PAIR PRODUCTION BY A CONSTANT EXTERNAL FIELD

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Solutions of the Dirac equation in a constant electromagnetic field are obtained and their properties are analyzed. The causal Green's function is constructed in the Feynman representation. The differential probability is found for production of a pair by an external field, and also the probability for annihilation of a pair in a field without the emission of photons.

1. INTRODUCTION

As Klein showed, an external field which produces pairs leads to a "paradox": a state with positive frequency before scattering goes over after scattering into a superposition of states of positive and of negative frequency; in the one-particle theory the total probability of scattering of an electron wave packet is smaller than unity. On this basis some physicists have supposed that we encounter a difficulty here, or even that the concept of external field cannot be applied in this case.[1-3] On the other hand, in the theory of Feynman[4-6] no limitations of this kind are seen, and moreover Feynman made the assertion that in his theory the total probability of scattering is equal to unity.[6] In the present paper we consider a constant electromagnetic field, construct the solutions of the Dirac equation, and find the causal propagation function. We obtain the probability for annihilation of a pair in the field, and the differential probability for pair production. The total pair-production probability per unit volume and unit time agrees with the imaginary part of the Lagrangian found by Schwinger.[7] This fact, and also the correct result for the total cross section, give us reason to suppose that in the Feynman theory we do not meet any formal difficulties in the case of external fields which produce pairs.

2. SOLUTIONS OF THE DIRAC EQUATION AND CAUSAL GREEN'S FUNCTION IN A CONSTANT EXTERNAL FIELD

It is convenient to obtain the solution of the Dirac equation from the squared equation[1)]

$$(\hat{\pi}^2 + m^2)Z = (\pi^2 + g + m^2)Z = 0,$$

$$\hat{\pi} = \pi_\mu\gamma_\mu, \quad \pi_\mu = -i\frac{\partial}{\partial x_\mu} + eA_\mu, \quad g = -\frac{ie}{2}F_{\mu\nu}\gamma_\mu\gamma_\nu, \tag{1}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}.$$

We look for the solution Z in the form

$$Z = f(x)\Gamma,$$

where f(x) is a scalar function and Γ is one of the eigenspinors of the matrix g,

$$g\Gamma_i = X_i\Gamma_i, \quad i = 1, 2, 3, 4.$$

It is sufficient to consider only i = 1, 2:

$$X_1 = -eH - ieE, \quad X_2 = eH - ieE,$$

$$\Gamma_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The corresponding spinors ψ_r, r = 1, 2, are orthogonal and can be determined from the relation

$$\psi_r = (m - i\hat{\pi}) f_r(x)\Gamma_r. \tag{2}$$

Let us first find ψ_r for a constant electric field:

$$A_1 = A_2 = A_0 = 0, \quad A_3 = -Et;$$

$$A_\mu = a_\mu(kx), \quad a_\mu = (0, 0, a, 0), \quad k_\mu = (0, 0, 0, ik_0), \quad E = ak_0. \tag{3}$$

Such a field produces pairs, and this fact manifests itself in the solutions: if for t → ±∞ we choose the solutions describing a particle, i.e., the "positive-frequency" functions, then for t → ±∞ "negative-frequency" terms also appear in these solutions. In this way we obtain two complete orthonormal sets of functions ${}_\pm\psi_{\mathbf{p},r}(x)$ and ${}^\pm\psi_{\mathbf{p},r}(x)$. The first set consists of functions that have only one sign of the frequency for t → −∞, and the second, for t → +∞. The conservation of the normalization (and orthogonality) of the functions follows from the Dirac equation. From (1)–(3) we get

$${}^{-}_{+}\psi_{1,2}(x) = [L^3 2eE\lambda]^{-1/2}\exp\left(-\frac{\pi\lambda}{8} + i\mathbf{px}\right)$$

$$\times\left\{u_{1,2}D_{i\lambda/2}[\mp(1-i)\xi] \mp u'_{1,2}(1-i)\sqrt{eE}\frac{\lambda}{2}D_{i\lambda/2-1}[\mp(1-i)\xi]\right\},$$

$$\xi = \frac{p_3 - eEt}{\sqrt{eE}}, \quad \lambda = \frac{p_1^2 + p_2^2 + m^2}{eE}, \tag{4}$$

$$u_1, u_1', u_2, u_2' = \begin{bmatrix} p_1 - ip_2 \\ m \\ p_1 - ip_2 \\ -m \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} m \\ -(p_1 + ip_2) \\ m \\ p_1 + ip_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Here $D_\nu(z)$ are the parabolic-cylinder functions, and L^3 is the normalization volume. The functions ${}^{+}_{-}\psi_{1,2}$ are obtained from ${}^{-}_{+}\psi_{1,2}$ by the replacements

$$\sqrt{\frac{2}{\lambda}}D_{i\lambda/2}[\mp(1-i)\xi] \to D_{-i\lambda/2-1}[\mp(1+i)\xi],$$

$$\sqrt{\frac{\lambda}{2}}D_{i\lambda/2-1}[\mp(1-i)\xi] \to -D_{-i\lambda/2}[\mp(1+i)\xi].$$

We can define the sign of the frequency either by adia-

1) We use the system of units ħ = c = 1, $\alpha = e^2 = 1/137$, and the same representation of the γ matrices as is used in [8].

batically turning off the field at $t \to \pm\infty$ (cf.[9]), or by comparing the asymptotic form of (4) for $\xi \to \pm\infty$ with the expression for the classical action:

$$S = \mathbf{px} + \frac{\lambda}{2}\left\{\operatorname{Arsh}\frac{\xi}{\sqrt{\lambda}} + \frac{\xi}{\sqrt{\lambda}}\sqrt{1+\frac{\xi^2}{\lambda}}\right\} + \text{const.} \quad (5)$$

Such a comparison shows that for $\xi^2 \gg \lambda^2$ (we shall be considering $\lambda \gtrsim 1$) the motion is quasiclassical, so that in these regions the states of the particles are well defined. For $\xi^2 \lesssim \lambda^2$ there is an admixture of the other frequency in the solution; i.e., production of a pair with quantum numbers $\mathbf{p}$ occurs in a time interval centered at $t = p_3/eE$ and of width $\Delta t_{\text{eff}} \sim \lambda/(eE)^{1/2}$. The quantum number p_3/eE is analogous to the coordinate of the center of the orbit in a magnetic field. The physical meaning of this number is that if a particle is described by a packet made up of the corresponding functions (4) with quantum numbers close to $\mathbf{p}$, then the longitudinal kinetic momentum π_3 is equal to zero at $t = p_3/eE$, of course to within the uncertainty characteristic of the packet. In classical language we can say that the particle is slowed down by the field for $t < p_3/eE$ and is accelerated for $t > p_3/eE$. It is clear from this, first, that a pair with $\pi_3 \sim m^2/(eE)^{1/2}$ is produced by the field, and second, that p_3 determines the time of production of the pair, so that in a constant field the probability cannot depend on p_3, and the integration of the probability over p_3 is equivalent to integration over the time:

$$\int dp_3 \to eET. \quad (6)$$

We now present the expression for the causal Green's function of an electron in a constant electric field (the derivation is given in the Appendix):

$$G(x', x) = \begin{cases} \sum_{\mathbf{p}, r} {}^{+}\psi_{\mathbf{p}, r}(x')\, {}_{+}\bar{\psi}_{\mathbf{p}, r}(x) N_{\lambda, r} & \text{for} \quad t' > t \\ -\sum_{\mathbf{p}, r} {}_{-}\psi_{\mathbf{p}, r}(x')\, {}^{-}\bar{\psi}_{\mathbf{p}, r}(x) N_{\lambda, r} & \text{for} \quad t' < t \end{cases}. \quad (7)$$

Here $N_{\lambda, r}$ is determined from the relation

$$N_{\lambda, r} \int {}_{+}\psi^{*}_{p, r}(x)\, {}^{+}\psi_{p, r}(x)\, d^3x = i,$$

$$N_{\lambda, r} = -\frac{1}{2}\Gamma\left(\frac{i\lambda}{2}\right)\sqrt{\frac{\lambda}{\pi}}\, e^{\pi\lambda/4},$$

$$|N_{\lambda, r}|^2 = [1 - e^{-\pi\lambda}]^{-1}. \quad (8)$$

If in addition to the electric field there is also a magnetic field, then in a suitable Lorentz coordinate system we can take $\mathbf{E}$ and $\mathbf{H}$ to be directed along the axis 3. Instead of the potential (3) we choose

$$A_1 = A_0 = 0, \quad A_2 = Hx_1, \quad A_3 = -Et. \quad (9)$$

Equations (4), (7), and (8) remain valid if we make the replacement

$$L^{-1/2}\exp(ip_1x_1)\{1, p_1 - ip_2, p_1 + ip_2\}$$

$$\to (eH)^{1/4}[2^l l!\sqrt{\pi}]^{-1/2}\exp\left(-\frac{\eta^2}{2}\right)\left\{H_l(\eta), -i\sqrt{eH}\frac{dH_l(\eta)}{d\eta}, i\sqrt{eH}H_{l+1}(\eta)\right\}, \quad (10)$$

$$\eta = \sqrt{eH}(x_1 + p_2/eH), \quad l = 0, 1, 2, \ldots$$

$$\lambda \to \lambda_r = \begin{cases} \lambda_1 = \dfrac{m^2 + 2eHl}{eE} \\ \lambda_2 = \dfrac{m^2 + 2eH(l+1)}{eE}. \end{cases} \quad (10')$$

It can be seen from this that the state with the smallest energy (the smallest $\lambda = m^2/eE$) is the state with $r = 1$.

3. THE PRODUCTION OF PAIRS. ANNIHILATION OF A PAIR IN THE FIELD

Having the causal Green's function and well defined states of particles for $t \to \pm\infty$, we can find the relative probabilities of processes in the usual way. The amplitude for the relative probability of production of a pair is

$$M_{\mathbf{p}, r} = -i\int {}^{+}\bar{\psi}_{\mathbf{p}', r'}(x')\beta G(x', x)\beta\, {}^{-}\psi_{-\mathbf{p}, r}(x)\, d^3x' d^3x$$

$$= -iN_{\lambda, r}\delta_{\mathbf{p}', -\mathbf{p}}\delta_{r', r}\exp(-\pi\lambda_r/2), \quad t', t \to +\infty, \quad (11)$$

i.e., the state of the pair is completely determined by the state of one component. The relative probability for production of a pair in the state $\mathbf{p}, r$ in all space and during all time is

$$w_{\lambda, r} = |N_{\lambda, r}|^2\exp(-\pi\lambda_r) = [\exp(\pi\lambda_r) - 1]^{-1}. \quad (11')$$

The absolute probabilities are obtained by multiplying the relative probabilities by a factor $c_{\mathbf{p}, r}$, which is the probability that no pairs have appeared in the state $\mathbf{p}, r$. The quantity $c_{\mathbf{p}, r}$ is determined from the relation

$$c_{\mathbf{p}, r}(1 + w_{\mathbf{p}, r}) = 1, \quad (12)$$

i.e.,

$$c_{\mathbf{p}, r} = 1 - \exp(-\pi\lambda_r). \quad (12')$$

The first and second terms in (12) are the probabilities for production of no pairs and one pair in the state $\mathbf{p}, r$; the other terms are equal to zero because of the Pauli principle. Accordingly, the absolute probability for production of a pair in the state $\mathbf{p}, r$ in all space and during all time is

$$c_{\mathbf{p}, r}w_{\mathbf{p}, r} = \exp(-\pi\lambda_r). \quad (13)$$

It is clear that this is also the probability of the inverse process—the annihilation of a pair in the state $\mathbf{p}$, r with transfer of the energy to the external field. It can be seen from (13) and (10′) that for $H \gg E$ the pair appears mainly in the state $l = 0$, $r = 1$. In principle this fact can be used for the production of polarized beams of electrons and positrons.

We shall now show that (12) and (13) give for the imaginary part of the Lagrangian density the value $W/2$ found by Schwinger. In fact, the probability for the vacuum to remain the vacuum can be written in the form[7]

$$c_v = e^{-WVT}. \quad (14)$$

In terms of the $c_{\mathbf{p}, r}$ we have

$$c_v = \prod_{\mathbf{p}, r} c_{\mathbf{p}, r}. \quad (14')$$

Therefore, if we define the quantity $W_{\mathbf{p}, r}$ by the relation[2)]

$$c_{\mathbf{p}, r} = \exp[-W_{\mathbf{p}, r}VT], \quad \text{i. e.} \quad W_{\mathbf{p}, r} = -\frac{1}{VT}\ln[1 - \exp(-\pi\lambda_r)], \quad (14'')$$

[2)] The writer is grateful to N. B. Narozhnyĭ for calling attention to the fact that for $c_{\mathbf{p},r}W_{\mathbf{p},r} \sim 1$ one can no longer obtain W by summing (13) over the final states; instead of $c_{\mathbf{p},r}W_{\mathbf{p},r}$ one must start from the $W_{\mathbf{p},r}$ defined in (14″).

then

$$c_v = \exp\Big[-\sum_{\mathbf{p},r} W_{\mathbf{p},r} VT\Big], \quad \text{i.e.,} \quad W = \sum_{\mathbf{p},r} W_{\mathbf{p},r}.$$

Summation of $W_{\mathbf{p},r}$ over p_2 and p_3 makes this quantity finite also for $VT \to \infty$, because the density of final states is $(2\pi)^{-2}dp_2\,dp_3\,L^2$ and the integration over p_2 and p_3 is equivalent to setting [cf. (6)]

$$\int dp_2 = eHL, \qquad \int dp_3 = eET.$$

Accordingly we have

$$W_{l,r} = \int \frac{dp_2\,dp_3\,L^2}{(2\pi)^2} W_{\mathbf{p},r} = -\frac{e^2HE}{(2\pi)^2}\ln[1-\exp(-\pi\lambda_r)]. \qquad (15)$$

Summing (15) over l and r, we get the imaginary part of the Lagrangian in agreement with the result of Schwinger[7]

$$W = \sum_{l=0}^{\infty}\sum_{r=1}^{2} W_{l,r} = \frac{e^2HE}{(2\pi)^2}\sum_{n=1}^{\infty} n^{-1}\exp\Big[-\frac{\pi m^2 n}{eE}\Big]\operatorname{cth}\pi\frac{nH}{E}. \qquad (16)$$

It is a curious fact that the presence of the magnetic field increases each term in the sum (16) by the factor $x \operatorname{cth} x$, with $x = \pi nH/E$.

It is clear that the results obtained are valid in an arbitrary Lorentz coordinate system. We need only take E and H to mean the invariants

$$\mathscr{E} = [\sqrt{\mathscr{F}^2+\mathscr{G}^2}-\mathscr{F}]^{1/2}, \quad \mathscr{H} = [\sqrt{\mathscr{F}^2+\mathscr{G}^2}+\mathscr{F}]^{1/2},$$
$$\mathscr{F} = \frac{1}{4}F_{\mu\nu}^2, \quad \mathscr{G} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu}^*, \quad F_{\mu\nu}^* = \frac{i}{2}\varepsilon_{\mu\nu\sigma\lambda}F_{\sigma\lambda}. \qquad (17)$$

We note, finally, that the relative probability of scattering, analogous to (11) and (11′), is equal to $|N_{\mathbf{p},r}|^2$, so that the absolute total scattering probability $c_{\mathbf{p},r}|N_{\mathbf{p},r}|^2$ is equal to unity, as Feynman expected.

In conclusion, the author expresses his gratitude to V. L. Ginzburg, N. B. Narozhnyĭ, and V. I. Ritus for a fruitful discussion.

APPENDIX

THE ELECTRON PROPAGATOR IN THE PRESENCE OF A CONSTANT ELECTRIC FIELD

We define the electron propagator in the presence of a field which produces pairs by analytic continuation of the corresponding propagation function in the case of a field which does not produce pairs. It is easy to obtain the causal function in the case of a magnetic field and to write it in invariant form:

$$G(x',x) = [m - i\hat{\pi}(x')]G(x',x|A),$$

$$\pi_\mu(x') = -i\frac{\partial}{\partial x_\mu'} + eA_\mu(x'), \qquad \hat{\pi} = \pi_\mu\gamma_\mu,$$

$$G(x',x|A) = (4\pi)^{-2}\exp\Big[-\frac{i}{2}e(ay)(k,x'+x)\Big]\int_0^\infty \frac{ds}{s}\frac{e\sqrt{-2\mathscr{F}}}{\operatorname{sh}\tau}\cdot$$
$$\exp\Big\{-ism^2+\frac{iy^2}{4s}+\frac{i}{4s}\frac{(F_{\mu\nu}y_\nu)^2}{2\mathscr{F}}(\tau\operatorname{cth}\tau-1)\Big\}\Big[\operatorname{ch}\tau - i\frac{\sigma F}{2\sqrt{-2\mathscr{F}}}\operatorname{sh}\tau\Big], \qquad (A.1)$$

$$y = x'-x, \quad y^2 = \mathbf{y}^2 - y_0^2, \quad (k, x'+x) = (kx') + (kx), \quad \tau = es\sqrt{-2\mathscr{F}};$$
$$\mathscr{F} = \frac{1}{4}F_{\mu\nu}^2 = \frac{1}{2}(H^2-E^2), \quad \sigma F = \sigma_{\mu\nu}F_{\mu\nu} = 2(\Sigma\mathbf{H} - i\alpha\mathbf{E}),$$
$$\sigma_{\mu\nu} = -\frac{i}{2}[\gamma_\mu, \gamma_\nu],$$
$$(\pi^2+m^2)G(x',x|A) = \delta^4(x'-x).$$

Assuming now that A_μ is as given in Eq. (3) of the text, and $\mathscr{F} = \frac{1}{2}E^2$, we get G(x, x′) in the Schwinger representation.[7] We need, however, to obtain the Feynman representation. It is easy to verify that the Fourier transform of G is

$$\int G(x',x|A)\exp(i\mathbf{px})d^3x = \frac{1}{2}\exp\Big[i\mathbf{px}'+\frac{3\pi i}{4}\Big](\pi eE)^{-1/2}\int_0 [\operatorname{sh}\tau\operatorname{ch}\tau]^{-1/2}$$
$$\times\Big[\operatorname{ch}\tau - i\frac{\sigma F}{2E}\operatorname{sh}\tau\Big]\exp\Big\{-i\tau\lambda-\frac{i}{4}(\xi'+\xi)^2\operatorname{th}\tau-\frac{i}{4}(\xi'-\xi)^2\operatorname{cth}\tau\Big\}d\tau. \qquad (A.2)$$

We must then use the equation

$$J(\xi,\xi') = \int_0^\infty\Big[\frac{1}{2}\operatorname{sh}2\tau\Big]^{-1/2}\exp\Big\{-(i\lambda\mp 1)\tau-\frac{i}{4}(\xi'+\xi)^2\operatorname{th}\tau$$
$$-\frac{i}{4}(\xi'-\xi)^2\operatorname{cth}\tau\Big\}d\tau = \Gamma\Big(i\frac{\lambda}{2}+\frac{1}{2}\mp\frac{1}{2}\Big)$$
$$\begin{cases} D_{-i\lambda/2-1/2\pm1/2}[-(1+i)\xi']D_{-i\lambda/2-1/2\pm1/2}[(1+i)\xi] & \text{for } \xi'<\xi, \\ D_{-i\lambda/2-1/2\pm1/2}[(1+i)\xi']D_{-i\lambda/2-1/2\pm1/2}[-(1+i)\xi] & \text{for } \xi'>\xi, \end{cases} \qquad (A.3)$$
$$\operatorname{Im}(\xi'-\xi)^2 < 0, \qquad \operatorname{Im}\lambda < 0,$$

which can be proved by using the fact that

$$\Big[\frac{d^2}{d\xi^2}+\xi^2+\lambda\pm i\Big]J(\xi,\xi') = 0,$$

and the symmetry of $J(\xi,\xi')$ with respect to the interchanges $\xi' \to -\xi$; $\xi' \to \xi$; $\xi' \to -\xi'$; and $\xi \to -\xi$. Then we can verify directly that (A.1) is identically the same as Eq. (7) of the text.

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141