PROPER FREQUENCIES OF A ROTATING RING RESONATOR

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Covariant equations describing the propagation of electromagnetic waves in an isotropic medium at rest in an arbitrary noninertial reference system (or located in a gravitational field) are derived on the basis of general relativity theory and electromagnetic field theory for continuous media. The dielectric permittivity and magnetic permeability tensor is introduced in order to formulate the covariant material equations that supplement the Maxwell equations. Resonance properties of a ring optical resonator rotating at constant angular velocity together with the medium in it are investigated with aid of the derived equations. The natural frequencies of the rotating ring resonator observed in the rotating reference system are expressed in the first approximation of perturbation theory in terms of the natural frequencies and modes of a stationary ring resonator.

INTRODUCTION

MANY recent papers are devoted to investigations of ring lasers. A correct analysis of the effects produced in a ring laser situated in a non-inertial (particularly, rotating) reference system, with allowance for the optical properties of the medium, must be based on general relativity theory. To this end, it is necessary first to obtain equations describing the propagation of electromagnetic waves in a material medium that is at rest in the non-inertial reference frame. Heer et al.[1] considered a ring resonator that rotates together with the medium that fills it, and obtained formulas for its resonant frequencies. Khromykh[2] took additional account of the motion of the medium relative to the rotating resonator. The material equations in[2] were obtained by a direct generalization of the Minkowski formulas for media moving with constant velocity in an inertial frame.

In this paper we investigate, by a different method, the resonant properties of a rotating ring resonator filled with an isotropic medium, and the dependence of the frequency shift of the opposing waves produced in such a resonator on the angular velocity of the resonator rotation. To determine the material equations that connect the vectors of the electromagnetic field in the medium, we use a general method first proposed by Mandel'shtam and Tamm[3], and show that the covariant form of the material equations, which gives correct relations between the field vectors in the inertial system, still does not ensure that these equations hold in a non-inertial system. It is necessary to justify physically the choice of one of the possible methods of writing down the covariant equations. The final formulas of our paper differ from analogous results obtained in[1,2], and from the results of[4], in which the connection between the tensors of the electromagnetic field in the material equations is established with the aid of a fourth-rank tensor.

In[3] they introduced a dielectric-constant and magnetic-permeability tensor of the medium, $S^{\alpha\beta\gamma\nu}$ or $S_{\alpha\beta\gamma\nu}$. With the aid of the tensor $S^{\alpha\beta\gamma\nu}$ and the tensors of the electromagnetic field, one writes down covariant material equations that establish the connection between the vectors of the electromagnetic field in the medium. If the coefficients $S^{\alpha\beta\gamma\nu}$ and $S_{\alpha\beta\gamma\nu}$ are transformed, on going over to other systems, in accordance with the rule for the transformation of contravariant and covariant tensors of fourth rank, then the obtained material equations express correctly the connection between the components of the electromagnetic-field tensors in any reference frame. The formulation of the material equations proposed in[3] makes it possible to obtain covariant or vector material equations for a medium that is at rest in an inertial reference frame and is considered from an arbitrary, not necessarily inertial reference frame. For a rotating ring resonator, this corresponds to the case in which the medium filling the resonator is not absolutely dragged upon rotation, and is at rest in an inertial reference frame. On the other hand, the case when the medium filling the resonator rotates with the resonator (or is partially dragged by it) is of greater practical importance. It is therefore necessary to see how the rotation of the medium influences its optical properties and how the resonant frequencies of the rotating ring resonator change as a result.

1. ELECTRODYNAMICS OF INERTIAL MEDIA

The propagation of electromagnetic waves in an arbitrary system of coordinates is described by the covariant Maxwell's equations[5,6]:

$$H^{\alpha\beta}{}_{;\,\beta} = -\frac{4\pi}{c} j^{\alpha}, \quad F_{\alpha\beta;\,\gamma} + F_{\beta\gamma;\,\alpha} + F_{\gamma\alpha;\,\beta} = 0, \tag{1}$$

where $H^{\alpha\beta}$ and $F_{\alpha\beta}$ are respectively the contravariant and covariant tensors of the electromagnetic field, j^{α} is the current four-vector, and c is the velocity of light. The indices separated by a semicolon denote the covariant derivatives with respect to the corresponding coordinates. Henceforth, the Green indices $\alpha, \beta, \gamma, \ldots$ will run through the value 0, 1, 2, 3, and the Latin indices i, j, k, ... through the values 1, 2, 3; summation is carried out over repeated indices.

Equations (1) can also be written in the form

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{-g}\,H^{\alpha\beta}\right)=-\frac{4\pi}{c}j^\alpha,\quad \frac{\partial F_{\alpha\beta}}{\partial x^\gamma}+\frac{\partial F_{\beta\gamma}}{\partial x^\alpha}+\frac{\partial F_{\gamma\alpha}}{\partial x^\beta}=0. \tag{2}$$

We introduce three-dimensional vectors in accordance with the scheme[7]

$$D_i=\sqrt{-g}\,H^{i0},\quad H_i=-{}^1/_2\varepsilon_{ikl}\sqrt{-g}\,H^{kl},$$
$$E_i=F_{0i},\quad B_i=-{}^1/_2\varepsilon_{ikl}F_{kl},\quad J_i=\sqrt{-g}j^i \tag{3}$$

and the charge density

$$\rho=\sqrt{-g}j^0/c.$$

here ϵ_{ikl} is a perfectly antisymmetrical unit pseudotensor of third rank, $g=|g_{\alpha\beta}|$ is the determinant of the four-dimensional metric tensor. Going over, on the basis of (3), to vector notation for equations (2), we obtain a system of equations that coincides with Maxwell's equations for the electric field in material media that are at rest in the inertial reference frame:

$$\operatorname{rot}\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\frac{4\pi}{c}\mathbf{J},\quad \operatorname{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0,$$
$$\operatorname{div}\mathbf{D}=4\pi\rho,\quad \operatorname{div}\mathbf{B}=0. \tag{4}$$

The quantities introduced in (3) are not independent. In vacuum, the connection between the components of the electromagnetic induction tensor $H^{\alpha\beta}$ and the field intensity tensor $F_{\alpha\beta}$ is established in simple manner with the aid of the metric tensor $g_{\alpha\beta}$[6,8]:

$$\sqrt{-g}\,H^{\alpha\beta}=\sqrt{-g}\,g^{\alpha\gamma}g^{\beta\nu}F_{\gamma\nu}\quad\text{or}\quad F_{\alpha\beta}=\frac{1}{\sqrt{-g}}g_{\alpha\gamma}g_{\beta\nu}\sqrt{-g}\,H^{\gamma\nu}, \tag{5}$$

where

$$g_{\alpha\gamma}g^{\gamma\beta}=\delta_\alpha{}^\beta.$$

In the inertial reference frame, in the absence of gravitational fields, the metric tensor is a diagonal matrix with elements:

$$g_{\alpha\beta}=\operatorname{diag}(1,-1,-1,-1),\quad g^{\alpha\beta}=\operatorname{diag}(1,-1,-1,-1). \tag{6}$$

In this case relations (5), written out in vector form with allowance for (3) and (6), take on the form that is customary for vacuum:

$$\mathbf{D}=\mathbf{E},\quad \mathbf{H}=\mathbf{B}.$$

In any other reference frame (or in the presence of a gravitational field) there are no such simple relations between the induction vectors (D and H) and the intensity vectors (E and B)[6-8].

Thus, if we introduce three-dimensional vectors in accordance with the scheme (3), then the equations of the electromagnetic field will coincide formally with Maxwell's equations for material media (see[6,7]), but a difference appears between the vectors D and E on one hand, and H and B on the other, even for vacuum in non-inertial reference frames or in the presence of gravitational fields. Naturally, when the potential of the gravitational field tends to zero or when we change over to an inertial reference frame, then the quantities E and B tend to their limiting values D and H, just as in the case of an inertial medium the vectors E and B take on the values D and H when the dielectric constant and magnetic susceptibility vanish. Consequently, we should retain for the vectors E and B the same resultant-field meanings that they have in a dielectric medium that is at rest in an inertial reference frame, and retain for the vectors D and H the meaning of inductions, to the values of which the vectors E and B tend in vacuum on going over to an inertial reference frame (in the absence of gravitational fields).

The medium exerts an influence only on the resultant electromagnetic fields E and B, i.e., on the components of the covariant tensor $F_{\alpha\beta}$ of the electromagnetic field. At specified charges and currents, the intensity E of the electromagnetic field decreases by a factor ϵ compared with its value in vacuum (see[5], p. 59). The presence of a magnetic medium leads to a change in the intensity of the magnetic field B by a factor μ, while the induction H remains unchanged (see, for example,[5], p. 162). Here ϵ and μ are respectively the dielectric constant and the magnetic permeability of the medium.

We introduce the mixed tensor S^α_β characterizing the electromagnetic properties of the medium. For a medium that is at rest in any reference frame with a space-time metric $g_{\alpha\beta}$ or located in a gravitational field, the material relations corresponding to equations (5) in the case of vacuum can then be rewritten as follows:

$$\sqrt{-g}\,H^{\alpha\beta}=\sqrt{-g}\,g^{\alpha\gamma}g^{\beta\nu}S^\lambda{}_\gamma S^\sigma{}_\nu F_{\lambda\sigma} \tag{7}$$

or

$$\frac{1}{\sqrt{-g}}g_{\alpha\gamma}g_{\beta\nu}\sqrt{-g}\,H^{\alpha\beta}=S^\lambda{}_\gamma S^\sigma{}_\nu F_{\lambda\sigma}.$$

Taking into account the physical considerations advanced above, it is easy to show that in the case of a linear isotropic medium that is at rest in an arbitrary reference frame the tensor S^α_β has the components

$$S^0{}_0=\varepsilon\sqrt{\mu},\quad S^1{}_1=S^2{}_2=S^3{}_3=1/\sqrt{\mu}, \tag{8}$$

and the remaining components of S^α_β vanish. Here ϵ and μ are the dielectric constant and the magnetic permeability of the medium, measured by local observers connected with the medium. For a medium that is at rest in an inertial reference frame, Eqs. (7) with the tensor (8) give the correct material relations:

$$\mathbf{D}=\varepsilon\mathbf{E},\quad \mathbf{H}=\frac{1}{\mu}\mathbf{B}.$$

In vacuum, we have $S^\alpha_\beta=\delta^\alpha_\beta$ and relations (7) take the form (5).

It should be noted that besides the tensor S^α_β (8) and the material equations in the form (7), it is also possible to introduce a tensor $\tilde{S}^\alpha_\beta$ with nonzero components

$$\tilde{S}^0{}_0=\frac{1}{\varepsilon\sqrt{\mu}},\quad \tilde{S}^1{}_1=\tilde{S}^2{}_2=\tilde{S}^3{}_3=\sqrt{\mu}. \tag{9}$$

With the aid of the tensor $\tilde{S}^\alpha_\beta$ (9) the material equations that give the correct relations between the vectors D, E, H, and B in the inertial medium at rest can be written in the form

$$\tilde{S}^\alpha{}_\gamma\tilde{S}^\beta{}_\nu\sqrt{-g}\,H^{\gamma\nu}=\sqrt{-g}\,g^{\alpha\lambda}g^{\beta\sigma}F_{\lambda\sigma},$$
$$\frac{1}{\sqrt{-g}}g_{\alpha\lambda}g_{\beta\sigma}\tilde{S}^\lambda{}_\gamma\tilde{S}^\sigma{}_\nu\sqrt{-g}\,H^{\gamma\nu}=F_{\alpha\beta}. \tag{10}$$

Here the tensor $\tilde{S}^\alpha_\beta$, which characterizes the electromagnetic properties of the medium, act on the components of the contravariant electromagnetic field tensor $H^{\alpha\beta}$, i.e., on the inductions D and H. However, by virtue of the physical considerations advanced above, preference should be given to Eqs. (7), although both

(7) and (10) give the correct relations between the electromagnetic field vectors for an isotropic medium that is at rest in an inertial reference frame.

In an inertial frame and in the absence of gravitational fields, the tensor $S^{\alpha\beta} = g^{\alpha\gamma}S^{\beta}_{\gamma}$ coincides with the tensor of the dielectric constant and magnetic permeability, encountered in the relativistic theory of moving inertial media[3].

Equations (7), together with the covariant Maxwell's equations (1), describe completely the process of propagation of electromagnetic waves in a medium for the general case of four-dimensional space-time of arbitrary metric. Consequently, to describe the electromagnetic processes in a material medium in the general case, it becomes necessary to introduce three tensors: the tensor S^{α}_{β}, characterizing the electromagnetic properties of the medium, and the electromagnetic field tensors $H^{\alpha\beta}$ and $F_{\alpha\beta}$. The material equations, which relate the components of the tensors of the electromagnetic field and depend on the properties of the four-dimensional space, are determined completely by the metric tensor $g^{\alpha\beta}$, which characterizes all the properties of the geometry of the space in each given system of coordinates.

Relations (7) between the field vectors in the material medium was obtained by us by starting from simple physical concepts concerning the "attenuation" of the resultant fields in the medium. The same equations can be obtained, however, more formally by generalizing the known concepts of polarization of an inertial medium by a field.

In vacuum, the tensor $H^{\alpha\beta}$ and $F^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\nu}F_{\gamma\nu}$ are equal to each other (see (5)). To write down the covariant equations of a macroscopic electromagnetic field in a material medium we define the antisymmetric polarization tensor of the medium $M_{\alpha\beta}$ by the relation

$$\sqrt{-g}H^{\alpha\beta} = \sqrt{-g}\,(F^{\alpha\beta} + 4\pi M^{\alpha\beta})$$

or

$$\sqrt{-g}H^{\alpha\beta} = \sqrt{-g}g^{\alpha\gamma}g^{\beta\nu}(F_{\gamma\nu} + 4\pi M_{\gamma\nu}), \qquad (11)$$

i.e., in perfect analogy with the definition for media at rest in an inertial frame[9].

We introduce the three-dimensional dielectric-polarization vectors of the medium **P** and of the magnetization of the medium **M**, in accordance with the scheme

$$P_i = M_{0i}, \qquad M_i = {}^1/_2\varepsilon_{ikl}M_{kl}.$$

If the medium is polarized isotropically and linearly over the field, then

$$P_i = \varkappa E_i = \frac{\varepsilon - 1}{4\pi}E_i, \quad M_i = \frac{\chi}{\mu}B_i = \frac{\mu - 1}{4\pi\mu}B_i, \qquad (12)$$

where κ is the coefficient of polarization of the medium (the dielectric susceptibility of the medium), ϵ is the dielectric constant of the medium, χ is the magnetic susceptibility, and μ is the magnetic permeability of the medium. The quantities κ, ϵ, χ, and μ are the characteristics of the medium measured by a local observer connected with the medium.

Taking relations (3), (8), and (12) into account, we see that the components of the summary tensor $F_{\alpha\beta} + 4\pi M_{\alpha\beta}$ coincides with the corresponding components of the tensor $S^{\lambda}_{\alpha}S^{\sigma}_{\beta}F_{\lambda\sigma}$ from (7). Thus, the material relations in the form (7) are equivalent to those of (11).

To establish the connection between the vectors of the electromagnetic field in accordance with formula (7) in a non-covariant form, it is necessary to know the components of the metric tensor.

2. ROTATING ISOTROPIC MEDIA

Assume that we have a material medium rotating with constant angular velocity **Ω** relative to the inertial reference system. The metric tensor in the rotating reference frame, the z axis of which is directed along the vector of angular velocity of rotation **Ω** of the medium, has the following components[6,10]:

$$g^{\alpha\beta} = \begin{pmatrix} 1 & \frac{\Omega y}{c} & -\frac{\Omega x}{c} & 0 \\ \frac{\Omega y}{c} & -1 + \frac{\Omega^2 y^2}{c^2} & -\frac{\Omega^2 xy}{c^2} & 0 \\ -\frac{\Omega x}{c} & -\frac{\Omega^2 xy}{c^2} & -1 + \frac{\Omega^2 x^2}{c^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \det|g_{\alpha\beta}| = -1. \qquad (13)$$

If we introduce three-dimensional vectors in accordance with the scheme (3), then the covariant Maxwell's equations (1) can always be written in the vector form (4). Further, substituting in the first equation of (7) the expressions for the components of the electromagnetic field tensors $H^{\alpha\beta}$ and $F_{\alpha\beta}$ from (3), the tensor S^{α}_{β} from (8), and the metric tensor from (13), we obtain non-covariant material equations connecting the field vectors in a medium that is at rest in the rotating reference frame:

$$\mathbf{D} = \varepsilon\mathbf{E} - \frac{1}{\mu}\left[\frac{[\Omega\mathbf{r}]}{c}\mathbf{B}\right], \quad \mathbf{H} = \frac{\mathbf{B}}{\mu} - \left[\frac{[\Omega\mathbf{r}]}{c}\mathbf{D}\right]. \qquad (14)^*$$

Equations (4) together with the material equations (14), written in the reference frame connected with the medium, describe completely the propagation of electromagnetic waves in a medium rotating with constant angular velocity **Ω** relative to the inertial reference frame.

3. FREQUENCY SHIFT OF OPPOSING WAVES IN A ROTATING RING RESONATOR

We note, first, that whereas in the first part of this paper we obtained exact equations, the results of this section have an approximate character and are valid only in the first approximation in $|\Omega \times \mathbf{r}/c|$. The space metric is Euclidean in this approximation.

To find the natural frequencies of a resonator at rest with the medium filling it in a uniformly rotating reference frame, we employ the method used in[11] to find the natural frequencies of a resonant volume.

Let ω'_p, $\mathbf{E}'_p$ and $\mathbf{H}'_p$ be the natural frequencies and modes of a resonator at rest in the initial inertial reference frame, normalized in the following fashion:

$$\int_V (\sqrt{\varepsilon}\mathbf{E}_p'\cdot\sqrt{\varepsilon}\mathbf{E}_q'^*)\,dV = \int_V (\sqrt{\mu}\mathbf{H}_p'\cdot\sqrt{\mu}\mathbf{H}_q'^*)\,dV = \delta_{pq},\ p, q = 0, 1, 2, 3, 4, \ldots$$

We expand Maxwell's equations (4), just as in[11], in terms of the eigenfunctions $\mathbf{E}'_p$ and $\mathbf{H}'_p$ of the rest-

*$[\Omega\mathbf{r}] \equiv \Omega \times \mathbf{r}$.

ing resonator, with allowance for the first approximation (with respect to $|\boldsymbol{\Omega} \times \mathbf{r}/c|$) of the material relations (14) and the boundary conditions for the functions **E** and **H**. Equating the coefficients of the series in both parts of the equations and neglecting the right sides of the equations, we obtain for the quantities **E**, **D**, **B**, and **H**, which are monochromatic with frequency ω, the following system of equations:

$$\begin{aligned} &\omega_p'\varepsilon \int_V \mathbf{E}\mathbf{E}_p'^* dV - \omega\mu \int_V \mathbf{H}\mathbf{H}_p'^* dV - \omega\varepsilon\mu \int_V \left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\mathbf{E}\right]\mathbf{H}_p'^* dV = 0, \\ &\omega_p'\mu \int_V \mathbf{H}\mathbf{H}_p'^* dV - \omega\varepsilon \int_V \mathbf{E}\mathbf{E}_p'^* dV + \omega \int_V \left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\mathbf{H}\right]\mathbf{E}_p'^* dV = 0. \end{aligned} \tag{15}$$

In the first approximation in $|\boldsymbol{\Omega} \times \mathbf{r}/c|$, we obtain from (15) by perturbation theory

$$\Delta\omega_p = 2\omega_p^{(1)} = \omega_p' \frac{\Omega}{c} \int_V [\mathbf{r}\{[\mathbf{H}_p'\mathbf{E}_p'^*] + \varepsilon\mu[\mathbf{H}_p'^*\mathbf{E}_p']\}]\, dV,$$

where E_p', H_p' and ω_p' are the modes and the natural frequencies of the resonator at rest in the inertial reference frame, $\omega_p^{(1)}$ is the first-approximation correction to the natural frequency $\omega_p' = \omega_p^{(0)}$.

Let us consider a plane electromagnetic wave propagating in a rotating ring resonator, having a contour perimeter equal to L. Let part of the resonator, of length l, be filled with a dielectric medium. The frequency shift of the opposing waves in such a resonator is

$$\frac{\Delta\omega}{\omega_p'} = \left[\frac{1+n^2}{2}\frac{4\Omega S_1}{c} + \frac{4\Omega S_2}{c}\right][L + l(n-1)]^{-1},$$

where

$$S_1 = \frac{1}{2}\int_0^l [\mathbf{r}\,d\mathbf{l}], \quad S_2 = \frac{1}{2}\int_l^L [\mathbf{r}\,d\mathbf{l}], \quad n = \sqrt{\varepsilon\mu},$$

S_1 is the area bounded by the contour section l filled with the medium and by the rays joining the ends of the section with the center of rotation, and S_2 is the remaining area of the contour.

4. COMPARISON WITH RESULTS BY OTHERS

It was already noted in the introduction that similar investigations of the dependence of the frequency shift of opposing waves in a ring resonator on the angular rotation velocity of the resonator were carried out in[1,2,4]. The deviations of the results of the indicated investigations from one another and from the present results are due to the fact that they employ different material equations for the electromagnetic field in a medium that is at rest in the rotating reference frame. Thus, in[1,2] the material equations for the medium at rest in a rotating reference frame are written in vector form, in first order approximation in $|\boldsymbol{\Omega} \times \mathbf{r}/c|$, in the form

$$\mathbf{D} = \varepsilon\mathbf{E} - \left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\mathbf{H}\right], \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} - \frac{1}{\mu}\left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\mathbf{E}\right]. \tag{16}$$

In[4], the material equations are written with the aid of the fourth-rank tensor $\chi^{\alpha\beta\gamma\nu}$, which describes the electromagnetic properties of the medium and the geometry of the corresponding reference frame. However, to determine the material equations for the medium at rest in the rotating reference frame (or moving in some other fashion), the tensor $\chi^{\alpha\beta\gamma\nu}$ is resolved in[4] into two components, one describing the vacuum and the other determined by the polarization of the medium, whereas in our case (and in other cases) the component tensors are transformed independently of each other. For a medium resting in an inertial reference frame, the following material equations were obtained in[4] (compare with (14) and (16):

$$\begin{aligned} \mathbf{D} &= \varepsilon\mathbf{E} - \left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\mathbf{B}\right], \\ \mathbf{H} &= \frac{1}{\mu}\mathbf{B} - \left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\mathbf{E}\right] + \left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\left[\frac{[\boldsymbol{\Omega}\mathbf{r}]}{c}\mathbf{B}\right]\right]. \end{aligned} \tag{17}$$

it is easy to write out equations relating the vectors **D**, **E**, **H**, and **B** in vacuum. Therefore Eqs. (17), (16), and (14) differ only in the fact that they contain the characteristics ϵ and μ of the medium in different manners. The indicated differences in the material equations lead to different values of the frequency shift of the opposing waves in a ring resonator rotating together with the medium. In this connection, it is very desirable to verify the results experimentally.

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