LINEAR WAVE CONVERSION IN AN INHOMOGENEOUS MAGNETOACTIVE PLASMA

A. D. PILIYA and V. I. FEDOROV

A. F. Ioffe Physico-technical Institute, U.S.S.R. Academy of Sciences

Submitted August 15, 1968

Zh. Eksp. Teor. Fiz. 57, 1198–1209 (October 1969)

Solutions of the wave equation are considered for an inhomogeneous magneto-active plasma, taking account of weak spatial dispersion under the assumption that there is point within the plasma at which the component ϵ_{xx} of the dielectric tensor vanishes (x is the coordinate on which the particle density depends; the magnetic field is assumed to make an arbitrary angle with the x axis). It is shown that in the region $|\epsilon_{xx}| \ll 1$ the solution of the equation is described by a standard function; the asymptotic form far from the point $\epsilon_{xx} = 0$ is a superposition of plasma waves and the solution of the equation with spatial dispersion neglected. In general, the coefficients in this superposition are different in different sectors of the complex plane of x. However, in the absence of an incident plasma wave, in a sector which includes the lower half plane and the real axis of x for a cold plasma the solution of the equations is given by a superposition with constant coefficients. It is shown that in lowest order in a parameter which takes account of the dispersion the magnitude of the energy carried away by the plasma waves coincides with the magnitude of the energy absorbed in the cold plasma. In the case of a highly inhomogeneous plasma, solutions are obtained for the "cold" waves in cases in which the equations do not break up into independent second-order equations. Making use of these solutions it is possible to obtain the boundary conditions at the vacuum-plasma interface which takes into account the presence of a transition region in which the density vanishes gradually.

It is well known, that in an inhomogeneous plasma at a finite temperature there is a coupling between electromagnetic waves and plasma waves with the possibility of conversion between wave modes. This linear transformation process is of interest in the analysis of absorption, emission, and scattering of waves in a plasma, and is also of interest in connection with stability and nonlinear interactions. Transformations of this kind have been studied by many authors, but up to the present time only certain limited cases have actually been examined. The most detailed analyses have been given to the transformation in isotropic media.[1-4] In a magneto-active plasma only that case has been considered in which the density gradient is perpendicular to the external magnetic field, with certain specialized conditions on the incident wave.[4-8] In many papers the transformation of waves has been discussed from the point of view of the geometric optics approximation, that is to say, under the assumption that the plasma inhomogeneity is weak.[4] However, the transverse dimensions of laboratory plasmas are frequently of the same order as the wavelength and for this reason it is of interest to consider the exact solutions of the wave equations.

We wish to consider the solution of the wave equation for a hot inhomogeneous plasma:

$$\operatorname{rot}\operatorname{rot}\mathbf{E} - \frac{\omega^2}{c^2}\mathbf{D} = 0, \qquad (1)$$

where **D** generally contains an integral operator which acts on **E**; in the general case this can be an extremely complicated problem. However, in most cases of practical interest the waves in a large region of the layer can be divided into long waves, for which spatial dispersion only introduces unimportant corrections, and plasma waves which can be considered as electrostatic waves to a high degree of accuracy.[1)] In other words, any solution of (1) **E** can be approximated by a linear combination

$$\mathbf{E} = A\mathbf{E}_c + B\mathbf{E}_p \qquad (2)$$

where the solutions $\mathbf{E}_c$ and $\mathbf{E}_p = -\nabla\varphi$ are governed by the simpler equations

$$\operatorname{rot}\operatorname{rot}\mathbf{E}_c - \frac{\omega^2}{c^2}\mathbf{D}_c = 0, \qquad (3)$$

where $D_{ci} = \epsilon_{ik}(x)E_{ck}$ and $\epsilon_{ik}(x)$ is the dielectric tensor, computed without taking account of the thermal motion of the particles, and

$$\operatorname{div}\mathbf{D}(\nabla\varphi) = 0. \qquad (4)$$

The approximation in (2) is known to be violated near singular points of the "cold" equation (3). In the plane case, in which the plasma density depends on the x coordinate (this is the only case which will be considered below) the singular points of (3) are defined by the condition

$$\varepsilon_{xx}(x_0) = 0. \qquad (5)$$

The exact wave equation (1) does not exhibit singularities at these points; on the other hand, in the vicinity of these points the field does not separate into the two modes indicated above. This means that we can call the

1) As will be made clear below, this separation does not hold if the characteristic scale of the inhomogeneity density is comparable with the Larmor (or Debye) radius of the particles; similarly it does not hold when the wave frequency is approximately equal to the cyclotron frequency or to the second harmonic of the cyclotron frequency.

vicinity of the roots of equation (5) regions of "mode interaction." Equation (2) represents an asymptotic representation of the solutions of (1) outside of these regions; obviously, it can have different forms on different sides of the interaction region. The construction of these representations requires a determination of the field inside the interaction region and appropriate matching of this field with the selected solutions $\mathbf{E}_c$ and $\mathbf{E}_p$.

The present work is devoted to an investigation of the behavior of the field inside these regions and the construction of asymptotic solutions of (1) for comparatively general assumptions. We will not give the actual form of the layer and will not use the explicit form of the tensor $\epsilon_{ik}(x)$. Moreover, we do not make any assumptions as to whether geometric optics can be used in (3).

1. FIELD IN THE MODE INTERACTION REGION

In the analysis of the field in the interaction region we can make use of the following simplifying circumstance. In this region there is a transition from plasma waves, for which dispersion is strong, to "cold" oscillations, for which dispersion is generally not important. Thus, under appropriate conditions we can assume that the dispersion in the interaction region is weak and that it can be taken into account by writing $\mathbf{D}$ in the form

$$D_i = \varepsilon_{ik}E_k + \frac{c^2}{\omega^2}\delta_{iklm}\frac{\partial^2 E_k}{\partial x_l \partial x_m}, \tag{6}$$

where $|\delta_{iklm}| \ll 1$.

Actually, (6) should contain terms of the form $(\gamma_{ikl}c/\omega)\partial E_k/\partial x_l$ but it can be shown that in the small region in which we use (6) these terms are unimportant and can be neglected. The conditions under which the approximation in (6) is valid will be clarified below.

We now consider the wave equation

$$\frac{\partial}{\partial x_i}\operatorname{div}\mathbf{E} - \Delta E_i - \frac{\omega^2}{c^2}\varepsilon_{ik}E_k - \delta_{iklm}\frac{\partial^2 E_k}{\partial x_l \partial x_m} = 0 \tag{7}$$

in the region of the point $x = x_0$, which is a root of (5). In this region we can write

$$\varepsilon_{xx} = (x_0 - x)/a, \tag{8}$$

where a is a constant, while the remaining coefficients can be assumed to be constants. We make use of the notation

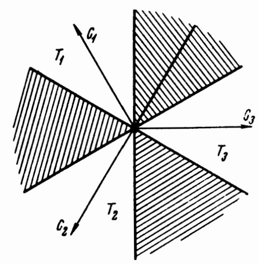
$$\delta_{xxxx}(x_0) \equiv \beta, \quad \left(\frac{c^2\beta}{\omega^2 a}\right)^{1/3} = \gamma \tag{9}$$

and will assume that β is so small that $\gamma \ll 1$. In order to be definite, for the time being we assume that $\beta > 0$. Then, introducing the new variable $\xi = (x - x_0)/a\gamma$ and considering waves of the form $\mathbf{E} = \mathbf{E}(x)\exp\{ik_y y + ik_z z\}$, after certain transformations, to lowest order in γ we find

$$E_x^{(4)} - \xi E_x'' - (2 - i\sigma)E_x' = 0, \tag{10}$$

$$E_y' = i\gamma k_y a E_x, \quad E_z' = i\gamma k_z a E_x, \tag{11}$$

where $\sigma = a[k_y\{\epsilon_{xy}(x_0) + \epsilon_{yx}(x_0)\} + k_z\{\epsilon_{xz}(x_0) + \epsilon_{zx}(x_0)\}]$ and the primes denote differentiation with respect to ξ. In view of the vector nature of the expressions in (2), the coefficients A and B can be determined from the matching conditions on any component of the field. It is clear from the form of the system in (10)–(11) that it is most convenient to treat the x component, which is determined by one equation of (10).



One of the solutions of this equation is a constant while the other three can be found by Laplace transform methods. These can be conveniently written in the form ($k = 1, 2, 3$)

$$W_k(\xi) = \exp\left\{i\frac{\pi}{3}(1 - i\sigma)(7 - 2k)\right\}\int_{C_k} t^{-i\sigma}\exp\left\{\xi t - \frac{t^3}{3}\right\}dt, \tag{12}$$

where the integration is carried out over the plane with a cut taken from the origin at an angle $\arg t = \pi/3$ along the contour shown in the figure. The functions W_k are related by expressions

$$W_1(\xi) = W_2(\xi e^{-2/3\pi i}), \quad W_3(\xi) = W_2(\xi e^{2/3\pi i}). \tag{13}$$

These do not have singularities for finite ξ and are expressible in terms of tabulated functions[9] when $\sigma = 0$. In the problem at hand greatest interest attaches to the asymptotic behavior of the function W_k when $|\epsilon| \gg 1$ for the entire complex plane of ξ. (The assumption $|\xi| \gg 1$ is compatible with condition $|x - x_0| \ll a$ if $\gamma \ll 1$.) We first consider the function W_2.

If $0 \le \arg\xi \le \pi/3$ the path of integration in (12) can first be taken along the line C_0 for the steepest descent of $\exp \xi t$ and then through the saddle point $t_0 = -\sqrt{\xi}$ into the sector T_2. In general, the main contribution to the integral comes from the beginning of the path in the vicinity of the saddle point. The contribution at the beginning of the contour is given by the integral

$$I_2 = \exp[i\pi(1 - i\sigma)]\int_{C_0} t^{-i\sigma}e^{\xi t}dt, \tag{14}$$

while the contribution due to the saddle point is of the form

$$\frac{i\sqrt{\pi}}{\xi^{1/4}}\exp\left[-\frac{2}{3}\xi^{3/2} - \frac{i\sigma}{2}\ln\xi\right]. \tag{15}$$

In the sector $0 \le \arg\xi < \pi/3$, $\operatorname{Re}\xi^{3/2} > 0$ the expression in (15) becomes asymptotically small and the asymptote of W_2 is determined by the initial path. In the present case the contour C_0 is represented by the ray $\arg t = -(\pi + \arg\xi)$. Thus

$$I_2 = \Gamma(1 - i\sigma)/\xi^{1-i\sigma}. \tag{16}$$

When $\arg\xi$ is approximately equal to $\pi/3$, $\operatorname{Re}\xi^{3/2}$ is reduced in absolute value and in a certain narrow range $\pi/3 - \delta < \arg\xi < \pi/3 + \delta$, $\delta \sim |\xi|^{-3/2}$ (16) and (15) are of the same order of magnitude. In this sector

$$W_2 = \frac{\Gamma(1 - i\sigma)}{\xi^{1-i\sigma}} + \frac{i\sqrt{\pi}}{\xi^{1/4}}\exp\left\{-\frac{2}{3}\xi^{3/2} - \frac{i\sigma}{2}\ln\xi\right\}.$$

Along the path $\xi = \pi/3$ the second term represents a wave traveling toward the origin.

When $\pi - \delta > \arg \xi > \pi/3 + \delta$ the expression in (15) becomes exponentially large and the contribution at the origin of the path can be neglected:

$$W_2 = \frac{i\sqrt{\pi}}{\xi^{1/4}} \exp\left\{-\frac{2}{3}\xi^{3/2} - \frac{i\sigma}{2}\ln\xi\right\}.$$

In the sector $\pi - \delta < \arg \xi < \pi + \delta$ both terms in the asymptote are again of the same order of magnitude but now the contour C_0 passes around the other side of the cut in the plane of t and

$$W_2 = \frac{e^{2\pi\sigma}\Gamma(1 - i\sigma)}{\xi^{1-i\sigma}} + \frac{i\sqrt{\pi}}{\xi^{1/4}} \exp\left\{-\frac{2}{3}\xi^{3/2} - \frac{i\sigma}{2}\ln\xi\right\}. \qquad (17)$$

Along the path $\arg \xi = \pi$ the second term describes a diverging wave.

In the region $\pi + \delta < \arg \xi < 4\pi/3$ the contribution of the saddle point again becomes negligibly small. When $\arg \xi = 4\pi/3$ the saddle point lies on the contour C_0 and at large values of $\arg \xi$ the contour C_0 passes between this point and the boundary of the sector T_2. In this case the path of integration in (12) can be extended, passing around the saddle point in such a way that the quantity $\mathrm{Re}\,(\xi t - t^3/3)$ falls off monotonically along the path. As a result

$$W_2 = \frac{e^{2\pi\sigma}\Gamma(1 - i\sigma)}{\xi^{1-i\sigma}}, \quad \pi + \delta < \arg\xi \leqslant 2\pi. \qquad (18)$$

It is immediately clear that the expression in (15) describes plasma waves while the integral in (14) describes the "cold" part of the solution. It should be emphasized that on the positive real axis the expressions in (18) and (16) are equal so that the cold part asymptotically becomes a continuous function of $\arg \xi$ over the entire sector bounded by the rays $-\pi - \delta < \arg \xi < \pi/3 + \delta$, that is to say including the entire real axis. It will be shown below that this circumstance implies that the cold part asymptotically represents the solution of the cold equation (3) both on the left and on the right of the mode interaction region. The asymptotic representation of the functions W_1 and W_3 can be obtained by means of (13).

2. ASYMPTOTIC REPRESENTATION OF THE SOLUTIONS

The system of equations (10)–(11) is equivalent to a sixth-order equation. Hence, the total array of characteristic functions of equations (1) must include six solutions for which the weak dispersion approximation is valid in the interaction region. The functions W_k considered above can be used to construct asymptotic representations of the form in (2) for all six normal waves.

In order to establish these asymptotic representations we must determine the limiting form of the solutions of equations (3) and (4) as $x \to x_0$.

We first consider (4). The geometric-optics approximation is valid for the plasma waves and in (6), which holds near the interaction region, the dispersion equation assumes the form

$$\beta k^4 - \frac{\omega^2}{c^2}\varepsilon_{xx}k^2 - \frac{\omega^2}{c^2}k\left[k_y(\varepsilon_{xy} + \varepsilon_{yx}) + k_z(\varepsilon_{xz} + \varepsilon_{zx})\right] = 0.$$

The plasma waves describe the roots of this equation which are largest in absolute value:

$$k^{\pm} = \pm\frac{\omega}{c}\sqrt{\frac{\varepsilon_{xx}}{\beta}} + \frac{\omega}{c}\,\frac{k_y(\varepsilon_{yx} + \varepsilon_{xy}) + k_z(\varepsilon_{zx} + \varepsilon_{xz})}{2\varepsilon_{xx}}.$$

In the region $|x - x_0| \ll a$ the x component of the appropriate solutions can be written in the form

$$E_{px}{}^{\pm} = \frac{i\sqrt{c}}{\sqrt{\omega a}\,\xi^{1/4}\gamma^{1-i\sigma}} \exp\left\{\pm\frac{2}{3}\xi^{3/2} - \frac{i\sigma}{2}\ln\xi - \frac{\pi}{2}\sigma\right\}, \qquad (19)$$

where the constant multiplier is chosen in such a way that the energy flux carried by the wave is equal to the Poynting vector of an electromagnetic wave of unit amplitude in vacuum. Obviously the expression in (19) represents the limiting value as $x \to x_0$ of the solutions of (4). We now consider the solutions E_c for the cold equation (3). When $x \to x_0$[10]

$$E_{cx} = A\left(\frac{a}{x_0 - x}\right)^{1-i\sigma} + B_x, \quad E_{cy,z} = -\frac{k_{y,z}a}{\sigma}\left(\frac{a}{x_0 - x}\right)^{-i\sigma} + B_{y,z}, \qquad (20)$$

where A and $B_{x,y,z}$ are constants so that x_0 it is a branching point of these solutions. In order to remove the ambiguity it is necessary to make a cut from the point x_0; depending on the position of this cut in the upper or lower half-plane of x we shall denote these solutions respectively by $E^{(u)}$ and $E^{(l)}$. The difference between these solutions lies in the fact that $E^{(u)}$ describes absorption while $E^{(l)}$ describes generation of energy at the singular point x_0. Actually, if one considers real x and requires that the solutions be continuous one obtains the branch $E^{(u)}$ if it is assumed that x_0 has a positive imaginary part $i\delta a$. In accordance with (8), the presence of this imaginary part corresponds to a positive antihermitian part $i\delta$ in ϵ_{xx}, which leads to energy absorption. Owing to the peculiarities of the field, when $\delta \ll 1$ this absorption is independent of δ and does not vanish in the limit $\delta \to 0$.[4] In the general case in which $\sigma \neq 0$ the power absorbed per unit area of surface at $x = x_0$ is given by

$$Q = \frac{\omega a}{8\pi}|A^{(u)}|^2\frac{1 - e^{-2\pi\sigma}}{2\sigma}, \qquad (21)$$

where $A^{(u)}$ is the value of the constant A for the solution $E^{(u)}$. In completely analogous fashion it can be shown that the solution $E^{(l)}$ represents generation of energy.

It will be evident that the solution $E^{(u)}$ is continuous in the lower half plane while $E^{(l)}$ is continuous in the upper half plane.

Having determined these properties of the solutions of (3), we can determine the coefficients in the asymptotic expression (2). We first consider the solutions in which no incident plasma waves are present. As will be evident from (17), the x component of this solution in the interaction region is of the form

$$E_x = C_2W_2 + C_1, \qquad (22)$$

where C_1 and C_2 are constants.

Since the cold part of the asymptotic form of W_2 is continuous in the lower half plane (22) can be only matched with $E_x^{(u)}$ so that outside the interaction region

$$\mathbf{E} = \mathbf{E}^{(u)} + \alpha_{12}\mathbf{E}_p{}^{-}, \quad x < x_0,$$
$$\mathbf{E} = \mathbf{E}^{(u)}, \quad x > x_0. \qquad (23)$$

Now, making use of (19), (20), (22) and (23) we find

$$C_1=B_x,\quad C_2=-\frac{A^{(u)}\exp[-\pi\sigma]}{\Gamma(1-i\sigma)\gamma^{1-i\sigma}},\quad \alpha_{12}=-\sqrt{\frac{\pi\omega a}{c}}\frac{A^{(u)}\exp[-\pi\sigma/2]}{\Gamma(1-i\sigma)}.$$

Substituting the four independent solutions $\mathbf{E}_i^{(l)}$ in these expressions we obtain four solutions $\mathbf{E}_i$ (i = 1, 2, 3, 4) which describe the transformation of the electromagnetic waves into plasma waves. It is convenient to classify these solutions in terms of the direction of incidence (from the right or from the left of x_0) of the incident waves. Each direction of incidence corresponds to two independent polarizations one of which can always be chosen in such a way that $A^{(u)} = 0$. The corresponding waves do not experience transformation while the waves polarized in the orthogonal direction are subject to maximum transformation. The energy carried by the plasma wave $c|\alpha_{12}|^2/8\pi$ is exactly equal to the quantity Q in (21). If the plasma wave is damped in the plasma layer the quantity $|\alpha_{12}|^2$ can be regarded as the effective absorption coefficient. It is determined completely by the cold solution $\mathbf{E}^{(u)}$. The conclusion reached here is a proof of this result, which has been noted in the literature[3, 4, 7] for various particular cases.

In the derivation of (22) and (23) we have not actually made use of the hermiticity of ϵ_{ik}; hence these results remain valid when collisions are taken into account, and in an unstable plasma. It should be emphasized that the proper choice of the branch of the cold solution, which asymptotically represents $\mathbf{E}_i$, is independent of the sign of the imaginary corrections to ϵ_{xx}, that is to say, the sign of the imaginary part of x_0. The asymptotic representation that has been obtained can also be used to take the limit $T \to 0$, that is to say, $\gamma \to 0$. When $\gamma \to 0$ the function W_2 approaches infinity in the sector $\pi/3 < \arg\xi < \pi$ and the cold limit in the remaining part of the complex plane (except for the rays $\arg\xi = \pi$, $\arg\xi = \pi/3$); it is then clear that in the presence of any arbitrary weak absorption $\mathbf{E}_i \to \mathbf{E}_i^{(u)}$ over the entire lower half-plane including the entire real axis of x. On the other hand, if it is assumed at the outset that T = 0 then the only physically meaningful solutions are the cold solutions of the equation $\mathbf{E}_i^{(u)}$, which are analytic in the lower half-plane.

In completely analogous fashion we can construct the solution that describes the transformation of plasma waves into electromagnetic waves. Outside of the mode interaction region

$$\mathbf{E}_5=\mathbf{E}_p^{+}+\alpha_{22}\mathbf{E}_p^{-}+\alpha_{21}(\mathbf{E}^{(l)}-\mathbf{E}^{(u)}),\quad x<x_0,$$
$$\mathbf{E}_5=\alpha_{21}(\mathbf{E}^{(l)}-\mathbf{E}^{(u)}),\quad x>x_0, \tag{24}$$

where $\mathbf{E}^{(u)}$ and $\mathbf{E}^{(l)}$ are any nonregular (with $A \neq 0$) solutions that satisfy the same boundary condition

$$\alpha_{21}=-i\sqrt{\frac{c}{\pi\omega a}}\frac{\Gamma(1-i\sigma)\exp[-\pi\sigma/2]}{A^{(l)}},\quad \alpha_{22}=-i\frac{A^{(u)}}{A^{(l)}}e^{-\pi\sigma}.$$

In the interaction region $|x_0 - x| \ll a$

$$E_{5x}=\frac{\sqrt{c}}{\sqrt{\pi\omega a}\,\gamma^{1-i\sigma}}\left\{\exp\left[-\frac{1}{6}\pi i(1-i\sigma)W_1-i\frac{A^{(u)}}{A^{(l)}}\exp\left[-\frac{3}{2}\pi\sigma\right]W_2\right\}. \tag{25}$$

It will be evident that the function $\mathbf{E}_5$ does not have a cold limit. Finally, the last sixth solution of equation (1) contains the exponentially diverging function W_3 when $x > x_0$. This solution will not be considered here.

Up to this point it has been assumed that $\beta > 0$. When $\beta < 0$ the ξ plane is rotated with respect to the x plane by an angle $\pi/3$. Along the ray $\arg\xi = -\pi/3$, the diverging wave is described by the function W_1 while the incident wave is described by W_3. Making use of (13) we can show that (22)–(25) remain valid when the following substitutions are made: $\gamma \to |\gamma|$, $\xi \to \xi'$, and $\xi' = (x_0 - x)/a|\gamma|$.

It has been assumed above that within the transition layer the density varies monotonically so that the quantity ϵ_{xx} vanishes at only one point in the plasma. The generalization to the case of an arbitrary density profile does not introduce any difficulty so long as the distance between the zeroes is large enough so that the interaction regions for the modes do not overlap. Under these conditions an expression such as (25) holds in the neighborhood of each such point.

Equation (7) is equivalent to a sixth-order equation with a small parameter multiplying the highest derivative. Hence, it is reasonable that there will be a significant similarity between the present results and the theory of asymptotic solutions of fourth-order equations given by Wasow.[11] In his case, however, the rapidly oscillating term (the analog of the plasma wave) appears at the boundaries of sectors with a small factor and hence is neglected.

In applying the analysis given above it is required, first of all, that the following condition be satisfied:

$$\gamma \ll 1,\quad \frac{\omega a}{c}|\varepsilon_{ik}|\gamma \ll 1, \tag{26}$$

which was used in the derivation of (10) and (11); secondly, we require the validity of the weak-dispersion approximation (6) in the mode-interaction region. It can be shown that the second condition is automatically satisfied when $\gamma \ll 1$ everywhere with the exception of frequencies close to the second harmonic of the electron-cyclotron frequency ω_{Be}, where the more stringent requirement

$$\gamma \ll \sqrt{\frac{\omega - 2\omega_{Be}}{\omega_{Be}}}. \tag{27}$$

must be satisfied. The second inequality in (26) guarantees that (20) is valid over a region that is broader than the mode-interaction region so that the solutions can be matched. The parameter β, which determines the value of γ in accordance with (9), is given by the following expression in a collisionless plasma:

$$\beta=\sum\frac{v^2\omega_0^2}{c^2}\left[\frac{3\omega^2\sin^4\alpha}{(\omega^2-\omega_B^2)(\omega^2-4\omega_B^2)}+\frac{6\omega^4-3\omega^2\omega_B^2+\omega_B^4}{(\omega^2-\omega_B^2)^3}\sin^2\alpha\cos^2\alpha+\frac{3\cos^4\alpha}{\omega^2}\right]$$

where the summation is taken over particle species and the plasma frequency ω_{0e}^2 is determined from the condition $\epsilon_{xx}(x_0) = 0$:

$$\omega_{0e}^2=\frac{\omega^2(\omega^2-\omega_{Bi}^2)(\omega^2-\omega_{Be}^2)}{\omega^2(\omega^2-\omega_{Be}\omega_{Bi})\sin^2\alpha+(\omega^2-\omega_{Bi}^2)(\omega^2-\omega_{Be}^2)\cos^2\alpha},$$
$$\omega_{0i}^2=\omega_{0e}^2\frac{m_e}{m_i}. \tag{28}$$

Wave conversion is possible at frequencies for which the left side of (28) lies between the limits 0 and $\omega_{0m}^2 = 4\pi n_0 e^2/m_e$ where n_0 is the maximum density in the layer. For a given angle α these frequencies form three isolated bands

$$0 \leqslant \omega \leqslant \Omega_1, \quad \omega_{Bi} \leqslant \omega \leqslant \Omega_2, \quad \omega_{Be} \leqslant \omega \leqslant \Omega_3.$$

Exact expressions for Ω_i can be found, for example, in [12] (in this case it is necessary to take $\omega_{0e} = \omega_{0m}$).

The parameter $a = -(d\epsilon_{xx}/dx)^{-1}_{x=x_0}$ can also be written in the form

$$a = \left(\frac{1}{n}\frac{dn}{dx}\right)^{-1}_{x=x_0}.$$

The conditions in (26) and (27) apply for this parameter and, consequently, the plasma dimensions are bounded from above and from below. An analysis of these conditions shows that these limitations are extremely stringent in the low-frequency region $\omega \lesssim \omega_{Bi}$. In general, when $\omega \sim \sqrt{\omega_{bi}\,\omega_{Be}}$ and $\alpha \neq \pi/2$ the parameter a must be large compared with the ion Larmor radius ρ_i (calculated for the electron temperature). When $\omega^2 \gg \omega_{Bi} \times \omega_{Be}$ (and also when $\omega^2 \sim \omega_{Bi}\omega_{Be}$ and $\alpha = \pi/2$) it is necessary that $a \gg \rho_e$; finally, when $\omega \gg \omega_{Be}$ the parameter a must be large compared with the Debye radius. It should be noted that in a layer with a maximum in the density the quantity a varies from zero to infinity as the frequency changes in the limits of the transformation band so that the analysis given here does not apply near the boundaries of these bands.

3. SOLUTION OF THE COLD EQUATION FOR A HIGHLY INHOMOGENEOUS PLASMA

We have shown in the preceding sections that when the conditions (26) and (27) are satisfied the problem of wave conversion reduces to the solution of the cold equation (3). In the general case, that is to say, for an arbitrary angle α, these equations represent a system of two coupled second-order equations; because of mathematical difficulties, the solutions have been obtained only in the geometric-optics approximation.[4] We now consider the opposite limiting case, in which the thickness of the layer l satisfies the condition

$$\frac{\omega l}{c} N \ll 1,$$

where N is the refractive index for the wave and does not have a singularity; N is computed for the characteristic density of the layer being considered. The problem has been solved in [7] $\alpha = \pi/2$. Since the components of the tensor $\epsilon_{ik}(x)$ are linear functions of n, these components can be written in the form

$$\varepsilon_{ik} = \delta_{ik} + a_{ik}(1 - \varepsilon(x)),$$

where $\epsilon(x) \equiv \epsilon_{xx}$ and a_{ik} is a constant (independent of x) tensor. Then, choosing the coordinate system so that $k_y = 0$ we can convert (3) to the form

$$E_z' = -\frac{i(\varepsilon - \varkappa^2)}{\varkappa\varepsilon} D_x - \frac{i\varkappa(\varepsilon_{xy}E_y + \varepsilon_{xz}E_z)}{\varepsilon}, \quad (29)$$

$$D_x' = -a_{zx}E_z' - \frac{i(1-\varkappa^2)a_{zx}}{\varkappa} D_x - i\varkappa(a_{zx}\varepsilon_{xy} + \varepsilon_{zy})E_y \cdot - i\varkappa(a_{zx}\varepsilon_{xz} + \varepsilon_{zz})E_z, \quad (30)$$

$$E_y'' = \left[\varkappa^2 - \varepsilon_{yy} + \frac{\varepsilon_{yx}\varepsilon_{xy}}{\varepsilon}\right] E_y + \left[\frac{\varepsilon_{yx}\varepsilon_{xz}}{\varepsilon} - \varepsilon_{yz}\right] E_z - \frac{\varepsilon_{yx}}{\varepsilon} D_x, \quad (31)$$

where

$$D_x = \varepsilon E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z. \quad (32)$$

Here, we have introduced the dimensionless quantities $s = \omega x/c$ and $\kappa = k_z c/\omega$ and the primes denote differentiation with respect to s. It is easy to show that (29)–(31) are equivalent to (3) if one notes that elimination of D_x from (29)–(31) leads to the same system of equations for E_y and E_z as is obtained by eliminating E_x from (3).

We will solve the system in (29)–(31) assuming formally that the thickness of the layer l is the small parameter. According to (20) the field components E_y and E_z as well as E_y' remain finite (or have logarithmic singularities for $\sigma = 0$) at the singular points x_{0i} at which $\epsilon = 0$. It follows from (32) that the quantity D_x also remains finite at these points. Now, integrating the right and left parts of (30) from the boundary of the layer s_1 (to be definite we assume that this is the interface between the plasma and the vacuum) to an arbitrary point s, within the layer, we see that if l is small the integral of the last three terms will converge and be small so that

$$D_x = \frac{i\varkappa}{1-\varkappa^2} E_{z1}' - a_{zx}(E_z - E_{z1}), \quad (33)$$

where $E'_{z1} = E'_z(s_1)$ and $E_{z1} = E_z(s_1)$.

Substituting (33) in (29), to the same accuracy, we have

$$E_z' - E_{z1}' + i\sigma_0\frac{1-\varepsilon}{\varepsilon}(E_z - E_{z1}) = i\varkappa A\frac{1-\varepsilon}{\varepsilon} + \frac{i(1-\varkappa^2)a_{zx}}{\varkappa}(E_z - E_{z1}), \quad (34)$$

where

$$A = \frac{i\varkappa}{1-\varkappa^2}E_{z1}' - i\varkappa a_{xy}E_{y1} - i\varkappa a_{xz}E_{z1}, \quad \sigma_0 = \frac{c}{\omega a}\sigma,$$

$$a = \sum_i \left|\frac{1}{n}\frac{dn}{dx}\right|^{-1}_{x=x_{0i}}. \quad (35)$$

We will solve (34) regarding the right side as an inhomogeneous term. Then, if $|s_1 - s|$ is small the contribution from the last two terms on the right side of (34) will be small, and to a first approximation

$$E_z = E_{z1} + \varkappa A f(s), \quad (36)$$

where

$$f(s) = \frac{1}{\sigma_0}\left[1 - \exp\left(-i\sigma_0\int_{s_1}^{s}\frac{1-\varepsilon}{\varepsilon}ds'\right)\right], \quad (37)$$

and the singular points must be traversed from below or from above depending on which type of solution ($E^{(u)}$ or $E^{(l)}$) is desired.

It is easy to show that in this same approximation

$$E_x = E_{x_1} + A\left[\frac{1-\varepsilon}{\varepsilon} - \left(\frac{\sigma_0}{\varepsilon} - \varkappa a_{xz}\right)f(s)\right], \quad E_y = E_{y_1}. \quad (38)$$

The power Q absorbed by the plasma (for the $E^{(u)}$ solution) is determined from (21), where A is given by (35).

In case of a thin layer, greatest interest attaches to the representation of the solution of the external problem, that is to say, the determination of the fields outside the layer. Equations (31), (34), (36) and (38) can be used to relate the values of the fields and their derivatives to boundary layers s_1 and s_2, that is to say, they can be used to obtain in the boundary conditions which determine the field in the external region. For the case in which the point s_1 is the interface between a vacuum and the plasma and in which the density $n_2 \neq 0$ at the second boundary, these boundary conditions can be written

$$E_{z_2} = E_{z_1} + \varkappa f_2 A,$$
$$E_{z_2}' = E_{z_1}' + iA\left[\frac{\varkappa(1-\sigma_0 f_2)(1-\varepsilon_2)}{\varepsilon_2} + f_2(1-\varkappa^2 a_{zx})\right], \quad (39)$$
$$E_{y_2} = E_{y_1},$$
$$E_{y_2}' = E_{y_1}' + i a_{yx} f_2 A,$$

where the subscript "2" denotes quantities at $s = s_2$.

In the frequency region $\omega \gg \omega_{Bi}$, when $\omega_0^2(x_0)/\omega_{Be}^2 \gg m_e/m_i$ we have

$$a_{xy} = \frac{\omega_{Be}}{\omega_{Be}^2\cos^2\alpha + \omega_{Be}\omega_{Bi} - \omega^2}[\omega_{Be}\sin\alpha\cos\alpha\sin\varphi + i\omega\sin\alpha\cos\varphi],$$
$$a_{xz} = \frac{\omega_{Be}}{\omega_{Be}^2\cos^2\alpha + \omega_{Be}\omega_{Bi} - \omega^2}[\omega_{Be}\sin\alpha\cos\alpha\cos\varphi - i\omega\sin\alpha\sin\varphi],$$

where φ is the angle between the xH and xz planes. In the same approximation in which (39) holds, the layer at hand can be replaced, in solving the external problem, by a plane (located at any point in the interval $s_2 - s_1$) at which these same boundary conditions must be satisfied. If $l \to 0$ then, when $f_2 \to 0$ (39) becomes the usual boundary conditions at a sharp vacuum-plasma interface.[13] Obviously (39) also applies when $n_2 = 0$.

An estimate of the terms that have been neglected in the derivations of (33), (36), and (39) shows that these terms are small when

$$|a_{xz}|\frac{\omega l}{c} \ll 1, \quad |a_{xy}|\frac{\omega l}{c} \ll 1, \quad |a_{zz}|\frac{\omega l}{c} \ll |a_{xz}|. \quad (40)$$

Using the explicit form of the dielectric tensor it is a simple matter to write the limitations on the thickness of the layer l in explicit form. These are found to be very stringent at low frequencies $\omega \lesssim \omega_{Bi}$. When $\omega \gg \omega_{Bi}$ we must have $l \ll c/\omega_{Be}$ and when $\omega \gg \omega_{Be}$ we must have $l \ll c/\omega$. If the condition in (40) is satisfied the real part of the integral in (37) is always small and we can write

$$f_2 = \frac{1 - \exp[-\sigma_0\pi\omega a/c]}{\sigma_0}.$$

In general, $dn/dx \sim n/l$ and the parameter a and consequently also f_2 is small. In this case terms that contain f_2 in (39) need be considered only in computing the absorption coefficient, which is a linear function of f_2 when f_2 is small. However, in a layer with a density peak $a \to \infty$ when $\omega \to \Omega_{1,2}$ and the parameter f_2 must also be large under these conditions. (When $\sigma_0 \le 0$, $f_2 \to \infty$ when $a \to \infty$.) Equation (39) applies for arbitrarily large values of f_2 because, as can easily be shown, the higher approximations do not give rise to terms which contain higher powers of f_2. This circumstance derives from the fact that even in the first approximation the field singularity is proper. In the frequency region in which $|\sigma_0\omega a/c| \sim 1$ it is natural to expect a strong deviation from the case of a sharp boundary $\sigma = 0$.

As an illustration we now compute, using (39), the absorption coefficient A_0 for a thin plasma layer which is bounded by vacuum on both sides:

$$A_0 = |\mathbf{e}\mathbf{e}_0|^2\frac{4f_2a_0\cos\theta}{[2\cos\theta + f_2a_0]^2},$$

where $\mathbf{e}$ is the unit polarization vector of the incident wave and $\mathbf{e}_0$ is a unit vector with components

$$e_{0x} = \frac{1}{\sqrt{a_0}}\sin\theta(\sin\theta + a_{zx}\cos\theta),$$
$$e_{0y} = -a_{yx}/\sqrt{a_0}, \quad e_{0z} = -e_{0x}\operatorname{ctg}\theta,$$

where θ is the angle between the wave vector of the incident wave and the x axis

$$a_0 = |a_{xy}|^2 + |\sin\theta + a_{zx}\cos\theta|^2.$$

If $\sigma_0 \le 0$, the parameter f_2 varies from 0 to ∞ within the limits of the absorption band. In this case the quantity $4f_2a_0\cos\theta\,[2\cos\theta + f_2a_0]^{-2}$ reaches a maximum value of 0.5 when $f_2 = 2\cos\theta/a_0$. When $\sigma_0 > 0$ the parameter f_2 varies within the limits $0 \le f_2 \le 1/\sigma_0$ and for large values of σ_0 the maximum absorption occurs at the boundary of the absorption band. For a layer with a single density maximum (at x = 0) the parameter a close to the upper boundary is

$$a = b\sqrt{\frac{\Omega_{1,2}}{\Omega_{1,2} - \omega}}, \quad b = l_0\left(\frac{\Omega_{1,2}^2 - \omega_{Be}^2}{2\Omega_{1,2}^2 - \omega_{0m}^2 - \omega_{Be}^2}\right)^{1/2},$$
$$\Omega_{1,2}^2 = \frac{1}{2}\{(\omega_{Be}^2 + \omega_{0m}^2) \mp$$
$$\mp[(\omega_{Be}^2 + \omega_{0m}^2)^2 + 4\omega_{0m}^2(\omega_{Be}^2\cos^2\alpha + \omega_{Be}\omega_{Bi}\sin^2\alpha)]^{1/2}\}$$

where

$$l_0^{-2} = \frac{1}{n}\frac{d^2n}{dx^2}$$

when x = 0. In this case the quantity a_0 can be regarded as being independent of frequency (taking $\omega = \Omega_{1,2}$). The maximum absorption occurs at a frequency

$$\omega = \Omega_{1,2}\left[1 - \frac{\pi^2b^2\omega^2\sigma_0^2}{c^2\ln^2(1 - 2\sigma_0\cos\theta/a_0)}\right], \quad \sigma_0 < \frac{a_0}{2\cos\theta}.$$

By hypotheses $\omega^2b^2/c^2 \ll 1$ so that the maximum absorption occurs near the boundary of the absorption region. Attention is directed to the strong dependence of the position of the maximum on the angle α near the frequency of the lower hybrid resonance.

The authors are indebted to V. E. Golant for useful discussion.

[1] V. V. Zheleznyakov and E. Ya. Zlotnick, Izv. vuzov, Radiofizika 5, 644 (1962).

[2] V. B. Gil'denburg, Zh. Eksp. Teor. Fiz. 45, 1978 (1963) [Sov. Phys.-JETP 18, 1359 (1964)].

[3] A. D. Piliya, Zh. Tekh. Fiz. 36, 818 (1966) [Sov. Phys.-Tech. Phys. 11, 609 (1966)].

[4] V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in Plasma), Nauka, 1967.

[5] S. S. Moiseev, Proc. VII International Conference on Phenomena in Ionized Gases, Belgrade, 1965.

[6] T. H. Stix, Phys. Rev. Letters 15, 878 (1965).

[7] O. Ya. Omel'chenko and K. N. Stepanov, Ukr. fiz. zhurn. 12, 1445 (1967).

[8] H. H. Kuehl, Phys. Rev. 154, 124 (1967).

[9] L. N. Nosova and S. A. Tumarkin, Tablitsy obobshchennykh funktsiĭ Éĭri dlya asimptoticheskogo resheniya differentsial'nykh uravneniĭ $\epsilon(py')' + (q + \epsilon r)y = f$ (Tables of Generalized Airy Functions for the Asymptotic Solution of Differential Equations $\epsilon(py') + (q + \epsilon r)y = f$), 1961.

[10] V. V. Dolgopolov, Zh. Tekh. Fiz. 36, 273 (1966) [Sov. Phys.-Tech. Phys. 11, 198 (1966)].

[11] W. Wasow, Annals of Mathematics **52**, 350 (1950).
[12] V. D. Shafranov, Reviews of Plasma Physics, Consultants Bureau, New York, 1967, Vol. 3.
[13] V. R. Allis, S. D. Buchsbaum, and A. Bers, Waves in Anisotropic Plasmas, MIT Press, 1965.

Translated by H. Lashinsky
140