MASS, CHARGE AND WAVE FUNCTION RENORMALIZATION IN A THEORY WITH PARITY NONCONSERVATION

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We consider a generalization of the renormalization procedure of quantum electrodynamics to a theory with parity nonconservation of the type of the weak interaction of leptons or quarks with an intermediate boson. Since such theories are nonrenormalizable, all calculations are made under the assumption that the theory involves a cutoff. The distinction from quantum electrodynamics consists, first of all in the appearance of pseudoscalars in the fermion propagators, and secondly, in the possibility of particle mixing (thus, the weak interaction of quarks generates a direct transition of the n-quark into the Λ-quark). The wave function renormalization in such a theory requires the introduction of a matrix that relates the renormalized and unrenormalized fields. The renormalization is treated not in the language of counterterms to be introduced in the initial Lagrangian, but in the language of skeleton diagrams[1,2].

1. INTRODUCTION

We shall consider a renormalization procedure for a system of N fermion fields $\Psi_1, \ldots, \Psi_N$, interacting with m boson fields (for definiteness, vector fields) $B_{1\mu}, \ldots, B_{m\mu}$:

$$L_{int} = \sum_{ikl} \bar{\Psi}_i \Gamma_\mu^{ikl} \Psi_k B_l^\mu, \quad \Gamma_\mu^{ikl} = a_{ikl}\gamma_\mu + b_{ikl}\gamma_5\gamma_\mu.$$

This is the form of the interaction Lagrangian for weak interactions in a model with intermediate bosons, as well as for the weak interaction of quarks.

It is well known that such an interaction is unrenormalizable in the sense that all divergences cannot be removed by introducing a finite number of renormalization constants. Therefore in this paper we shall understand by renormalization a procedure of calculating renormalized propagators, vertices, etc. as functions of a cutoff parameter Λ. The unrenormalizable character of the theory manifests itself in the fact that the renormalized Green's functions do not have good limits for $\Lambda \to \infty$.

The model under consideration differs from electrodynamics in two respects. First, the Fourier transforms of the fermion propagators involve not only the two invariants 1 and $\hat{p}$ (=p), but four invariants 1, $\hat{p}$, γ_5 and $\gamma_5\hat{p}$ in their expansion (if CP is conserved, then the term with γ_5 is forbidden, but we consider the general situation). This is a consequence of parity nonconservation. Therefore, the usual requirement in quantum electrodynamics on the renormalized mass operator[1,2]

$$M_{Ren}\big|_{\hat{p}=m} = 0, \quad \frac{dM_{Ren}}{d\hat{p}}\bigg|_{\hat{p}=m} = 0 \qquad (1)$$

or, what amounts to the same

$$M_{Ren} \sim (\hat{p} - m)^2$$

cannot be applied in the case considered here, and must be reformulated, taking into account the appearance of pseudoscalars in the mass operator.

Secondly, the interaction under consideration can lead to particle mixing both for bosons and for fermions, if the appropriate nondiagonal components of the matrix of Green's functions are different from zero. Thus, in the quark model, the n-quark can become a Λ-quark as a consequence of the fact that the interaction Lagrangian does not conserve strangeness. Therefore the wave function renormalization requires the introduction of a matrix which connects the renormalized and unrenormalized fields.

2. THE RENORMALIZATION PROCEDURE

We introduce the notation $\Psi \equiv (\Psi_1, \ldots, \Psi_N)$, $B \equiv (B_1, \ldots, B_m)$. The propagators $S = \langle 0|T(\Psi\bar{\Psi})|0\rangle$, $D = \langle 0|T(BB^+)|0\rangle$ are matrices in the isospin indices.

The wave function (field) renormalization is realized by means of the introduction of matrices relating the renormalized and unrenormalized fields:

$$\Psi^{ren} = A\Psi, \quad B^{ren} = A'B.$$

In the sequel we shall be concerned only with the renormalization of the fermion fields and will set $A' = 1$, since the boson field renormalization has already been considered in the literature.[3-6]

A is a matrix in isospin, and its matrix elements are linear combinations of 1 and γ_5, or otherwise $A = A_1P_1 + A_2P_2$, where $A_{1,2}$ are numerical matrices and $P_1 = (1 + \gamma_5)/2$, $P_2 = (1 - \gamma_5)/2$. Then $\psi^{ren} = \bar{\psi}\bar{A}$, where $\bar{A} = \gamma_0 A^+ \gamma_0$.

The mass and charge renormalization are essentially the same as usual: to each entering fermion line in the complete Green's function (with external lines) there corresponds one matrix A, acting from the left, and to each outgoing line there corresponds a matrix $\bar{A}$ acting from the right. For compact Green's functions (without external lines) each entering fermion line acquires a matrix $\bar{A}^{-1}$ from the left, and each outgoing one, acquires a matrix A^{-1} from the right.

The renormalized Green's function is represented as usual as a sum of renormalized skeleton diagrams.

We denote by Γ^{ren} the compact renormalized vertex function. We have $\Gamma^{unren} = \Gamma_{0u} + \Lambda^{unren}$, where Γ_{0u} is the system of unrenormalized coupling constants and Λ^{unren} is the sum of higher-order diagrams. According to the general rule

$$\Gamma^{ren} = \bar{A}^{-1}\Gamma^{unren}A^{-1} \equiv \Gamma_{0r} + \Lambda_{ren},$$

where Γ_{0r} is the system of renormalized coupling constants, defined as the values of Γ^{ren} at a point M, such that $\Lambda^{ren}|_M = 0$ (Γ is labelled by three indices Γ_{ikl}, and the normalization points may be chosen differently for the different indices). The directly calculable quantity is $\hat{\Lambda} \equiv \bar{A}^{-1}\Lambda^{unren}A^{-1}$ equal to the sum of skeleton renormalized diagrams of the compact vertex, in terms of which can be expressed the renormalized quantity $\Lambda^{ren} = \tilde{\Lambda} - \hat{\Lambda}|_M$ and the renormalization constant (more precisely, the system of such constants) Z, defined by $\Gamma_{0r} = Z^{-1}\bar{A}^{-1}\Gamma_{0u}A^{-1}$. It is easy to show that $Z = 1 - \tilde{\Lambda}|_M\Gamma_{0r}^{-1}$ (here Γ_{0r}^{-1} is the inverse matrix with respect to isospin indices, for fixed boson index).

Only the renormalization of fermion propagators becomes considerably more complicated. The purpose of this renormalization is the computation of the renormalized propagator $S^{ren} = AS\bar{A}$, or of the renormalized mass operator M^{ren}

$$S_{ren}^{-1} = i^{-1}(\hat{p} - m - M_{ren})$$

(m is the diagonal matrix of physical masses). We expand M_{ren}

$$M_{ren} = \sum_{i=1}^{4} M_i Q_i,$$

where $Q_1 = P_1\hat{p}$, $Q_2 = P_2\hat{p}$, $Q_3 = P_1$, $Q_4 = P_2$ and M_i are matrices in isospin, with matrix elements depending on p^2.

If one writes down a Källén-Lehmann representation for S and S^{ren}, it is easy to see that in the general case the matrix elements of S (including the off-diagonal ones) contain poles at all physical masses (if there is indeed particle mixing), whereas S^{ren} will have poles only on the diagonal, and each diagonal matrix element will have a pole only at its mass (this shows the necessity of introducing a renormalization matrix):

$$S^{ren}|_{p^2=m_k^2} = i\frac{1}{\hat{p} - m_k}P_k + \text{nonpole terms}, \qquad (2)$$

where P_k is the projection operator onto the field of mass m_k. Starting from (2) one can obtain a generalization of the requirement (1) for the renormalized mass operator to the case under consideration. By means of simple algebra, taking into account that the presence in a scalar matrix $A(p^2)$ of a pole of the form $(p^2 - m_k^2)^{-1}P_k$ is equivalent to the following requirement on the inverse matrix

$$P_k(A^{-1})^{0k} = (A^{-1})^{0k}P_k = 0, \quad P_k(A^{-1})^{1k}P_k = P_k,$$

where

$$(A^{-1})^{0k} \equiv A^{-1}\Big|_{p^2=m_k^2}, \qquad (A^{-1})^{1k} \equiv \frac{dA^{-1}(p^2)}{dp^2}\Big|_{p^2=m_k^2},$$

we obtain the following system of requirements on the invariants M_i of the renormalized mass operator:

$$\begin{gathered} P_k[M_2^{0k} + m^2(M_1^{1k} + M_2^{1k}) + m(M_3^{1k} + M_4^{1k})]P_k = 0, \\ M_2{}^r m = -M_4{}^r, \quad M_1{}^r m = -M_3{}^r, \\ mM_1{}^l = -M_4{}^l, \quad mM_2{}^l = -M_3{}^l, \end{gathered} \qquad (3)$$

where $M_i^{r(l)}$ is

$$(M_i{}^r)_{\alpha\beta} = (M_i(p^2))_{\alpha\beta}|_{p^2=m_\beta^2}, \qquad (M_i{}^l)_{\alpha\beta} = (M_i(p^2))_{\alpha\beta}|_{p^2=m_\alpha^2}.$$

It is easy to check that in the absence of mixing and for parity conservation (i.e., when $M_1 = M_2$, $M_3 = M_4$) these formulas go over into (1).

Making use of (3) one can indicate a procedure for the computation of the renormalized mass operator. We write as usual[1,2]

$$\begin{gathered} S_{unren} = S_{0u} + S_{0u}\Sigma_{unren}S_{unren}, \quad M_{unren} = i\Sigma_{unren}, \\ S_{unren}^{-1} = S_{0u}{}^{-1} - \Sigma_{unren} = i^{-1}(\hat{p} - m_0 - i\Sigma_{unren}), \\ S_{ren}^{-1} = i^{-1}(\hat{p} - m - M_{ren}), \quad i\Sigma_{ren} = M_{ren} \equiv M. \end{gathered}$$

We have

$$\bar{A}^{-1}(S_{0u}{}^{-1} - \Sigma_{ren})A^{-1} = S_{0r}{}^{-1} - \Sigma_{ren}. \qquad (4)$$

For Σ_{unren} we write as usual[1,2] the Schwinger equation $\Sigma = \Gamma_0 S\Gamma D$. Denoting $\tilde{\Sigma} = \bar{A}^{-1}\Sigma_{unren}A^{-1}$, we have

$$\tilde{\Sigma} = \bar{A}^{-1}\Gamma_{0u}S_{unren}\Gamma_{unren}A^{-1}D = \bar{A}^{-1}\Gamma_{0u}A^{-1}S_{ren}\Gamma_{ren}D.$$

The quantity $\bar{A}^{-1}\Gamma_{0u}A^{-1}$ can be expressed in terms of the zeroth approximation of the renormalized vertex function and the charge renormalization constant Z; this yields $\tilde{\Sigma} = Z\Gamma_{0r}S_{ren}\Gamma_{ren}D$. The quantity $\tilde{\Sigma}$ (or $\tilde{M} = i\tilde{\Sigma}$) is the initial calculable quantity, in terms of which we express the renormalized mass operator.

Introducing $A^{-1} \equiv B = B_1P_1 + B_2P_2$, we rewrite (4) in the form

$$S_{ren}{}^{-1} = i^{-1}[Z_2Q_1 + Z_1Q_2 - HQ_3 - H^+Q_4 - \tilde{M}],$$

where $B_1^+B_1 \equiv Z_1$, $B_2^+m_0B_1 \equiv H$. Denoting the invariants of $\tilde{M}_i$ by $\tilde{M}$ we obtain a relation between M and $\tilde{M}$:

$$\begin{gathered} M_1 = \tilde{M}_1 + 1 - Z_2, \quad M_2 = \tilde{M}_2 + 1 - Z_1, \\ M_3 = \tilde{M}_3 + H - m, \quad M_4 = \tilde{M}_4 + H^+ - m. \end{gathered} \qquad (5)$$

Imposing on M_i the requirements (3), we obtain expressions for $Z_{1,2}$ and H in terms of the directly calculable quantity M and then from (5) we determine the M_i.

From (3) we obtain the expressions for the off-diagonal elements of Z_i and H:

$$(Z_1)_{ik} = (m_k^2 - m_i^2)^{-1}(R(Z_1))_{ik}$$

etc., where

$$\begin{aligned} R(Z_1) &= m(\tilde{M}_1{}^r - \tilde{M}_1{}^l)m + \tilde{M}_2{}^r m^2 - m^2\tilde{M}_2{}^l \\ &\quad + (\tilde{M}_4{}^r - \tilde{M}_4{}^l)m + m(\tilde{M}_3{}^r - \tilde{M}_3{}^l), \\ R(Z_2) &= m(\tilde{M}_2{}^r - \tilde{M}_2{}^l)m + \tilde{M}_1{}^r m^2 - m^2\tilde{M}_1{}^l \\ &\quad + (\tilde{M}_3{}^r - \tilde{M}_3{}^l)m + m(\tilde{M}_4{}^r - \tilde{M}_4{}^l), \\ R(H) &= m^2(\tilde{M}_1{}^r - \tilde{M}_1{}^l)m + m(\tilde{M}_2{}^r - \tilde{M}_2{}^l)m^2 \\ &\quad + m^2\tilde{M}_3{}^r - \tilde{M}_3{}^l m^2 + m(\tilde{M}_4{}^r - \tilde{M}_4{}^l)m, \\ R(H^+) &= m^2(\tilde{M}_2{}^r - \tilde{M}_2{}^l)m + m(\tilde{M}_1{}^r - \tilde{M}_1{}^l)m^2 \\ &\quad + m^2\tilde{M}_4{}^r - \tilde{M}_4{}^l m^2 + m(\tilde{M}_3{}^r - \tilde{M}_3{}^l)m. \end{aligned} \qquad (6)$$

The diagonal matrix elements of these matrices are determined from (3):

$$\begin{gathered} (Z_1)_{kk} = 1 + (\tilde{M}_2{}^{0k})_{kk} + X_k, \quad (Z_2)_{kk} = 1 + (M_1{}^{0k})_{kk} + X_k, \\ (H)_{kk} = m_k - (\tilde{M}_3{}^{0k})_{kk} + m_kX_k, \quad (H^+)_{kk} = m_k - (\tilde{M}_4{}^{0k})_{kk} + m_kX_k, \end{gathered} \qquad (7)$$

where we have used the notation $X_k \equiv m_k^2(M_1^{1k} + M_2^{1k})_{kk} + m_k(M_3^{1k} + M_4^{1k})_{kk}$.

In conclusion we discuss the computation of the renormalization matrix A and the matrix of unrenormal-

ized masses m_0 or of $\delta m = m - m_0$ in terms of the directly computable matrices $Z_{1,2}$ and H, H^+. Generally speaking, all matrix elements are expressed only in terms of the renormalized propagators and vertices, which are determined completely by $Z_{1,2}$, H and H^+. Nevertheless, the explicit expressions for δm and the renormalization matrices may present interest. The matrices $Z_i = B_i^+ B_i$ determine the corresponding B_i up to a unitary transformation: $B_i = U_i B_{i0}$, where $B_{i0} = (Z_i)^{1/2}$ (the square root of a positive definite matrix is well defined).

Thus, the problem reduces to the determination of two unitary matrices U_i and of an hermitean matrix m_0 (we consider m_0 Hermitean, assuming that all particles are stable). For the determination of these three matrices we have at our disposal only one equation: $B_2^+ m_0 B_1 = H$, or $U_2^+ m_0 U_1 = A$, where $A = B_{20}^{-1+} H B_{10}^{-1}$. It is clear that this equation cannot uniquely determine all three unknown matrices, since the equation is invariant with respect to the transformation

$$U_1 \to VU_1, \quad U_2 \to VU_2, \quad m_0 \to Vm_0V^{-1}, \quad V^+V = 1.$$

It is also clear that this arbitrariness always allows one to diagonalize the Hermitean matrix m_0.

If one requires the diagonal character of m_0, the arbitrariness in the solution is partially removed. We show that the remaining arbitrariness allows one to consider m_0 a positive matrix (i.e., that all unrenormalized masses may be chosen positive). Indeed, the equation $U_2^{-1} m_0 U_1 = A$ is invariant with respect to the transformation $m_0 \to m_0\lambda$, $U_1 \to \lambda U_1$, where λ is an arbitrary diagonal matrix the square of which is the unit matrix. It is clear that by appropriate choice of the matrix λ one can always make the eigenvalues of the matrix $m_0\lambda$ positive.

If m_0 is considered to be a positive diagonal matrix, the problem reduces to the determination of the eigenvalues of this matrix and the determination of the unitary matrices U_1 and U_2. We have:

$$U_2^{-1} m_0^2 U_2 = AA^+, \quad U_1^{-1} m_0^2 U_1 = A^+A.$$

These equations determine the eigenvalues of m_0 uniquely: the eigenvalues of m_0 are the square roots of the eigenvalues of the positive matrix AA^+ (or A^+A, which is the same). At the same time these equations show that U_1 and U_2 are unitary operators, which diagonalize respectively the matrices A^+A and AA^+. If one assumes that the matrix m_0 is nondegenerate, the preceding considerations determine the matrices U_1, U_2 up to arbitrary unitary diagonal operators, denoted respectively by V_1 and V_2: $U_1 = U_{10}V_1$, $U_2 = U_{20}V_2$; here U_{10} and U_{20} are some fixed particular solutions for U_1 and U_2. Thus, the problem reduces to the determination of two unitary diagonal matrices V_1 and V_2.

We write

$$V_2^{-1} U_{20}^{-1} m_0 U_{10} V_1 \equiv V_2^{-1} B V_1 = A.$$

The matrices A and B are known, V_1 and V_2 have to be determined. Denoting $(V_i)_{kk} = \exp\{i\delta_k^{(i)}\}$ and taking the matrix elements of the above equality, we obtain $B_{ik} \exp\{i(\delta_k^{(1)} - \delta_i^{(2)})\} = A_{ik}$. Setting $i = k$, we determine all phase differences, in other words $V_2^{-1}V_1$. The possibility of obtaining additional information relative to the phases $\delta_k^{(1)}$ depends on whether the matrix A has nonvanishing off-diagonal matrix elements. If A is diagonal (and consequently, so is B), it is impossible to obtain any additional information and the matrix V_1 remains arbitrary. The presence of a nonvanishing A_{ik} allows one to fix the phase difference $\delta_i^{(1)} - \delta_k^{(1)}$. If A has sufficiently many nonvanishing off-diagonal matrix elements, one can determine all phase differences $\delta_i^{(1)} - \delta_k^{(1)}$ and the remaining arbitrariness consists in the possibility of adding to each of the phases the same constant, corresponding to a common gauge transformation of the field.

3. CONCLUSION

From the preceding considerations it can be seen that the renormalization procedure in the case under consideration shows no differences of principles from the corresponding procedures of quantum electrodynamics. All arising complications have a purely technical character.

In the same manner as in quantum electrodynamics the renormalization can be formulated not in the language of skeleton diagrams but rather in the language of counterterms to be introduced into the initial Lagrangian. For this purpose it is necessary to rewrite the initial Lagrangian in terms of the renormalized field:

$$L = L_{0B} + \overline{\Psi}_{\text{ren}} \overline{B} (i\hat{\partial} - m + \delta m) B \Psi_{\text{ren}} + \sum_{ikl} \overline{\Psi}_{i\text{ren}} Z_{ikl} \Gamma_{0p}^{ikl} \Psi_{k\text{ren}} B_l \quad (8)$$

and then in each order of perturbation theory one selects the renormalization constants in such a manner that the Green's functions calculated in terms of the unrenormalized diagrams of the interaction (8) are already renormalized.

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