SOVIET PHYSICS JETP VOLUME 30, NUMBER 5 MAY 1970

PARAMETRIC EXCITATION OF A QUANTUM OSCILLATOR. II

V. S. POPOV and A. M. PERELOMOV

Institute for Theoretical and Experimental Physics

Submitted May 23, 1969

Zh. Eksp. Teor. Fiz. 57, 1684–1690 (November, 1969)

An exact solution of the Schrödinger equation is obtained for a quantum oscillator with a varying frequency $\omega(t)$, which is acted upon by an external force f(t). The time dependence of $\omega(t)$ and f(t) may be arbitrary. The population distribution at the end of the process is calculated for the case when at $t \to -\infty$ the oscillator was in the ground state $|0, \omega_-\rangle$. The quasi-energy spectrum for an oscillator with periodically varying parameters is determined.

1. THE quantum oscillator is the usual representation of a specified mode of excitation of the electromagnetic field. Recently, the oscillator model has been widely used in considering the quantum theory of the laser, photostatistics of laser radiation, relaxation of coherent light in weakly absorbing media, and other questions of quantum optics (see, e.g., [1,2]; more detailed references to the literature may be found in [2,3]). In this connection, a number of new works[2-6] devoted to the oscillator have appeared.

Excitation of a quantum oscillator is possible both by the action of an external force, and by variation of the frequency (parametric resonance). The problem of an oscillator of constant frequency, acted on by a force f(t) which depends in an arbitrary fashion on time, has been considered in connection with problems of quantum field theory in the well-known works of Feynman[7] and Schwinger.[8] In those works, formulae were obtained which describe the excitation of the quantized electromagnetic field (equivalent to an assembly of oscillators) by a classical current. The oscillator with frequency varying with time was the subject of the works.[3,9,10] It is important to emphasize that in both cases one succeeds in finding an exact solution of the problem.

In the present work, we consider the general case, when a quantum oscillator of variable frequency $\omega(t)$ is acted on by a (classical) external force f(t):

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \left\{\frac{1}{2}\omega^2(t)x^2 - f(t)x\right\}\psi \tag{1}$$

($\hbar = m = 1$ everywhere). We shall suppose that the dependence of $\omega(t)$ and f(t) is arbitrary,[1)] assuming only (everywhere except in Sec. 4) that the natural boundary conditions:

$$f(t) \to 0 \quad \text{as} \quad t \to \pm\infty, \tag{2a}$$

$$\omega(t) \to \begin{cases} \omega_- & \text{as} \quad t \to -\infty \\ \omega_+ & \text{as} \quad t \to +\infty \end{cases} \tag{2b}$$

are fulfilled (the limits may be different). We list the basic results obtained.

In Sec. 2 we obtain formula (11), describing the evolution in time of an arbitrary initial state. The wave function $\psi(x, t)$ at any moment of time is expressed in terms of the wave function ψ_- of the initial state, and the solution $\xi(t)$ of the equation of motion for the classical oscillator. One might say that for an oscillator, quantum mechanics reduces to classical mechanics (cf. [7,9]). However, it should be noticed that the solution $\xi(t)$ is complex, and therefore cannot be regarded simply as a trajectory of the classical oscillator. In Sec. 3, the transition probabilities from the ground state $|0, \omega_-\rangle$ into final states $|n, \omega_+\rangle$ are calculated. These probabilities depend on the three parameters: ρ, ν, and φ introduced in Sec. 2. Here, ρ depends only on the law of variation of the frequency $\omega(t)$, ν characterizes the excitation of the oscillator by an external force, and φ is a phase angle. The case of non-periodic variation of frequency is dealt with in Sec. 4, where the quasi-energy spectrum of the oscillator is found. The nature of the spectrum of the quantum oscillator is determined once that of the classical oscillator with the same law of variation of frequency is found in the region of stability. It is independent of the form of the force f(t).

The problem discussed, apart from possible applications in quantum optics, is of interest as a quite rare example in which a non-stationary problem in quantum mechanics admits an exact solution in analytic form.

3. We shall find the general form of wave function that satisfies the Schrödinger equation (1). Suppose f(t) = 0 initially. Denoting by $\xi(t)$ the solution of the classical equation of motion

$$\ddot{\xi} + \omega^2(t)\xi = 0 \tag{3}$$

with the initial condition

$$\xi(t) \to e^{i\omega_- t} \quad \text{as} \quad t \to -\infty, \tag{3a}$$

we set

$$\xi(t) = r(t)e^{i\gamma(t)}, \quad r(t) = |\xi(t)|. \tag{4}$$

Then r(t) gives a scale of length at the instant t, and the quantity $\tau = \gamma(t)/\omega_-$ is the corresponding scale of time. It is natural to look for $\psi(x, t)$ in the form

$$\psi(x, t) = [r(t)]^{-1/2}e^{-i\Phi(x, t)}\chi_-(y, \tau), \quad y = x/r(t) \tag{5}$$

[1)] The Schrödinger equation (1) describes the most general linear quantum system with continuous parameters, possessing one degree of freedom (the time dependence of the mass parameter m(t) can be excluded, as was shown in [3]). According to Eq. (1), the force acting on the particle is $f(t) - \omega(t)x$, and any complication of it leads to nonlinearity. It is understood that the variable x can have the meaning not of a space coordinate, but, for example, the intensity of the electric or magnetic field.

(the factor $r^{-1/2}$ ensures that normalization is maintained, and $\Phi(x, t)$ is a real phase). χ_- denotes an arbitrary solution of the Schrödinger equation for an oscillator with constant frequency ω_-. We substitute Eq. (5) in Eq. (1) and require that the resulting equation does not contain terms $\sim \partial\chi_-/\partial x$. This determines the function Φ:

$$\Phi(x,t) = -\frac{\dot{r}}{2r}x^2 \tag{6}$$

and the equation for χ_- assumes the form

$$i\frac{\gamma}{\omega_-}\frac{\partial\chi_-}{\partial\tau} = -\frac{1}{2r^2}\frac{\partial^2\chi_-}{\partial y^2} + \frac{1}{2}\left[\omega^2 + \left(\frac{\dot{r}}{r}\right)^2 + \frac{d}{dt}\left(\frac{\dot{r}}{r}\right)\right]x^2\chi_-. \tag{7}$$

Supposing next that

$$\dot{\gamma} = \frac{\omega_-}{r^2}, \qquad \frac{\ddot{r}}{r} + \omega^2(t) = \dot{\gamma}^2 = \frac{\omega_-^2}{r^4}, \tag{8}$$

we see that Eq. (7) reduces to the required form:

$$i\frac{\partial\chi_-}{\partial\tau} = \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + \omega_-^2 y^2\right)\chi_-. \tag{7a}$$

It remains to be shown that the conditions (8) imposed above do not contradict Eqs. (3) and (4). To this end, we introduce the quantity

$$a(t) = a_1 + ia_2 = -i\frac{\dot{\xi}(t)}{\xi(t)} = \dot{\gamma} - i\frac{\dot{r}}{r} \tag{9}$$

($a \to \omega_-$ as $t \to -\infty$), and notice that

$$a_1 = -i\frac{\Delta(\xi,\xi^*)}{2|\xi|^2} = \frac{\omega_-}{r^2}, \quad a_2 = -\frac{\xi\dot{\xi}^* + \dot{\xi}\xi^*}{2|\xi|^2} = -\frac{\dot{r}}{r}, \tag{10a}$$

$$\gamma(t) = \frac{1}{2i}\ln\frac{\xi(t)}{\xi^*(t)}. \tag{10b}$$

The Wronskian $\Delta(\xi, \xi^*) = \dot{\xi}\xi^* - \dot{\xi}^*\xi$ is independent of time, and has the value $2i\omega_-$, whence we immediately obtain the first of the relations (8). Further, since

$$\frac{\ddot{r}}{r} = \frac{d}{dt}\left(\frac{\dot{r}}{r}\right) + \left(\frac{\dot{r}}{r}\right)^2,$$

we find, taking account of Eq. (10a), that $\ddot{r}/r = a_2^2 - \dot{a}_2 = a_1^2 - \omega^2$ which leads to the second of the equations (8). Thus, we have shown that the general solution of the Schrödinger equation for an oscillator with variable frequency $\omega(t)$ has the form (5), with the phase $\Phi(x, t) = \frac{1}{2}a_2(t)x^2$. Since $r \to 1$, $a_2 \to 0$ and $\gamma(t) = \omega_- t$ as $t \to -\infty$, χ_- in Eq. (5) is simply the initial wave function of the oscillator. If χ_- is taken to be the n-quantum state $|n, \omega_-\rangle$, Eq. (5) goes over into Eq. (9) of [3].

The force $f(t)$ can then be taken into account following Husimi (see formula (2.6) of [3]). As a result, we obtain for the general solution of Eq. (1):

$$\psi(x,t) = \frac{1}{\sqrt{r(t)}}\chi_-\left(\frac{x-\eta}{r}, \tau\right) \times\exp\{i[\dot{\eta}(x-\eta) - {}^1\!/_2 a_2(x-\eta)^2 + \sigma]\}. \tag{11}$$

Here, r, a_2 and τ have the values given above,

$$\sigma(t) = \int_{-\infty}^{t} L(t')\,dt', \quad L = {}^1\!/_2(\dot{\eta}^2 - \omega^2\eta^2) + f\eta \tag{12}$$

(the phase $\sigma(t)$ is inessential to determine transition probabilities w_{mn}), and $\eta(t)$ is the real solution describing forced oscillations of the classical oscillator:

$$\ddot{\eta} + \omega^2(t)\eta = f(t), \qquad \eta = \dot{\eta} = 0 \quad \text{as} \quad t = t_0 \to -\infty. \tag{13}$$

The quantity $\eta(t)$ can be expressed in terms of f and ξ. The Green's function for Eq. (3) is:

$$G(t,t') = \frac{1}{\Delta(\xi,\xi^*)}[\xi(t)\xi^*(t') - \xi^*(t)\xi(t')]\theta(t-t'),$$

where $\theta(t - t')$ is the usual step function. Hence, taking account of the initial conditions (13), we find

$$\eta(t) = (2\omega_-)^{-1/2}(\xi d^* + \xi^* d). \tag{14}$$

Here, we have introduced the quantity $d = d(t)$:

$$d(t) = \frac{i}{\sqrt{2\omega_-}}\int_{-\infty}^{t} f(t')\xi(t')\,dt'. \tag{15}$$

which will be important subsequently. The physical meaning is the following. Change from coordinate x and momentum p to the dimensionless variables

and set
$$X = (\omega/2)^{1/2}x, \quad P = (2\omega)^{-1/2}p, \qquad \alpha = X + iP = (2\omega)^{-1/2}(\omega x + ip) \tag{16}$$

(the complex α-plane will be called the phase plane). A state of the classical oscillator at each moment of time is represented as a point of the phase plane, and $d(t)$ is the displacement of this point under the action of the external force. For an oscillator of constant frequency ω_0, we have $\xi(t) = \exp i\omega_0 t$, and Eq. (15) reduces to the expression already encountered in the works.[7, 8]

In what follows, we shall be interested in the transition probability w_{mn} for $t \to \infty$, when the frequency $\omega(t)$ tends to the constant value ω_+. In this case, the expressions for $\xi(t)$ and $\eta(t)$ simplify. From Eq. (3) we have

$$\xi(t) = C_1 e^{i\omega_+ t} - C_2 e^{-i\omega_+ t}, \tag{3b}$$

where C_1 and C_2 are constants, for the determination of which we need to solve Eq. (3) for the classical oscillator in the whole of the interval $-\infty < t < \infty$. We remark also that $\xi(t)$ can be interpreted as the wave function for the problem of reflection by a barrier (see [3, 10] for more details). With this interpretation, the amplitude of the reflected wave R is

$$R = \frac{C_2}{C_1^*} = \sqrt{\rho}\, e^{2i\delta} \quad (0 \leqslant \rho < 1); \tag{17}$$

here we have introduced the reflection coefficient ρ and the phase 2δ of the amplitude R.

Given ρ and the ratio ω_+/ω_-, the moduli of the coefficients C_1 and C_2 are completely determined.

$$C_1 = e^{i\delta_1}\sqrt{\frac{\omega_-}{\omega_+(1-\rho)}}, \quad C_2 = e^{i\delta_2}\sqrt{\frac{\omega_-\rho}{\omega_+(1-\rho)}}, \tag{17a}$$

with $\delta_1 + \delta_2 = 2\delta$. The amplitude of the forced oscillations $\eta(t)$ satisfies the relation

$$\frac{\omega\eta + i\dot{\eta}}{\sqrt{2\omega}}\underset{t\to\infty}{=}\sqrt{\frac{\omega_+}{\omega_-}}(C_1^* d - C_2 d^*)e^{-i\omega_+ t} \tag{14a}$$

(see (14)), where

$$d = \lim_{t\to\infty} d(t) = \sqrt{\nu}\, e^{i\beta} \tag{18}$$

(the quantity ν characterizes the perturbation of the oscillator by the external force). As will be shown in the next section, the transition probabilities w_{mn} depend only on the three quantities: ρ, ν, and the phase angle $\varphi = \delta - \beta$.

3. We proceed to the calculation of the transition probabilities w_{mn}. We consider the case of greatest physical interest, when as $t \to -\infty$ the oscillator is in the ground state $|0, \omega_-\rangle$. Substituting in Eq. (11) the initial wave function

$$\chi_-(x,t) = \left(\frac{\omega_-}{\pi}\right)^{1/4} \exp\left\{-\frac{\omega_-}{2}(x^2 + it)\right\},$$

we find

$$\psi(x,t) = (\xi(t))^{-1/2}\left(\frac{\omega_-}{\pi}\right)^{1/4} \exp\left\{-\frac{a}{2}(x-\eta)^2 + i[\dot\eta(x-\eta)+\sigma]\right\}. \tag{19}$$

From this, we obtain for the probability of the transition $|0, \omega_-\rangle \to |n, \omega_+\rangle$ [2)]

$$w_{00} = (1-\rho)^{1/2}\exp\{-\nu[1-\rho^{1/2}\cos 2\varphi]\},$$
$$w_{n0} = w_{00}\frac{\rho^{n/2}}{2^n n!}|H_n(s)|^2, \tag{20}$$

where $H_n(s)$ is a Hermite polynomial, and

$$s = d\left(\frac{1-\rho}{2R}\right)^{1/2} = \left[\frac{(1-\rho)\nu}{2\sqrt{\rho}}\right]^{1/2} e^{-i\varphi} \tag{20a}$$

(for the calculation we make use of the values of the integral (A.3) set down in Appendix A of [3]). We consider limiting cases of the distribution (20).

1) If the external force vanishes, $s = \nu = 0$, whence

$$w_{2n,0} = \sqrt{\frac{1-\rho}{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\rho^n, \qquad w_{2n+1,0} = 0, \tag{21}$$

which is in agreement with [3,9].

2) For an oscillator with constant frequency $\omega(t) = \omega_0$, we have $\rho = 0$, $s \to \infty$, and the formula (20) becomes the well-known[7] Poisson distribution:

$$w_{n0} = e^{-\nu}\frac{\nu^n}{n!}, \quad \nu = \left|(2\omega_0)^{-1/2}\int_{-\infty}^{\infty} f(t)e^{i\omega_0 t}dt\right|^2. \tag{22}$$

3) If the frequency $\omega(t)$ varies adiabatically (i.e., $\omega^{-2}d\omega/dt \ll 1$; this need not imply that the ratio ω_+/ω_- is near unity), then the reflection coefficient ρ is exponentially small. In this case, the distribution of transition probabilities differs from (22) by a term $\sim\sqrt{\rho}$. The corresponding formula for w_{n0} can be obtained from Eq. (19) in [3] by the change $(\varphi - \delta) \to \varphi$.

4) Suppose next that $\rho \to 1$ (limiting case of abrupt change of frequency). Then the oscillator is strongly excited, and the transition probabilities w_{n0} are essentially concentrated in the region $n \gg [\nu(1-\rho)]^{1/2}$. To simplify Eq. (20) we use the asymptotic expansion for the Hermite polynomials in the case when the argument is much smaller than the index (see [11], line 254). Suppose $\varphi = 0$ initially. Introducing the variable $\kappa = 1 - \sqrt{\rho}$ ($\kappa \to 0$), we find $s = \sqrt{\nu\kappa}$,

$$w_{00} = (2\kappa)^{1/2}e^{-\nu\kappa}, \quad w_{n0} = 2(\kappa/\pi n)^{1/2}e^{-n\kappa}\cos^2\Phi_n, \tag{23}$$

where $\Phi_n = \sqrt{2n\nu\kappa} - n\pi/2$ is a rapidly oscillating (for $n \gg 1$) quasi-classical phase. The distribution w_{n0}, averaged over this oscillation, has the form

$$\overline{w}_{n0} = (\kappa/\pi n)^{1/2}e^{-n\kappa} \quad (\varphi = 0). \tag{24}$$

The case $\varphi \neq 0$ is considered similarly. In summary, we arrive at a single formula describing the behavior of the transition probabilities w_{n0} (averaged over the rapid oscillations):

$$\overline{w}_{n0} = (\kappa/\pi n)^{1/2}e^{-(n+n_0)\kappa}\,\mathrm{ch}\,(2\kappa\sqrt{nn_0}), \tag{25}$$

where $n_0 = (\nu/\kappa)(1-\cos 2\varphi)$. If $\nu(1-\cos 2\varphi) \gg 1$, the expression (25) becomes a Gaussian distribution with center at $n = n_0$ and dispersion $\overline{\Delta n^2} = 2n_0/\kappa$, with $(\overline{\Delta n^2})^{1/2} \ll n_0$. For $\nu(1-\cos 2\varphi) \ll 1$, it reverts to the exponential distribution (24).

In calculating the transition probability w_{mn} with $m, n > 0$, a straightforward use of the expression (11) for $\psi(x, t)$ leads to too complicated an integral. However, it can be shown that the probabilities w_{mn} for arbitrary numbers m and n depend, as before, only on the parameters ρ, ν and φ introduced above.

4. Finally there remains the question of the quasi-energy spectrum of the oscillator with variable $\omega(t)$ and $f(t)$. In those cases when the Hamiltonian depends periodically on time, there exist solutions of the Schrödinger equation which change back into themselves after a period T, multiplied only by a phase factor

$$\psi_\varepsilon(t+T) = e^{-i\varepsilon T}\psi_\varepsilon(t) \tag{26}$$

(for any value of t). The quantity ϵ is called the "quasi-energy".[12–14] For processes proceeding under the action of a field that is periodic in time,[3)] the quasi-energy plays the same role as the energy in the stationary case.

Consider the quantum oscillator with periodically varying frequency and force:

$$\omega(t+T) = \omega(t), \quad f(t+T) = f(t). \tag{27}$$

According to Floquet's well-known theorem of mechanics, linearly independent solutions ξ_1 and ξ_2 of Eq. (3) can be constructed such that

$$\xi_1(t) = e^{i\lambda t}u_1(t), \quad \xi_2(t) = e^{-i\lambda t}u_2(t), \tag{28}$$

where u_1 and u_2 are periodic functions with period T, and λ is either real (zone of stability) or pure imaginary (zone of instability).

In the zone of stability, $\lambda > 0$ and $\xi_2(t) = \xi_1^*(t)$. Choosing χ_- in Eq. (11) as the n-quantum state of an oscillator with instantaneous frequency $a_1(t)$ (see Eq. (9)), and $\eta(t)$ as a periodic solution of Eq. (13), we obtain the wave function of a state with a definite value of quasi-energy:

$$\varepsilon_n = (n + 1/2)\lambda + \Delta\varepsilon \quad (n = 0, 1, 2, \ldots), \tag{29}$$

where

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2) The transition probability w_{n0} gives also the population of the n-th level for $t \to +\infty$.

3) For example, the ionization of atoms, and also the displacement and splitting of atomic levels by an intense light wave (from a laser). In this connection see [14], in which the quasi-energy spectrum for the hydrogen atom is determined in perturbation theory (under the condition that the wave field is significantly weaker than the atomic field).

$$\Delta\varepsilon = -\frac{1}{T}\int_0^T L(t)\,dt = -\frac{1}{2T}\int_0^T f(t)\eta(t)\,dt. \qquad (30)$$

Thus, the quasi-energy spectrum of the oscillator is evenly spaced, and the magnitude of the "quantum" λ is completely determined by the form of $\omega(t)$, and is independent of the external force. In order to determine λ, it is necessary to solve the equation of motion (3) of the classical oscillator. Thus, for example, if the frequency changes according to the law

$$\omega(t) = \omega_0\sqrt{1+4h\cos\omega t},$$

where $\omega = 2\pi/T$ and $-1 < 4h < 1$, then Eq. (3) leads to Mathieu's equation

$$\ddot{x} + (a - 2q\cos 2\tau)x = 0$$

with parameters

$$a = \left(\frac{2\omega_0}{\omega}\right)^2, \quad q = -8h\left(\frac{\omega_0}{\omega}\right)^2, \quad \tau = \frac{\omega t}{2}. \qquad (31)$$

The theory of Mathieu functions is established in every detail,[15] and the value of parameter λ as a function of a and q is known.

In particular, as $h \to 0$ we have

$$\lambda = \omega_0 \times \begin{cases} \sqrt{\varepsilon^2 - h^2}, & \text{if } \omega = 2\omega_0(1+\varepsilon) \text{ and } \varepsilon \to 0 \\ 1 - \dfrac{4\omega_0^2 h^2}{4\omega_0^2 - \omega^2}, & \text{if } \omega \text{ is not close to } 2\omega_0 \end{cases} \qquad (32)$$

(the case of frequencies ω close to $2\omega_0$ corresponds to parametric resonance). If the point (a, q) approaches the boundary of the zone of stability, $\lambda \to 0$.

The inclusion of a periodic force f(t) leads to an overall displacement of the quasi-energy spectrum by an amount $\Delta\epsilon$. As is well-known, the periodic solution of an equation of the type (13) is unique, so the displacement $\Delta\epsilon$ is uniquely determined.

The quasi-energy spectrum of the quantum oscillator in the zone of stability is described by the formula (29). If the law of variation of frequency $\omega(t)$ is such that the corresponding classical oscillator lies in the zone of instability, then the quasi-energy spectrum is continuous, and the magnitude of the displacement $\Delta\epsilon$ is of no interest.

[1] Y. R. Shen, Phys. Rev. **155**, 921 (1967).
[2] B. Ya. Zel'dovich, A. M. Perelomov, and V. S. Popov, Zh. Eksp. Teor. Fiz. **55**, 589 (1968) [Sov. Phys.-JETP **28**, 308 (1969)].
[3] V. S. Popov and A. M. Perelomov, Zh. Eksp. Teor. Fiz. **56**, 1375 (1969) [Sov. Phys.-JETP **29**, 719 (1969)].
[4] B. R. Mollow, Phys. Rev. **162**, 1256 (1967).
[5] L. E. Estes, T. H. Keil, and L. M. Narducci, Phys. Rev. **175**, 286 (1968).
[6] H. R. Lewis, Phys. Rev. Lett. **18**, 510, 636 (1967); Phys. Rev. **172**, 1313 (1968).
[7] R. P. Feynman, Phys. Rev. **80**, 440 (1950); **84**, 108 (1951).
[8] J. Schwinger, Phys. Rev. **91**, 728 (1953).
[9] K. Husimi, Progr. Theor. Phys. **9**, 381 (1953).
[10] A. M. Dykhne, Zh. Eksp. Teor. Fiz. **38**, 570 (1960) [Sov. Phys.-JETP **11**, 411 (1960)].
[11] W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer (1966).
[12] Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **51**, 1492 (1966) [Sov. Phys.-JETP **24**, 1006 (1967)].
[13] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776 (1964) [Sov. Phys.-JETP **19**, 529 (1964)].
[14] V. I. Ritus, Zh. Eksp. Teor. Fiz. **51**, 1544 (1966) [Sov. Phys.-JETP **24**, 1041 (1967)].
[15] N. V. MacLachlan, The Theory and Application of Mathieu Functions, Oxford (1947) [Russ. Transl., IIL, M. 1953].

Translated by C. M. Hindle
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