the present case the particles cannot be treated as test bodies, since the fluctuations themselves must appear as a result of an instability of the particle velocity distribution within the shock-wave front.

Thus, for example, in wave propagation along a weak magnetic field ($\beta \gg 1$) plasma compression and the subsequent increase in the longitudinal pressure $p_{\parallel}$ leads to the so-called firehose instability[3].

Therefore it is natural to attempt to construct a model of the shock wave front by transferring part of the ordered energy to the energy of the magnetic-field fluctuations, which then create an effective viscosity. A highly idealized (essentially hydrodynamic) model of this kind was constructed by one of the authors (R.Z.S.) and C. Kennel for solar-wind shock waves, using shock-wave propagation along a weak magnetic field based on Chew-Goldberger-Low equations, i.e., precisely in a case where, strictly speaking, these equations are not applicable. To avoid this difficulty, the authors introduced an additional assumption $T_e \gg T_i$ (electron temperature significantly higher than ion temperature). When this condition is violated hydrodynamics does not hold and, as is well known, the kinetic equation is not easy to solve even for the case of an ordinary real gas.

For the case of turbulence induced by the firehose instability, however, it is possible to construct a solvable kinetic model.

2. THE MODEL

The difficulty of reducing the problem of a hydrodynamic model is due to the presence of particles moving with velocities close to the velocity of the shock wave front (we call them resonant particles). When $T_e \gg T_i$, the number of these ions is exponentially small and their contribution may be neglected, as was done in[3]. Further on we shall consider the opposite limiting case $T_V \ll T_i$, where resonant ions play the main role. We begin with an exact quasilinear equation for the firehose instability[4], written in the coordinate system moving with the wave velocity

$$(v_{\parallel} - u_0)\frac{\partial f}{\partial s} = \sum_{\mathbf{k}} (u - u_0)\frac{d}{ds}\frac{|H_{\mathbf{k}}|^2}{H_0^2}\left\{\frac{[v_{\parallel} - u(s)]^2}{v_{\perp}}\frac{\partial}{\partial v_{\perp}}v_{\perp}\frac{\partial f}{\partial v_{\perp}} - 2\frac{\partial}{\partial v_{\parallel}}v_{\perp}[v_{\parallel} - u(s)]\frac{\partial f}{\partial v_{\perp}} + \frac{\partial}{\partial v_{\parallel}}v_{\perp}^2\frac{\partial f}{\partial v_{\parallel}}\right\}, \quad (1)$$

where s is the coordinate along the unperturbed magnetic field H_0, $u(s)$ is the average ion velocity at point s, $v_{\parallel}$ and $v_{\perp}$ are the ion velocity components along and across the unperturbed magnetic field, $f(v_{\parallel}, v_{\perp}, s)$ is the ion distribution function, and $|H_K|^2/8\pi$ is the spectral density of the energy of the magnetic field fluctuations.

Substituting for the coordinate s a new variable

$$h = \sum_{\mathbf{k}} |H_{\mathbf{k}}(s)|^2/H_0^2, \quad (2)$$

we reduce the quasilinear equation to the diffusion equation in velocity space with time h. The choice of the initial condition for this problem corresponds to the choice of a definite particle velocity distribution ahead of the shock wave front. It is quite natural to assume that it is Maxwellian:

$$f \xrightarrow[s\to-\infty]{} F(v_\parallel, v_\perp) = \frac{N_0}{\pi^{3/2} v_T^3} \exp(-v^2/v_T^2), \tag{3}$$

where $v_T = \sqrt{2T/m}$ is the thermal velocity of the ions, and the distribution function is normalized with density N_0.

To simplify the analytic investigation of the problem we consider only weak shock waves, where it is possible to seek a solution to (1) in the form of expansion in powers of the energy of the fluctuating magnetic field. Strictly speaking, we are not interested in the function f itself, but in its second moments $p_\parallel$ and $p_\perp$, which we shall compute. The contributions made to these moments by resonant particles with velocities $v_\parallel \approx u_0$ and by nonresonant ones will be computed separately.

It is simpler to determine the effect of the nonresonant particles. For this purpose it is sufficient to substitute in the right side of (1) the unperturbed distribution function of the ions ahead of the shock wave front and integrate with respect to h. Obviously, the change in the nonresonant-particle pressure, due to the magnetic field fluctuations, is linear with respect to the fluctuation energy

$$\frac{p_\parallel - p_\perp}{4p_0 h} = -\frac{u_0}{4p_0} \int \frac{d^3v}{v_\parallel - u_0} \left[m v_\parallel^2 - \frac{1}{2} m v_\perp^2 \right] \left\{ v_\parallel^2 \frac{\partial}{v_\perp \partial v_\perp} v_\perp \frac{\partial}{\partial v_\perp} - 2 \frac{\partial}{\partial v_\parallel} v_\perp v_\parallel \frac{\partial}{\partial v_\perp} + v_\perp^2 \frac{\partial^2}{\partial v_\parallel^2} \right\} F \equiv -x(1-2x^2) - 2x(1 - 2x^2 + 2x^4) e^{-x^2} \int_0^x e^{t^2} dt, \tag{4}$$

where $x = -u_0/v_T$ is the dimensionless shock-wave velocity, and $p_0 = N_0 T$ is the plasma pressure ahead of the shock wave front. It should be noted here that in the calculation of the second moments from the distribution function, the integrals containing a pole-type singularity were understood in the sense of the principal value. The latter assertion is justified, since rearranging the resonant particle distribution (cf. below) leads to a "smearing" of the singularity.

For concreteness, in what follows, we will assume that the shock wave moves towards the negative semi-axis of s, so that x has only positive values. The dependence of the excess of longitudinal pressure of the nonresonant particles over the transverse pressure on x is illustrated in Fig. 1. We see that the appearance of magnetic field fluctuations always moves this quantity toward the more stable side, because the compression of nonresonant particles in the wave front is very small.

On the other hand, it turns out that when the resonant particles are slowed down the increase in longitudinal pressure is larger than the change of the nonresonant-particle pressure. Therefore, the development of the instability is connected precisely with the resonant particles whose distribution undergoes the greatest changes in the shock wave front. The nature of this change will be calculated accurate to the first two terms of the expansion of the contribution of these particles to the pressure in terms of the small quantity $h^{2/3}$.

To calculate the first term of this expansion, it is sufficient to take into account the two fundamental terms in the kinetic equation (1):

$$(v_\parallel - u_0) \frac{\partial f}{\partial h} = -u_0 v_\perp^2 \frac{\partial^2 f}{\partial v_\parallel^2}. \tag{5}$$

Using the Laplace transformation, we reduce this equation to an Airy equation having a right side

$$\frac{\partial^2 \tilde{f}(p,z,w)}{\partial z^2} - \frac{pz}{xw} \tilde{f}(p,z,w) = -\frac{z}{wx} F(z,w), \tag{6}$$

where

$$z = (v_\parallel - u_0)/v_T, \quad w = v_\perp^2/v_T^2. \tag{7}$$

The solution of((6) can be written in the form of the corresponding Green's function and the initial value of the distribution function. However if we attempt to use the inverse transform in this solution, i.e., if we attempt to write the solution in the form of a function of "time" h, then we obtain an integral over p which generally diverges at large p:

$$f(z,w,h) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{ph} \tilde{f}(p,z,w)\, dp. \tag{8}$$

This is connected in turn with the rapid oscillations of the solution to the left of point z = 0 at the initial moment (cf. Fig. 2). In other words, our solution has an essential singularity at zero.

We can avoid this difficulty if we use the inverse transform not of the function itself, but of its second moments. In practice, in the calculation of the second moments we average over the rapid oscillations of the distribution function, whereupon the integrals of (8) converge.

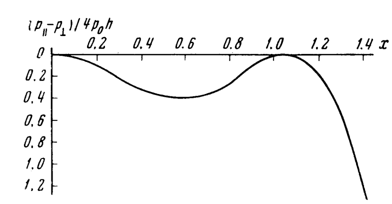

FIG. 1. Pressure anisotropy of nonresonant particles behind the wave front vs. wave velocity.

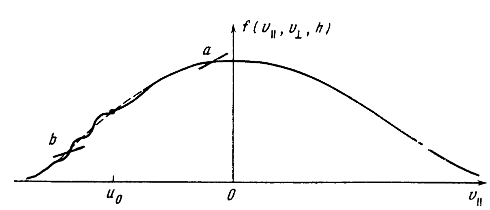

FIG. 2. Propagation of nonresonant particles behind the wave front.

After a long but straightforward calculation of tabulated integrals, we obtain for the change of the resonant-particle pressure

$$(p_\parallel - p_\perp) = \frac{4p_0}{3\pi^{1/2}} \Gamma\left(\frac{4}{3}\right) (9hx)^{2/3} e^{-x^2} \left[x^2 - \frac{5}{6} - \frac{\pi\sqrt{3}}{\Gamma^3(1/3)} \right]. \tag{9}$$

We see that the excess of longitudinal pressure due to plasma compression is formed only at sufficiently large shock-wave velocities:

$$u_0^2 > c_s^2, \qquad (10)$$

where

$$c_s^2 = \left(\frac{5}{3} + \frac{2\pi\sqrt{3}}{\Gamma^3(1/3)}\right)\frac{p_0}{\rho_0} \approx 2.24 p_0/\rho_0.$$

The presence of a velocity at which no variation of the resonant particle pressure takes place is due to the nature of the relaxation of the resonant-particle distribution. In contrast to the usual "quasi-linear plateau" parallel to the abscissa axis, here a "plateau" is formed with a definite inclination to the axis (cf. Fig. 2). Therefore, at small wave velocities the particles give part of their longitudinal energy to the wave, and at large velocities they draw energy (the position of the plateau for these cases is denoted in Fig. 2 by indices a and b).

Furthermore, the Mach number is determined by

$$M = \frac{u_0^2}{c_s^2} \equiv x^2 \Big/ \left[\frac{\pi\sqrt{3}}{\Gamma^3(1/3)} + \frac{5}{6}\right]. \qquad (11)$$

Thus it is to be expected that at Mach numbers close to unity the energy of the fluctuating magnetic field in the shock wave front will be small due to the smallness of the excess of longitudinal pressure of the resonant particles compared to the transverse pressure. A comparison of (4) and (9) shows that owing to random causes the difference between longitudinal and transverse pressures of nonresonant particles at small Mach numbers is also small, and therefore the relaxation of nonresonant particles is not capable of arresting the instability. As a result, we must calculate the variation of the resonant particle pressure in the second order of the expansion in powers of $h^{2/3}$. To calculate the latter in all the derivatives, lower than the second, of the distribution function with respect to the longitudinal velocity $v_\|$, we substitute in (1) the solution of the simplified equation (5) and again solve the transformed equation (1)[1]. Taking into account, in addition, the pressure variation due to the presence of fluid flow with a velocity $u \sim h^{2/3} u_0$, we find in the second order of the expansion with respect to $h^{2/3}$:

$$(p_\| - p_\perp) = -(98 p_0 / 9\pi^{1/2})\Gamma(2/3)(9hx)^{1/3} x e^{-x^2}\{(7/6 - x^2)(x^2 - 1) + \Gamma^2(1/3) / 126\Gamma^3(2/3) + 72\Gamma^2(1/3) x^3 e^{-x^2} / 7\pi^{1/2}\Gamma(2/3)\}. \qquad (12)$$

Summarizing the results of (4), (9), and (12) for Mach numbers close to unity, we obtain the following dependence of the plasma pressure anisotropy on the magnetic-field fluctuation level:

$$(p_\| - p_\perp) / p_0 = 0.92(M-1)h^{2/3} - 0.5h^{4/3} - O(0.04h). \qquad (13)$$

We see that with increasing fluctuation energy the anisotropy reaches a certain maximum and then again vanishes at the following value of the fluctuation energy:

$$h_1 \approx 1.5(M-1)^{3/2} \ll 1. \qquad (14)$$

Therefore, it is to be expected that at shock wave velocities greater than critical the small magnetic field fluctuations grow in the shock wave front only up to this finite level.

[1]Of course, we omit the expansion terms proportional to h, as they were already taken into account in (4).

For the energy of the growing fluctuations, we use an equation from linear theory:

$$(u - u_0)\frac{d}{ds}\frac{|H_{\mathbf{k}}|^2}{H_0^2} \approx |k_\|| \sqrt{[p_\|(h) - p_\perp(h)]/\rho_0}\,\frac{|H_{\mathbf{k}}|^2}{H_0^2}. \qquad (15)$$

The quasilinear equation (13), together with the equation for the oscillation amplitudes, completely solves the problem of the structure of the weak shock wave front. To be sure, whereas the quasilinear equation remains valid up to the limiting amplitude h_1, the validity of applying the second equation at such amplitudes is doubtful. This is due to the fact that the rate of nonlinear interaction between different modes at such amplitudes becomes comparable to the linear increment of instability[2]. Therefore (15) generally is even less justified, and we must add terms to its right side to describe the energy redistribution among the different modes. If we put aside the fine points of the energy distribution over the wave number spectrum and consider the change of the total oscillation energy, then it is obvious that fluctuation energy ceases to grow at the total oscillation amplitude determined by (14). We can describe the qualitative behavior of the fluctuation amplitude in the wave front if we assume that the mode with the largest growth rate has the largest amplitude. The wavelength of this mode is bounded from above by the stabilizing effect of the finite Larmor radius[6]

$$k_\|^2 r_H^2 = (p_\| - p_\perp) / 2p_0,$$
$$r_H^2 = v_T^2 / 2\omega_H^2.$$

Since the shock wave front thickness is large (cf. (17)), the wavelength of the oscillation has time to adjust itself to the local value of the pressure anisotropy, and we can rewrite (15) in the simpler form:

$$r_H \frac{dh}{ds} = 0.84(M-1)^2\left(\frac{h}{h_1}\right)^{2/3}\left[1 - \left(\frac{h}{h_1}\right)^{2/3}\right] h. \qquad (16)$$

The solution of (16) can be written explicitly:

$$\ln\frac{(h/h_1)^{1/3}}{1-(h/h_1)^{2/3}} - \left(\frac{h_1}{h}\right)^{2/3} \approx 0.56(M-1)^2 s/r_H. \qquad (17)$$

The decrease of the fluctuation amplitude ahead of the wave front ($s \to -\infty$) has a power-law character, and the instability saturation behind the front occurs according to an exponential law (cf. Fig. 3).

In conclusion we should mention that within the framework of a kinetic description of plasma, the small sound-type oscillations always resonate with a sufficiently large group of ions and are damped. Therefore, the existence of the shock wave is due to the finiteness of its amplitude, and its velocity obviously is not con-

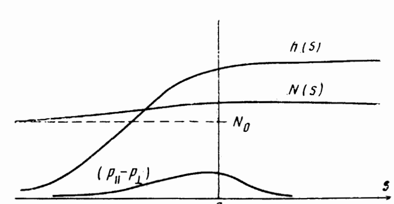

FIG. 3. Profiles of density, magnetic field, and pressure anisotropy in the wave front.

[2]In [4] the opposite was suggested, owing to a mistake in calculating the nonlinear mode interaction.

nected with the velocity of the infinitesimally small damped plasma perturbations.

It is also interesting to note that an analogous calculation, conducted within the framework of the quasihydrodynamic CGL equations[5], shows that the velocity of weak shock waves is determined not by the one-dimensional, but by the three-dimensional sound velocity (i.e., the effective adiabatic exponent is equal to $^5/_3$ and not to 3). In other words, the firehose instability of the CGL model plays the role of the collisions that equalize the longitudinal and transverse particle temperatures.

APPENDIX

A COMPARISON WITH THE CGL MODEL

In order to make a comparison with the results of quasilinear theory, we now determine the change of the difference between the longitudinal and transverse pressures in the shock wave front within the framework of quasihydrodynamic Chew-Goldberger-Low (CGL) equations[5]. In the coordinate system moving with the wave, these equations are easy to integrate, and we obtain

$$\rho(u-u_0) = -\rho_0 u_0,$$
$$\rho(u-u_0)^2 + p_\| = \rho_0 u_0^2 + p_0, \quad (u-u_0)^3 p_\| (1+h) = -p_0 u_0^3,$$
$$(u-u_0) p_\perp = -p_0 u_0 \sqrt{1+h}.$$

Solving this system of equations by expansion in powers of h, we find

$$(p_\| - p_\perp) = p_0 (1+h)^{-1/2} \left[\frac{5p_0 - 3\rho_0 u_0^2}{2[\rho_0 u_0^2 - 3p_0]} h - \frac{p_0}{(\rho_0 u_0^2 - 3p_0)^2} \times \left(3p_0 - \frac{6(p_0 + \rho_0 u_0^2) p_0}{[\rho_0 u_0^2 - 3p_0]} - \rho_0 u_0^2 \right) h^2 - \frac{3h^2}{8} \right].$$

The most interesting feature of this expression is that the longitudinal pressure exceeds the transverse pressure at wave velocities greater than the three-dimensional sound velocities. In other words, the firehose instability plays the role of the effective collisions that equalize the longitudinal and transverse pressure of the particles. Therefore, in the presence of this instability, even one-dimensional compression is described by an adiabatic curve with adiabatic exponent $^5/_3$ and not 3.

The pressure change in the weak shock wave front can be described by an approximate equation that follows from the expression introduced earlier

$$(p_\| - p_\perp)/\rho = 9p_0[(M_* - 1) - 3{,}3h]h/8\rho_0,$$

where $M_* = 3\rho_0 u_0^2/5p_0$ is the Mach number.

Thus, both in the CGL model and in the kinetic model plasma pressure anisotropy increases in the wave front at sufficiently large propagation velocity.

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