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KINETICS OF NONLINEAR WAVES IN DISPERSIVE MEDIA

G. M. ZASLAVSKIĬ and N. N. FILONENKO

Novosibirsk State University

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Solutions of the Korteweg—de Vries equation are considered in the form of nonlinear periodic waves with very large periods. It is shown that under certain conditions, superposition of such nonlinear waves can be regarded as the zeroth approximation of the solution. Corrections to the solution are found by perturbation theory. The perturbation is the interaction of resonant nonlinear waves. It is shown that if the number of nonlinear waves is large, then their phases are random functions of time. A kinetic equation is found which describes the ensemble of nonlinear waves. Some of its particular solutions are obtained. A power dependence of the mean wave energy on the wavelength is obtained and corresponds to the Kolmogorov turbulence theory.

1. INTRODUCTION

In recent years, a number of researches have been devoted to the theoretical and numerical investigation of the Korteweg—de Vries (KV) equation:

$$\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial x}+\frac{\partial^3 v}{\partial x^3}=0, \quad (1.1)$$

where v is the physical variable of the problem (for example, the velocity), x the coordinate, and t the time. Interest in the KV equation is associated with the fact that many problems describing nonlinear motion in a dispersive medium reduce to it. We refer here to waves in a channel of finite depth, ion sound, magnetic sound, etc. (see, for example,[1]). In a simplified variant, the KV equation also describes the characteristic properties of the initial equations.[1)] The stationary solution of the KV equation

$$v = v(x - ut) \quad (1.2)$$

has the form of a nonlinear periodic wave moving with velocity u (in what follows, we shall call this solution the S wave). A degenerate case of the solution in the form (1.2) is the solitary wave (soliton). A fundamental result for the KV equation at the present time is the solution of the Cauchy problem for an arbitrary smooth, initial profile v(x) bounded in space, obtained by Gardner, Greene, Kruskal and Miura.[2] As $t \to \infty$, the solution v(x, t) represents a set of a finite number of solitons which do not interact with one another, and whose parameters are determined from the corresponding equations. A rephrasing of this result is the existence of an asymptotic (as $t \to \infty$) principle of superposition for solutions of the type of solitons and the possibility of the determination of their parameters with the aid of a finite number of integrals of the KV equation of motion.[3]

We now proceed to the exposition of another viewpoint on the possible solutions of (1.1), associated with the Fermi-Past-Ulam problem.[4] The hypothesis enunciated by Fermi was that a system (even a one-dimensional one) of a large number of coupled nonlinear oscillators should possess statistical properties in its completely regular (non-stochastic) excitation. As a consequence, the energy included in the excited modes of oscillation should be distributed in the mean uniformly over all (or almost all) degrees of freedom, similar to what occurs for the Maxwellian distribution. Numerical analysis and analytic estimates, carried out by Izraĭlev and Chirikov,[5] confirmed the considerations of Fermi. However, in the approximation of a continual distribution and not very strong nonlinearity of the system of oscillators considered in[4], it is described by the KV equation.[6] If the approximation of such a description is not important, then solutions of the KV equation of a different type than sound in[2] should exist. That is, such solutions are possible in which v(x, t) is a random function of the variables and, furthermore, a function F(v, t) exists which represents the statistical distribution function in v which, as $t \to \infty$, tends to a stationary value, which ensures thermodynamic equilibrium. The considerations just stated can be called the problem of finding the stochastic solutions of the KV equation.

Finally, there is one more feature of the solutions of the KV equation, closely associated with the Fermi-Past-Ulam problem. Inasmuch as Eq. (1.1) describes a definite class of hydrodynamic motions, the stochastic solutions should also describe the turbulent motion of the medium. The problem of the existence of stochastic solutions for the KV equation with strong nonlinearity and without the introduction of additional random sources remains open at the present time.

The method of investigation set forth below will be applied to an arbitrary nonlinear medium with dispersion, in which solutions of the S-wave type exist. We shall show the range of parameters in which one can construct stochastic solutions, we shall obtain the kinetic equation for the distribution function of states of the system, and we shall make clear the properties of some solutions of this equation. The resultant stochastic solutions correspond to motions with a large Reynolds number.

[1)] An exception is "wave breaking," which is absent in the KV equation.

2. THE FUNDAMENTAL EQUATIONS

We shall write down in brief form the properties of a stationary solution of the KV equation which will be used essentially. Moreover, we shall also study the case in which the length of the S-wave $\lambda \to \infty$. Therefore, all the expressions will be written down in the corresponding approximation.

The periodic solution describing the S wave in the case under discussion has the form

$$v = 3\alpha \cdot \mathrm{cn}^2\left[\frac{\sqrt{3\alpha}}{2}(x-ut);\, 1\right], \tag{2.1}$$

where cn is the elliptic cosine (here its modulus is unity); the quantity $\alpha = u - 1$ is the difference of the S-wave velocity and the characteristic sound velocity for the problem considered, the latter velocity taken as unity. In the following, $\alpha > 0$ and $\alpha \ll 1$, the latter inequality is not necessary; however, the KV equation usually has physical meaning only for small α. By virtue of the periodicity of (2.1), it can be expanded in a Fourier series:

$$v = \sum_{n=-\infty}^{\infty} a_n e^{ink(x-ut)}, \quad k = \frac{2\pi}{\lambda} = \frac{\sqrt{\alpha}}{N}. \tag{2.2}$$

Here N is the characteristic number of harmonics a_n in the S-wave spectrum. For $n > N$, the amplitudes of the expansion fall off exponentially as $\exp(-n/N)$. As $k \to 0$, we have $N \to \infty$. The number N can be given still another meaning. According to (2.1), the width of the humps in Fig. 1 is equal to $\sim 1/\sqrt{3\alpha}$, and their separation is λ. According to the last equality in (2.2), N is equal to the ratio of the distance between humps to their width. We also write down the value of the Fourier amplitude a_n:

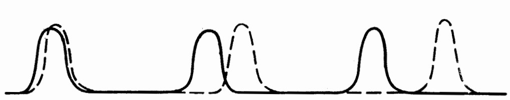

FIG. 1. Profile of two S waves and their overlapping.

$$a_n \approx 3\alpha/N \quad (n \lesssim N). \tag{2.3}$$

The Hamiltonian formalism for the KV equation looks like the following. We introduce the Hamiltonian

$$\mathscr{H} = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} dx\left[\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - \frac{1}{6}v^3 - \frac{1}{6}v^2\right]$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} dq(1-q^2)v(q)v(-q)$$

$$-\frac{1}{6}\int_{-\infty}^{\infty} dq_1\, dq_2\, dq_3\, v(q_1)v(q_2)v(q_3)\delta(q_1+q_2+q_3), \tag{2.4}$$

where the expansion

$$v = \int_{-\infty}^{\infty} dq\, e^{iqx} v(q), \quad v(-q) = v^*(q). \tag{2.5}$$

is used. Equation (1.1) is equivalent to the following canonical equations of motion:

$$\frac{dv(q)}{dt} = iq\frac{\delta\mathscr{H}}{\delta v(-q)}, \quad \frac{dv(-q)}{dt} = -iq\frac{\delta\mathscr{H}}{\delta v(q)}, \tag{2.6}$$

and the time derivative of the arbitrary functional P is computed from the formula

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + i\int q\left(\frac{\delta P}{\delta v(q)}\frac{\delta\mathscr{H}}{\delta v(-q)} - \frac{\delta P}{\delta v(-q)}\frac{\delta\mathscr{H}}{\delta v(q)}\right)dq. \tag{2.7}$$

3. STATEMENT OF THE PROBLEM AND THE EXPANSION PARAMETER

Let $H_S = H_S(k_S, u_S)$ be the Hamiltonian of the S wave $v_S(x - u_S t)$. In view of the nonlinearity of the KV equation, the superposition

$$v = \sum_{s=1}^{S} v_s \tag{3.1}$$

is not a solution of Eq. (1.1). Below, we shall show that if the condition

$$\alpha_s \ll 1, \quad N_s \gg 1 \tag{3.2}$$

is satisfied for all s (these relations denote the "closeness" of v_S to solutions of the "soliton" type), then Eq. (3.1) can be used as a zeroth approximation, the corrections to which are computed with the help of the corresponding perturbation theory.[2)]

We apply the following limitations to the packet (3.1) of S waves:

1. $S \ll N$.
2. Different values k_S of the wave numbers differ from one another by the quantity Δk:

$$k \gg \Delta k \gtrsim k/N, \tag{3.3}$$

where k and N are certain values of k_S and N_S and are characteristic for the packet. The meaning of the inequality (3.3) will be revealed below. For simplicity, we also take $\Delta\alpha/\alpha \sim \Delta k/k$.

3. The numbers k_S are incommensurate, so that to each q of the discrete series of values entering in the expansion of the quantity v in zeroth approximation one can ascribe one and only one index r, which indicates that the given wave number belongs to the spectrum of the r-th S wave.

We write down the Hamiltonian (2.4) in the form

$$\mathscr{H} = \sum_s H_s + H_I, \tag{3.4}$$

where

$$H_s = -\frac{1}{2}\int dq_s(1-q_s^2)v(q_s)v(-q_s) -$$

$$-\frac{1}{6}\int dq_s dq_s' dq_s'' v(q_s)v(q_s')v(q_s'')\delta(q_s+q_s'+q_s''),$$

$$H_I = -\frac{1}{6}\sum_{s_1,s_2,s_3}{}' \int dq_{s_1}dq_{s_2}dq_{s_3} v(q_{s_1})v(q_{s_2})v(q_{s_3})\delta(q_{s_1}+q_{s_2}+q_{s_3}),$$

$$v_s = \int dq_s e^{iq_s x} v(q_s). \tag{3.5}$$

The prime on the sum indicates exclusion of the number with $s_1 = s_2 = s_3$. Integration is carried out in H_S over the values of q belonging to one and the same s; conversely, the indices for q in H_I show that the volumes of q refer to different s. We estimate H_I, taking v_S as given by Eq. (2.1). Let the periods of the S waves differ by

$$\Delta\lambda \gtrsim 1/\sqrt{\alpha}, \tag{3.6}$$

i.e., by more than the width of the hump. In this case,

2)The expansion in the exact stationary solutions of the nonlinear equation was used in [7]. The possibility of application of the weak coupling theory to problems of strong turbulence is connected with such a form of the expansion, as we shall see below. [8]

as is seen from Fig. 1, coincidence of the humps belonging to different s takes place roughly every N periods. This means that the overlap integrals that determine H_I have the order of S/N, which is small in view of the limitation 1. Limitation 2 is equivalent to (3.6), since $\Delta k = k^2 \Delta\lambda \gtrsim k^2/\sqrt{\alpha}$. Finally, limitation 3 allows us to write down the canonical equations for the Fourier harmonics $v(q_s)$ belonging to a particular S wave, similarly to (2.6):

$$\frac{dv(q_s)}{dt} = iq_s \frac{\delta\mathscr{H}}{\delta v(-q_s)}, \quad \frac{dv(-q_s)}{dt} = -iq_s \frac{\delta\mathscr{H}}{\delta v(q_s)}. \qquad (3.7)$$

Thus, in accord with (3.4), the Hamiltonian of the problem is equal, in zeroth approximation, to the sum of the energies of each of the S waves, and the perturbation H_I takes into account their interaction, which is of the order of $S\Delta k/k$:

$$S/N \lesssim S\Delta k/k \ll 1.$$

Actually, the right hand inequality should be the more stringent, and we shall make it more precise later.

We set

$$v(q_s) = a(n_s)e^{in_s\vartheta_s}, \quad a(-n_s) = a^*(n_s), \quad q_s = n_s k_s. \qquad (3.8)$$

In the zeroth approximation, in accord with (2.2),

$$d\vartheta_s/dt = \omega_s = u_s k_s, \qquad (3.9)$$

and $u_s = u_s(H_s, k_s)$. It is convenient in what follows to consider each of the S waves as a degree of freedom of the packet (3.1), characterized by the variables energy and phase, (H_s, ϑ_s).

4. RESONANT PROCESSES AND THE CONDITION OF RANDOM PHASE FOR S WAVES

We consider the change in the variables (H_s, ϑ_s) associated with account of interaction of the S waves. According to (2.7) and (3.4),

$$\frac{dH_s}{dt} = i\int q_s \left(\frac{\delta H_s}{\delta v(q_s)}\frac{\delta H_I}{\delta v(-q_s)} - \frac{\delta H_s}{\delta v(-q_s)}\frac{\delta H_I}{\delta v(q_s)}\right) dq_s. \qquad (4.1)$$

In view of the smallness of H_I, we can use the zeroth approximation for $\delta H_s/\delta v(\pm q_s)$. Taking into account the expressions (3.7)–(3.9), we get the following:

$$\frac{dH_s}{dt} = -iu_s \int q_s \left[v(-q_s)\frac{\delta H_I}{\delta v(-q_s)} - v(q_s)\frac{\delta H_I}{\delta v(q_s)}\right] dq_s. \qquad (4.2)$$

In (4.2), we discard those terms which can lead to resonances and, consequently, are of higher order. For this, we note that

$$\frac{\delta H_I}{\delta v(q_s)} = -\frac{1}{2}\sum_{s_1,s_2}\int dq_s dq_{s_1} dq_{s_2} v(q_{s_1}) v(q_{s_2}) \delta(q_s + q_{s_1} + q_{s_2}) \qquad (4.3)$$

and exclude the case $s = s_1 = s_2$. With each factor $v(q_s)$ there enters in (4.2) the factor $\exp(-iu_s q_s t)$. It then follows that the simplest resonances take place when the following conditions are simultaneously satisfied:

$$q_s + q_{s_1} + q_{s_2} = 0,$$
$$u_s q_s + u_{s_1} q_{s_1} + u_{s_2} q_{s_2} = 0. \qquad (4.4)$$

However, the latter is possible only when all the numbers of the triplet (s, s_1, s_2) are different.

A resonance of the type

$$q_s + q_{s_1} + q_{s_2} + q_{s_3} = 0,$$
$$u_s q_s + u_{s_1} q_{s_1} + u_{s_2} q_{s_2} + u_{s_3} q_{s_3} = 0$$

can occur even when a pair of the numbers s_i are identical; however, it is connected with the interaction in second order in H_I and gives a contribution that is smaller than (4.4). Thus, in (4.2), we should discard the terms corresponding to a resonance of the type (4.4), in which harmonics of three different S waves enter.

Substituting (4.3) in (4.2) and taking the foregoing into account, we get

$$\frac{dH_s}{dt} = -\frac{i}{2}\beta k_s u_s \sum_{s_1,s_2}\sum_{n_s,n_{s_1},n_{s_2}} n_s\{2a(n_s)a(n_{s_1})a^*(n_{s_2})$$
$$\times \exp\{-i(n_s\vartheta_s + n_{s_1}\vartheta_{s_1} - n_{s_2}\vartheta_{s_2})\}\delta_{n_s k_s + n_{s_1}k_{s_1} - n_{s_2}k_{s_2}}$$
$$+ a^*(n_s)a(n_{s_1})a(n_{s_2})\exp\{i(n_s\vartheta_s - n_{s_1}\vartheta_{s_1} - n_{s_2}\vartheta_{s_2})\}\cdot$$
$$\delta_{n_s k_s - n_{s_1}k_{s_1} - n_{s_2}k_{s_2}} + \text{c.c.}\}. \qquad (4.5)$$

Here β is a normalization constant connected with the transition from integration to summation; the numbers n_{s_i} change from 1 to ∞; c.c. means terms that are the complex conjugates of the preceding terms; the function $\delta_{n_s\lambda_s \pm n_{s_1}k_{s_1} - n_{s_2}k_{s_2}}$ must be understood here and below as the Kronecker symbol. The system (4.4) must be supplemented by equations for the change in phase ϑ_s, which, in first order perturbation theory, retain the form (3.9), with the only difference that now the H_s depend on t in correspondence with (4.5).

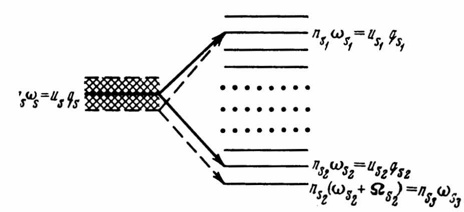

FIG. 2. Scheme of overlapping of resonances between three S waves.

We investigate the change of the quantities (H_s, ϑ_s) as a consequence of the resonances, similar to what was done in[9]. Change of H_s under action of process (4.4) leads to a change in the frequency ω_s because of nonlinearity and, consequently, to a violation of the condition (4.4). In this connection, the maximum possible change of energy takes place because of the resonance, which change we shall denote by δH_s, and the maximum change in frequency associated with it is:

$$\delta\omega_s = \frac{d\omega_s}{dH_s}\delta H_s. \qquad (4.6)$$

The solid arrows connect the frequencies of the harmonics entering in the resonance (4.4).

We now introduce the distance between the resonances Ω_s, equal to the value by which one must change ω_s in order that a resonance of type (4.4) be possible with the possible values of the wave numbers in the spectrum closest to q_{s_1} and q_{s_2} (for example, the process shown by dashed arrows in Fig. 2). Under the condition

$$K_s = (n_s\delta\omega_s/\Omega_s)^2 \gg 1 \qquad (4.7)$$

the harmonic with number n_s is constantly in resonance with some pair of harmonics (n_{s_1}, n_{s_2}) and a process of type (4.4) is constantly sustained. In this case (see[5,9]) the change in phase ϑ_S with time can be regarded as random.

We shall make an estimate of the parameter K_S. For this purpose, we note that the process (4.4) can be rewritten in the form

$$n_s k_s \pm n_{s_1} k_{s_1} - n_{s_2} k_{s_2} = 0,$$
$$n_s \nu_s \pm n_{s_1} \nu_{s_1} - n_{s_2} \nu_{s_2} = 0 \quad (n_i > 0),$$
$$\nu_i = k_i \alpha_i = k_i (u_i - 1). \tag{4.8}$$

In the sum (4.5) we remove the term corresponding, for example, to the minus sign in (4.8). This term is proportional to

$$\exp i(n_s \theta_s - n_{s_1} \theta_{s_1} - n_{s_2} \theta_{s_2}),$$

where

$$d\theta_s / dt = \nu_s (H_s). \tag{4.9}$$

From (4.5) and (4.6) we obtain the estimate

$$n_s \delta\omega_s = n_s \delta\nu_s \sim \left[n_s^2 S^2 k_s \frac{d\nu_s}{dH_s} a_{n_s}^3 \right]^{1/2}.$$

Transforming in (3.5) from integration to summation, assuming that $\alpha_S \ll 1$, we have

$$dH_s / d\nu_s \sim \sqrt{\alpha_s}. \tag{4.10}$$

We then get, by using the expression (2.3) for $a(n_S)$,

$$n_s \delta\omega_s \sim \frac{S}{N_s} n_s k_s \alpha_s \quad (n_s \lesssim N_s). \tag{4.11}$$

Inasmuch as, in accord with (4.8),

$$\Omega = \alpha \Delta k + k \Delta\alpha,$$

it is then easy to write the criterion (4.7) in explicit form:

$$K = \left(\frac{S}{N} n \frac{k}{\Delta k} \right)^2 \gg 1 \quad (n \lesssim N), \tag{4.12}$$

where the subscripts are omitted for convenience. It follows from (4.12) that the criterion for stochasticity for some $n < n_0$ cannot be satisfied and, conversely, for sufficiently large n, close to N, it is always satisfied. In the optimal case, substituting (3.3) in (4.12), we have simply

$$K = S^2 n^2 \gg 1.$$

This means that in fact, all the harmonics of the S wave have random phases.

The time of randomization of the phase τ_n of the harmonic with frequency $n\omega$ is determined from[9]:

$$\tau_n \sim (NS\Omega \ln K)^{-1}.$$

It follows from (4.7) and (4.12) that the number of overlapping resonances, equal to $\sqrt{K}$, leads to an effective increase in the small parameter of the problem. The condition of smallness of H_I should now have the form

$$S^2 / N \ll 1.$$

Actually, the change in α is equal to

$$\Delta\alpha \sim \sqrt{K}\, \delta\alpha = \alpha \sqrt{K} S \frac{\Delta k}{k} \sim \alpha S^2 / N,$$

and the inequality written down above immediately follows from the condition $\Delta\alpha \ll \alpha$.

5. THE KINETIC EQUATION FOR S WAVES

The results of the previous section allow us to proceed to the derivation of the fundamental kinetic equation ("master equation") for S waves in the random phase approximation.

We introduce the distribution function $f(t, H, \theta)$, where (H, θ) denote the set of all variables $(H_1, \ldots, H_S;\ \theta_1, \ldots, \theta_S)$. The Liouville equation for f has the form

$$\frac{\partial f}{\partial t} + \sum_s \nu_s \frac{\partial f}{\partial \theta_s} \frac{\partial \mathcal{H}}{\partial H_s} - \sum_s \nu_s \frac{\partial f}{\partial H_s} \frac{\partial \mathcal{H}}{\partial \theta_s} = 0$$

or, after substituting (3.4) for H,

$$\frac{\partial f}{\partial t} + \sum_s \nu_s \frac{\partial f}{\partial \theta_s} = \sum_s \nu_s \left(\frac{\partial H_I}{\partial \theta_s} \frac{\partial f}{\partial H_s} - \frac{\partial H_I}{\partial H_s} \frac{\partial f}{\partial \theta_s} \right). \tag{5.1}$$

We expand f in a Fourier series:

$$f(H, \theta, t) = \sum_{\{m\}} \left(f^{(m)}(H, t) \exp\left[i\left(m, \theta - \int^t \nu dt\right) \right] + f^{-(m)}(H, t) \exp\left[-i\left(m, \theta - \int^t \nu dt \right) \right] \right)$$
$$f^{(m)} = (f^{-(m)})^*, \tag{5.2}$$

where the summation is carried out over all possible sets of numbers $\{m\}$ and

$$(m, \varphi) \equiv \sum_r m_r \varphi_r.$$

In the substitution of (5.2) in (5.1), we take it into account that the frequencies ν change, according to (4.11), by an amount

$$\Delta\nu = S\nu / N \tag{5.3}$$

as a consequence of the resonances. Equation (5.3) means that the effect of the resonances is revealed in the next order of smallness of the parameter S/N, according to which the expansion is carried out in the derivation of the kinetic equation. This allows us to use in what follows a diagram technique of the same type as in the work of Prigogine-Brout.[10]

From (5.1), (5.2), and (3.5), we have

$$\frac{\partial F}{\partial t} = -\frac{3}{2} \sum_{s, s_1, s_2} \sum_{n_s, n_{s_1}, n_{s_2}} \sum_{\{m\}} (2\pi)^{-S} \int d\theta_1 \ldots d\theta_S \Big(i \exp\Big\{ i[n\theta] + i \sum_r m_r \Big(\theta_r - \int^t \nu_r dt \Big) \Big\} a^*(n_s) a(n_{s_1}) a(n_{s_2}) \left[n\nu \frac{\partial}{\partial H} \right] f^{(m)}$$
$$- i \Big\{ \left[n\nu \frac{\partial}{\partial H} \right] a^*(n_s) a(n_{s_1}) a(n_{s_2}) \Big\} \exp\Big\{ i[n\theta] + i \sum_r m_r \Big(\theta_r - \int^t \nu_r dt \Big) \Big\} f^{(m)} \Big) \delta_{[nk]} + \text{c.c.}$$
$$= -\frac{3}{2} i \sum_{s, s_1, s_2} \sum_{n_s, n_{s_1}, n_{s_2}} \Big\{ \left[n\nu \frac{\partial}{\partial H} \right] a^*(n_s) a(n_{s_1}) a(n_{s_2}) \times \exp\Big(-i \Big[n \int^t \nu dt \Big] \Big) \delta_{[nk]} f^{(n_s), (n_{s_1}), (n_{s_2})} \Big\} + \text{c.c.}$$
$$F \equiv f^{(0)}(H, t), \tag{5.4}$$

where the notation

$$[nQ] \equiv n_s Q_s - n_{s_1} Q_{s_1} - n_{s_2} Q_{s_2}$$

is used. Each exponent of the type $\exp(i n_S \theta_S)$ we shall represent by thin lines included within two heavy lines. Thus, for example, the transition

$$\langle 0 | \rightarrow | n_s \rangle$$

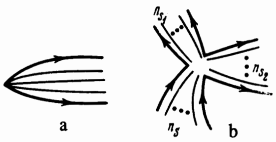

FIG. 3. Diagrams of the simplests transitions: a—$\langle 0| \to |n_s\rangle$, b—$\langle n_s| \to |n_{s_1}\rangle + |s_2 s$.

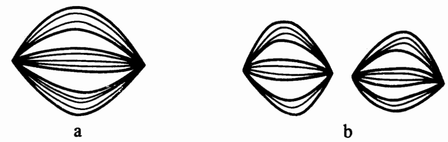

FIG. 4. Examples of diagrams which determine the principal terms of a perturbation theory series for Eq. (5.4).

is shown in Fig. 3a for $n_s = 3$. The process (4.8) is shown in Fig. 3b. The set of thin lines, bounded by the two heavy lines, refers to a fixed S wave and can be called the strong coupling element, since a change of any of the Fourier components $a(n_s)$ leads to a change in all the remaining Fourier components of the same S wave. Summation in (5.4) over n_i takes into account the effect of the strong coupling within the i-th S wave. The vertex on the diagram 3b takes into account the weak coupling ($\sim S/N$) between three strong coupling elements. We shall further express F from (5.4) in the form of a formal series in powers of H_I, discard the principal terms of the series and consider the initial conditions

$$f^{(m)}(t=0) = F(t=0)\,\delta_{m,0},$$

corresponding to the random phases θ. An example of the principal terms in second and fourth orders is shown in Figs. 4a and b, respectively. Omitting the standard contributions, we obtain the following kinetic equation:

$$\frac{\partial F}{\partial t} = 6\pi \sum_{s,s_1,s_2} \sum_{n_s,n_{s_1},n_{s_2}} \left[n\nu\frac{\partial}{\partial H}\right] |a^*(n_s)a(n_{s_1})a(n_{s_2})|^2 \times \delta([n\nu])\,\delta_{[nk]}\left[n\nu\frac{\partial}{\partial H}\right]F, \tag{5.5}$$

where the $a(n_i)$ are expressed as functions of the H_I.

We introduce the action variation I_s with the help of the relation

$$\frac{\partial}{\partial I} = \nu(H)\frac{\partial}{\partial H} = k\alpha(H)\frac{\partial}{\partial H}. \tag{5.6}$$

With accuracy up to a numerical coefficient of order unity, we obtain the following

$$\alpha = (H/k)^{2/3}. \tag{5.7}$$

Substitution of (5.7) in (5.6) gives the coupling

$$H = kI^3, \quad \alpha = I^2. \tag{5.8}$$

Moreover, in accord with (2.3),

$$a_n = \alpha / N = \sqrt{\alpha}k = kI. \tag{5.9}$$

We now assume that the amplitudes a_n are exponentially small for $n > N$. This allows us to limit the summation over n_i in (5.5). Substituting (5.6), (5.8), (5.9), in (5.4), we finally obtain

$$\frac{\partial F}{\partial t} = 6\pi \sum_{s,s_1,s_2} \sum_{n_s,n_{s_1},n_{s_2}}^{N_s,N_{s_1},N_{s_2}} \left[n\frac{\partial}{\partial I}\right] G_{ss_1s_2} \times \delta([nkI^2])\,\delta_{[nk]}\left[n\frac{\partial}{\partial I}\right]F, \tag{5.10}$$

where $N_i = \sqrt{\alpha_i}/k_i = I_i/k_i$,

$$G_{ss_1s_2} = (k_s k_{s_1} k_{s_2} I_s I_{s_1} I_{s_2})^2, \quad [nkI^2] = n_s k_s I_s^2 - n_{s_1}k_{s_1}I_{s_1}^2 - n_{s_2}k_{s_2}I_{s_2}^2, \tag{5.11}$$

In the derivation of the kinetic equation (5.10), we have actually assumed that the set of S waves remains closed. This imposes the following limitations, in addition to those shown in Sec. 2: 1) we consider such a class of initial conditions, in which the statistical weight of solutions of the S-wave type is very large in comparison with the weight of the other possible solutions; 2) the deviation of the initial distribution function F(t = 0) from equilibrium, which causes the right hand side of (5.10) to vanish, is rather small. The latter leads to small relaxation times in comparison with the time of spreading out of the S waves, which is associated with corrections of the next order of smallness in Eqs. (4.5) and (4.9).

6. PARTICULAR PROPERTIES OF THE KINETIC EQUATION AND TURBULENT MOTION

The equilibrium solution of the kinetic equation is

$$F = F\Big(\sum_s H_s\Big), \tag{6.1}$$

which causes the right-hand side of (5.5) to vanish. It is of interest to discuss other possibilities connected with the Kolmogorov theory of turbulent motion. As has already been noted in the Introduction, the velocity of nonlinear waves in real problems has an upper limit, above which the wave breaks. We denote by α_c the critical value of the parameter α, which does not depend on s. The motion of the individual S wave with random phase can be described in the same way as a Brownian particle, which leads on average to an increase in α_s.[11] This process is accompanied by a narrowing of the humps of the S wave and the appearance of high values of wave numbers q at the expense of an increase in N. When α_s reaches the value α_c, the wave breaks and its energy is dissipated into another (multistream) form of motion. If the number of breaking waves is small in comparison with S, then their reaction on the process described by Eq. (5.10) can be neglected. Here the stationary picture corresponds to a constant energy flow

$$\frac{d}{dt}\langle H_r\rangle = \mathrm{const} \equiv C, \tag{6.2}$$

where C does not depend on r, and the brackets denote averaging over F(H, t). The expression (6.2) can be rewritten in the form

$$\int dH_1 \ldots dH_r \ldots dH_S H_r \frac{\partial F(H_1, \ldots, H_r, \ldots, H_S, t)}{\partial t} = C. \tag{6.3}$$

The dependence of the average energy

$$E_r(k_r) = \langle H_r\rangle$$

on k_r can be estimated from the considerations of dimensionality.

We note that a change in energy over the wave spectrum takes place in quite a different fashion than in the theory of weak coupling. In the reversal of the r-th S wave, the Fourier harmonics corresponding to wave numbers $n_r k_r$ disappear; here n_r changes from unity to

N_r. Thus the energy is added uniformly over nearly the entire spectrum.

We multiply (5.10) by I_r^m and integrate the equation over all H_s. This gives

$$\frac{d}{dt}\langle I_r{}^m\rangle = 6\pi m \sum_{s,\, s_1,\, s_2} \Big\langle \sum_{n_s,\, n_{s_1},\, n_{s_2}}^{N_s,\, N_{s_1},\, N_{s_2}} (n_s I_s^{m-1}\,\delta_{rs} - n_{s_1} I_{s_1}^{m-1}\,\delta_{rs_1}$$

$$- n_{s_2} I_{s_2}^{m-1}\,\delta_{rs_2})\, G_{ss_1s_2}\delta([nkI^2])\,\delta_{[nk]}\left[n\frac{\partial}{\partial I}\right]\Big\rangle$$

$$= 6\pi m \sum_{s_1,\, s_2} \Big\langle \sum_{n_r,\, n_{s_1},\, n_{s_2}}^{N_r,\, N_{s_1},\, N_{s_2}} n_r I_r{}^{m-1}\Big\{G_{rs_1s_2}\delta(n_r k_r I_r{}^2 - n_{s_1}k_{s_1}I_{s_1}{}^2$$

$$- n_{s_2}k_{s_2}I_{s_2}{}^2)\,\delta_{n_rk_r - n_{s_1}k_{s_1} - n_{s_2}k_{s_2}}\left(n_r\frac{\partial}{\partial I_r} - n_{s_1}\frac{\partial}{\partial I_{s_1}} - n_{s_2}\frac{\partial}{\partial I_{s_2}}\right)$$

$$- 2G_{s_1rs_2}\delta(n_{s_1}k_{s_1}I_{s_1}{}^2 - n_r k_r I_r{}^2 - n_{s_2}k_{s_2}I_{s_2}{}^2)\,\delta_{n_{s_1}k_{s_1} - n_rk_r - n_{s_2}k_{s_2}}$$

$$\left(n_{s_1}\frac{\partial}{\partial I_{s_1}} - n_r\frac{\partial}{\partial I_r} - n_{s_2}\frac{\partial}{\partial I_{s_2}}\right)\Big\}\Big\rangle.$$

With the help of δ functions, we carry out summation over the n_i:

$$k_r\frac{d}{dt}\langle I_r{}^m\rangle = 6\pi m\sum_{s_1,\, s_2}\Big\langle\sum_{n_r}^{N_r} n_r{}^2 I_r^{m-3}\Big\{G_{rs_1s_2}\left(\frac{\partial}{\partial I_r} - Q_1\frac{\partial}{\partial I_{s_1}} - Q_2\frac{\partial}{\partial I_{s_2}}\right)$$

$$- 2G_{s_1rs_2}\left(P_1\frac{\partial}{\partial I_{s_1}} - \frac{\partial}{\partial I_r} - P_2\frac{\partial}{\partial I_{s_2}}\right)\Big\}\Big\rangle \approx 2\pi m k_r{}^{-3}\Big\langle I_r{}^m\sum_{s_1,\, s_2}\Big\{G_{rs_1s_2}\Big[3\frac{\partial}{\partial I_r}$$

$$- (Q_1 + 2P_1)\frac{\partial}{\partial I_{s_1}} - (Q_2 - 2P_2)\frac{\partial}{\partial I_{s_2}}\Big]\Big\}\Big\rangle, \tag{6.4}$$

where we use the notation

$$Q_1 = \frac{k_r}{k_{s_1}}\frac{I_r{}^2 - I_{s_2}{}^2}{I_{s_1}{}^2 - I_{s_2}{}^2}, \quad Q_2 = \frac{k_r}{k_{s_2}}\frac{I_{s_1}{}^2 - I_r{}^2}{I_{s_1}{}^2 - I_{s_2}{}^2},$$

$$P_1 = \frac{k_{s_1}}{k_r}\frac{I_{s_1}{}^2 - I_{s_2}{}^2}{I_r{}^2 - I_{s_2}{}^2}, \quad P_2 = \frac{k_{s_1}}{k_{s_2}}\frac{I_r{}^2 - I_{s_1}{}^2}{I_r{}^2 - I_{s_2}{}^2}.$$

For m = 3, we adjust the right side of (6.4) by a constant which is independent of k, in correspondence with (6.2), we seek a solution of the resultant equation in the form

$$E_r = E_0 k_r{}^{\mu}. \tag{6.5}$$

From dimensional considerations, we easily find

$$\langle I_r\rangle \sim k_r{}^{-5/8}.$$

Hence $\mu = -7/8$.

In conclusion, we call attention to the following circumstance. The case considered in the research corresponds to weak nonlinearity, the value of which can be characterized by the number N of "internal" degrees of freedom of the S waves, which correspond in ordinary hydrodynamics to a large Reynolds number. We can establish the latter in the following way.[3)] We construct the analog of the Reynolds number for Eq. (1.1):

$$R = v / k^2 \sim \alpha / k^2 = N^2.$$

Thus the expansion in the reduced perturbation theory is made over powers of $R^{-1/2}$.

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[1] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nuclear Fusion 1, 82 (1961).

[2] C. Gardner, J. Greene, M. Kruskal, and R. Miura, Phys. Rev. Lett. 19, 1095 (1967).

[3] Yu. A. Berezin and V. I. Karpman, Zh. Eksp. Teor. Fiz. 51, 1557 (1966) [Sov. Phys.-JETP 24, 1049 (1967)].

[4] S. Ulam, Unsolved Mathematical Problems (Russ. transl.), Nauka, 1964).

[5] F. M. Izraĭlev and B. V. Chirikov, Dokl. Akad. Nauk SSSR 166, 57 (1966) [Sov. Phys.-Doklady 11, 30 (1966)].

[6] N. J. Zabusky, Proc. Confer. on Math. Models in Phys. Sciences, New York, 1963, p. 99.

[7] M. P. Rabinovich, Izv. vyssh. uch. zav. Radiofizika 10, 214 (1967).

[8] B. B. Kadomtsev, in: Voprosy teorii plazmy (Problems of Plasma Theory), 4, Atomizdat, 1964.

[9] G. M. Zaslavskii and R. Z. Sagdeev, Zh. Eksp. Teor. Fiz. 52, 1081 (1967) [Sov. Phys.-JETP 25, 718 (1967)].

[10] R. Brout and I. Prigogine, Physica 22, 621 (1956).

[11] G. M. Zaslavskii and N. N. Filonenko, Zh. Eksp. Teor. Fiz. 56, 1064 (1969) [Sov. Phys.-JETP 29, 572 (1969)].

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3) We are indebted to B. V. Chirikov for this remark.