THEORY OF ELECTROMAGNETIC EXCITATION OF SPIN WAVES IN FERROMAGNETS IN NON-UNIFORM STATIONARY MAGNETIC FIELDS

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It is shown that in the problem of high-frequency electromagnetic field excitation of spin waves in an infinite ferromagnetic sample in a non-uniform stationary magnetic field, the high-frequency field cannot be regarded simply as a constant-amplitude driving force as in[1]. The problem should be considered as that of interaction of mixed electromagnetic-spin spectrum branches. A parameter Γ defining the interaction is pointed out. The approach used in[1] is valid only for $\Gamma \ll 1$, when the coefficient (η) of transformation of the electromagnetic wave into a spin wave is small. The value of η is calculated for $\Gamma \ll 1$ for two cases. If the electromagnetic wave is incident from the side of large stationary magnetic fields, then $\eta \approx 1$. When it is incident from the opposite side, the electromagnetic wave is almost completely reflected and the coefficient η is exponentially small.

1. QUALITATIVE CONSIDERATION

A method of excitation of spin waves by the electromagnetic field in a nonuniform stationary magnetic field H was proposed by Schlomann.[1,2] The wave vector of the spin wave with frequency ω is

$$k_c^2 = D^{-1}[\omega/\gamma - H(r)] \quad (1)$$

(D is the nonuniform-exchange constant) and depends on the coordinate z, while the spin wave is excited by the electromagnetic field at the point z = 0 (see Fig. 1, dotted curve), where $k_c^2 = k_e^2 = (\omega/c)^2 \approx 0$. The expression obtained in[1] for the energy flux of the excited spin wave has the form

$$S_c = \pi\frac{\omega M}{D\alpha^3}h^2, \quad \alpha^3 = \left|\frac{dk_c^2}{dz}\right|_{z=0}. \quad (2)$$

Here M is the saturation magnetization, h the amplitude of the exciting high-frequency magnetic wave of circular polarization. It is not difficult to see, however, that Eq. (2) leads to an absurdity under certain conditions. Actually, if S refers to the energy flux in the incident electromagnetic wave, then the transformation coefficient η thus obtained is equal to

$$\eta = \pi\Gamma, \quad \Gamma = k_e q^2/\alpha^3, \quad q^2 = 4\pi M D^{-1}. \quad (3)$$

The only approximation used in[1] in the derivation of (2) lies in the applicability of the WKB method for the characteristic values of k_c^2. Here, however, η can be both smaller and greater than unity—the latter being quite meaningless. In the present work, we shall show that this paradox is associated with the fact that the exciting electromagnetic field is regarded as the driving force with fixed amplitude. Meanwhile, it is clear that because of the pumping of energy of the electromagnetic field into the spin system, the amplitude of the electromagnetic wave should change as it propagates, and the process of excitation of the spin wave should therefore be treated in terms of the transformation of waves from one mode to another in the propagation process. It is also clear from what has been said that the result (2) obtained by Schlomann is valid only for a small transformation coefficient, i.e. for $\Gamma \ll 1$. In the present work, we set up the problem so that it is valid for all values of Γ and make clear the physical meaning of this parameter and compute the transformation coefficient in the other limiting case $\Gamma \gg 1$. As in[1], an infinite ferromagnetic sample is considered, in which the magnetization and the stationary magnetic field are directed along the z axis and H increases in the direction of positive z. The incident and the excited waves propagate along the z axis.

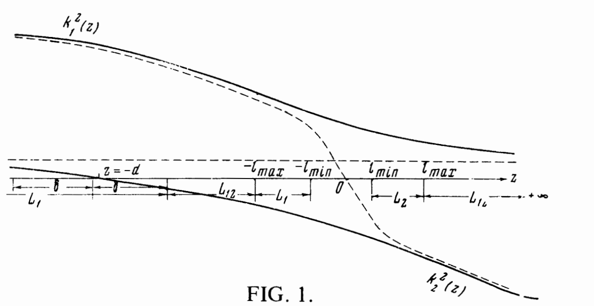

FIG. 1.

FIG. 1. Spatial variation of the wave vectors of two branches of the electromagnetic-spin spectrum. The dashed lines show the branches of the non-interacting spin and electromagnetic waves. The WKB approximation is valid only for the branch with $k^2 = k_i^2$ in the regions L_i, and for both branches in the regions L_{12}.

Our approach is based on the simultaneous solution of the equations of Maxwell and those of Landau-Lifshitz, which gives two modes of coupled electromagnetic-spin waves $k^2 = k_{1,2}^2(\omega, H)$. In the nonuniform magnetic field for a fixed frequency, k^2 depends on the coordinates and the form of this dependence is shown qualitatively in Fig. 1. These modes most closely approach one another and most strongly interact with one another at the point of intersection of the noninteracting branches (z = 0), where $k_c^2(z) = k_e^2 \approx 0$. Here $k_1^2(0) = -k_2^2(0) = qk_e$. In all real cases,

$$\beta = \alpha^3/a^3 \gg 1, \quad a = (qk_e)^{1/2}, \quad (4)$$

i.e., at the point of intersection, the condition of applicability of the WKB method is not satisfied, which generally makes possible the transition between the branches at this point, which we shall call the interaction point. For example, we consider an electromagnetic wave incident (along the branch k_2) from $z = -\infty$. In order to go over to the upper branch, the wave must reach the point of interaction. However, the branch k_2 has a turning point ($z = -d$), after which the incident wave penetrates a distance

$$\delta \approx (dk_2^2/dz)_{z=-d}^{-1/3} \approx \alpha^{-1}(q/k_3)^{2/3}.$$

The ratio of this penetration depth to the distance d from the turning point to the point of interaction ($d \approx q^2\alpha^{-3}$) is $\Gamma^{-2/3}$.

If $\Gamma \ll 1$, then the incident wave reaches the point of interaction and departs practically completely to $z = +\infty$ along the upper branch. Therefore, in such a situation, the result (3) obtained by Schlomann for the transformation coefficient must be valid; in our scheme, this is the reflection coefficient on the upper branch. In the opposite case $\Gamma \gg 1$, the electromagnetic wave is virtually entirely reflected from the turning point and only an exponentially small part of it reaches the point of interaction and proceeds on the upper branch, partially passing through and being partially reflected, i.e., transforming into a spin wave. Thus, for $\Gamma \gg 1$, the coefficient of transformation of the electromagnetic wave (incident from $z = -\infty$) into a spin wave is exponentially small. In this same situation, the electromagnetic wave incident from $z = +\infty$ (along the branch k_1) remains practically completely on the upper branch (as is shown below), transforming itself into a spin wave.

Finally, we note the following. For an estimate of the penetration depth δ, we have neglected the presence of the upper branch. This is possible, inasmuch as for $z = -d(k_c^2(-d) = q^2)$ the WKB condition is satisfied for the upper branch by a large margin, since we always have

$$q^3/\alpha^3 \gg 1. \tag{5}$$

By formulating the inequalities (4) and (5), we have assumed that $4\pi M = 2\times 10^3$ g, $dH/dz \approx 10^3$ Oe/cm, $D \approx 5\times 10^{-9}$ Oe-cm^2, so that $q^3/\alpha^3 = 10^6$, $k_e = 1$ cm^{-1}, $\beta = 10^3$. Here $\Gamma = 2$, and therefore, in view of the estimate character of this equality, both the case $\Gamma \ll 1$ and the case $\Gamma \gg 1$ can be regarded as real.

2. BASIC EQUATIONS AND THEIR SOLUTION

Let the waves be polarized circularly ($m_x - im_y = me^{-i\omega t}$, $h_x - ih_y = he^{-i\omega t}$). For the problem formulated above, the set of equations of Maxwell and Landau-Lifshitz has the form (after elimination of the electric field)[1)]

$$\begin{aligned} \tilde{h}'' + k_3^2\tilde{h} + a^2\tilde{m} &= 0,\\ \tilde{m}'' + k_c^2\tilde{m} + a^2\tilde{h} &= 0. \end{aligned} \tag{6}$$

Here $\tilde{m} = (\omega D M^{-1})^{1/2} m$, $\tilde{h} = c(4\pi\omega)^{-1/2}h$, and the prime in (6) and below denotes differentiation with respect to z. In such variables, the energy flux is

$$S = \mathrm{Im}\,(\tilde{h}^*\tilde{h}' + \tilde{m}^*\tilde{m}'). \tag{7}$$

We reduce the set (6) to a single equation of fourth order:

$$\tilde{h}^{IV} + [k_c^2(z) + k_3^2]\tilde{h}'' + k_3^2[k_c^2(z) - q^2]\tilde{h} = 0. \tag{8}$$

In the WKB approximation, the four linearly independent solutions of this equation have the form

$$\begin{aligned} \tilde{h}_{1,2} &= \left(\frac{k_1^2 - k_c^2}{k_1(k_1^2 - k_2^2)}\right)^{1/2} \exp\left(\pm i\int^z k_1\,d\xi\right),\\ \tilde{h}_{3,4} &= \left(\frac{k_c^2 - k_2^2}{k_2(k_1^2 - k_2^2)}\right)^{1/2} \exp\left(\pm i\int k_2\,d\xi\right),\\ \tilde{m}_i &= (k^2 - k_3^2)a^{-2}\tilde{h}_i,\\ k_{1,2}^2 &= \frac{1}{2}[k_c^2 + k_3^2 \pm ((k_c^2 - k_3^2)^2 + 4a^4)^{1/2}]. \end{aligned} \tag{9}$$

Normalization in (9) is performed so that the flux (7) is equal to unity. We shall explain in what regions the WKB solution is valid. For the branch with the larger value of $|k^2|$ (i.e., k_1^2 for $z > 0$ and k_2^2 for $z < 0$) the ratio $(k^2)'/k^3 \ll 1$ for $|z| \gg \alpha^{-1} = l_{min}$, i.e., for $k_c^2 \gg \alpha^2$. We note that, by virtue of (5), the WKB solution for the larger of k^2 is suitable when $k_c^2 \ll q^2$, i.e., for $|z| \ll d$. For the branch with the smaller value of $|k^2|$, the situation is more complicated. For $|z| \gg l_{min}$, the solutions corresponding to it are slowly changing and are solutions of an equation of second order, which is obtained if we discard the term in Eq. (8) with the fourth derivative, i.e.,[2)]

$$\tilde{h}'' + k^2(z)\tilde{h} = 0, \quad k^2(z) = k_3^2[1 - q^2/k_c^2(z)]. \tag{10}$$

Equation (10) is of standard type[3,4] with one turning point (for $z = -d$). The WKB approximation for it is suitable at a distance from the turning point much greater than the penetration depth δ. In the vicinity of the point of interaction, as follows from (10), for $k_c^2 \ll q^2$, we have $(k^2)'/k^3 \approx \alpha^{3/2}(a^2z^{1/2})^{-1}$. Therefore, the WKB for the smaller of the $|k^2|$ is valid for $|z| \gg \alpha^3 a^{-4} = l_{max} = l_{min}\alpha^4 a^{-4} \gg l_{min}$. The ratio $l_{max}d^{-1} = \Gamma^{-2}$, and $\delta d^{-1} = \Gamma^{-3/2}$. Consequently, for $\Gamma \ll 1$, the WKB approximation on these portions of the branches is suitable only for $k_c^2 \gg q^2$ (i.e., for $k^2 \approx k_e^2$, see (10)). In particular, it is not applicable on the lower branch anywhere between the turning point and the point of interaction.

As already noted, we shall consider the case $\Gamma \gg 1$. From the estimates given above, it follows that the WKB approximation is satisfied on both branches of the spectrum when $k_c^2 \ll q^2$ (an exception is the vicinity of the turning point, where we can, however, use the standard procedure,[3,4] developed for second order equations of the type (10). We shall find a solution of Eq. (8) in the following way. We find its exact solution in the vicinity of $z = 0$; we continue the asymptotes of these equations then for $l_{max} \ll |z| \ll d$, which are superpositions of the WKB solutions (9), in this same region, into the accessible region for the lower branch ($z < -d$) with the

1) We note that the equation considered in [1] is the second equation of the set (6) with h = const.

2) A similar situation existed in [5] in the transition from Eq. (2.2) to (2.7).

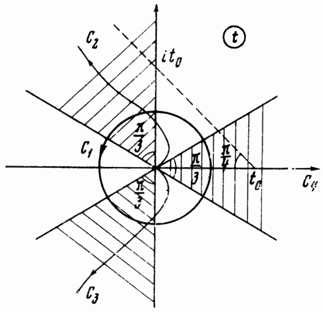

FIG. 2. Integration contours in the solutions (12). The contours C_{2-4} can be deformed only in such a way that their ends do not leave the corresponding shaded sector. The contours $C_{2,3}$ should approach the point t = 0 from the right (from the side Re t > 0).

aid of Eq. (10).[3)] Inasmuch as we shall seek exact solutions of Eq. (8) for $|z| \ll d$, where $k_c^2 \ll q^2$, then, with account of the fact that $k_e^2 \ll k_c^2$, expanding $k_c^2(z)$ near z = 0 ($k_c^2 = -\alpha^3 z$), we can write (8) in the form

$$\hbar^{IV} - \alpha^3 z\hbar'' - a^4\hbar = 0. \quad (11)$$

The four linearly independent solutions u_i of Eq. (11), found by the method of Laplace,[3,6] have the form

$$u_i(z) = \left(\frac{\beta}{\pi a}\right)^{1/2} v_i = \left(\frac{\beta}{\pi a}\right)^{1/2} \int_{C_i} \frac{dt}{t^2} \exp\left(xt - \frac{1}{t} - \frac{t^3}{3\beta^4}\right) \quad (12)$$

where $x = a^4 z\alpha^{-3} = z/l_{max}$. The factor before the integral in (12) is chosen for convenience and the contours C_i are drawn on Fig. 2. Calculation of the asymptote of these solutions (see the Appendix) leads to the following result for $|z| \gg l_{max}$ (i.e., $|x| \gg 1$):

$$u_1 = \begin{cases} 2ik_1^{-1/2}\cos(w_1 - 3\pi/4), & z > 0, \\ -i|k_2|^{-1/2} e^{-w_2}, & z < 0, \end{cases}$$

$$u_3 = u_2^* = \begin{cases} k_1^{-1/2} e^{-i(w_1 - \pi/4)}, & z > 0, \\ |k_2|^{-1/2} e^{w_2} + a^2 k_1^{-5/2} e^{-i(w_1-\pi/4)}, & z < 0, \end{cases} \quad (13)$$

$$w_1(z) = \int_0^z k_1\, d\xi, \qquad w_2(z) = \int_0^z |k_2|\, d\xi.$$

For z > 0, $u_4 \sim e^{w_2}$ and inasmuch as this is the only solution which increases without limit as $z \to +\infty$, it will not be needed by us below.

It was taken into account in (13) that for $\Gamma \gg 1$ one can choose z so that $k_c^2 \ll q^2$ and simultaneously $|z| \gg l_{max}$. Then, for example, for z < 0, $k_1^2 \approx k_c^2 \approx -\alpha^3 z$, $k_2^2 = a^4/\alpha^3 z \ll k_1^2$, and $k_1^2 - k_c^2 \approx -k_2^2$. Inasmuch as the WKB is applicable on both branches for $|z| \gg l_{max}$, then Eqs. (13) are naturally expressed only in terms of the WKB solution (9).

3. CALCULATION OF THE TRANSFORMATION COEFFICIENT OF THE WAVES

Let us consider the solutions of Eq. (8) which correspond to excitation of spin waves by the electromagnetic wave.

A. The electromagnetic wave is incident from z = −∞. The general solution of (8) has the form

$$\hbar = \sum_{i=1}^{4} C_i u_i. \quad (14)$$

The boundary conditions are the following: 1) the solution should be bounded as $z \to +\infty$; 2) as $z \to -\infty$, there should be no waves on the upper branch propagating in the direction of positive z; 3) as $z \to +\infty$, there should be no waves on the upper branch propagating in the direction of negative z. Finally, condition 4) is the normalization. We normalize the incident wave so that the energy flux in it is equal to unity. From the first two conditions, it follows that $C_2 = C_4 = 0$. The third condition yields $C_3 = C_1$, so that the solution (14) has the form

$$\hbar = C_1(u_1 + u_3). \quad (15)$$

Far from the turning point, for negative z > −d, as follows from (13),

$$\hbar \approx -iC_1\left\{\frac{1}{|k_2|^{1/2}}\exp\left(-\int_{-d}^{z}|k_2|\,d\xi + \tilde{\Gamma}\right) + \frac{a^2}{k_1^{5/2}}\exp\left[-i\left(\int_0^z k_1\,d\xi - \frac{3\pi}{4}\right)\right]\right\},$$

$$\tilde{\Gamma} = \int_{-d}^{0}|k_2|\,d\xi \sim \Gamma \gg 1. \quad (16)$$

Upon the continuation of (16) beyond the turning point in the accessible region (z < −d), the expression

$$\exp\left(-\int_{-d}^{z}|k_2|\,d\xi\right)$$

transforms into[4)] [4]

$$2\cos\left(-\int_{-d} k_2\,d\xi - \frac{\pi}{4}\right),$$

so that the fourth boundary condition gives $C_1 = e^{-\tilde{\Gamma}}$. Since the WKB solutions of (9) are normalized to unit flux, it follows that $|C_1|^2$, in accord with (16) gives the reflection coefficient along the upper branch. Inasmuch as for z > 0 this branch is a spin one, the coefficient of transformation of the electromagnetic wave into a spin wave is

$$\eta = |C_1|^2 = e^{-2\tilde{\Gamma}}. \quad (17)$$

By considering the solution (15) for z > 0, it can be verified that the transmission coefficient of the electromagnetic wave beyond the point of interaction is equal to $|C_1|^2$ [Eq. (17)].

B. The electromagnetic wave is incident from z = +∞. In this case, the two boundary conditions coincide with the first two boundary conditions in case A, so that $C_2 = C_4 = 0$ again. Normalizing the incident wave to unit flux, we obtain $C_3 - C_1 = 1$. One boundary condition should consist in the absence of a wave traveling along the lower branch in the direction of positive z as $z \to -\infty$. However, if we are not interested in the exponentially small (order of $e^{-\tilde{\Gamma}}$) amplitude of the previous (at z = −∞) electromagnetic wave, then we can avoid the joining of the solutions at the turning point, proceeding in the following way. We "forget" that the solutions of (13) are valid only to the right of the turning point and then require the boundedness of the solution as

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3)We note that for $\Gamma \ll 1$ one should find the asymptote of the exact solution for $|z| \gg l_{min}$ and then join it with the WKB approximation on the upper branch and the exact solutions of Eq. (10).

4)Corrections of the order of $e^{-2\tilde{\Gamma}}$ to the amplitude of the reflected electromagnetic wave in such a procedure do not arise because of the asymptotic character of the expressions used for the cylindrical functions, which do not take into account the exponentially small terms.

$z \to -\infty$. This gives $C_1 = 0$, so that $C_3 = 1$. Therefore, as is seen from the structure of the solution u_3, the wave as a whole goes into the region of negative z along the upper branch, i.e., in this case the electromagnetic wave, its completely transformed into a spin wave.

In conclusion, we note that in a bounded sample, the spectrum shown in Fig. 1 cannot be fully realized. For example, there can be a boundary of the sample between the point of interaction and the turning point. In this case, instead of the conditions as $z \to \pm\infty$, we have the boundary conditions on the surface of the sample, on which the transformation coefficient can depend materially.

APPENDIX

We calculate v_i (12) for the conditions (4) and

$$|z| \gg l_{min},\ \text{i. e.}\ |x|^3\beta^4 \gg 1. \tag{A.1}$$

1. **Calculation of $v_2 = v_3^*$.** a) $x > 0$. Inasmuch as we have Re $t < 0$, the integral converges for $|t| \lesssim x^{-1}$, so that for the condition (A.1) we can assume

$$v_2|_{x>0} \approx \int_{C_2} \frac{dt}{t^2} e^{xt-t^{-1}} = x^{1/2} \lim_{\varepsilon\to+0} \int_{C_2} \frac{dt}{t^{2-\varepsilon}} e^{x^{1/2}(t-t^{-1})}.$$

In the latter integral, the contour C_3 can be "straightened" on the positive imaginary semiaxis (for $\epsilon > 0$, there is no longer a divergence as $t \to 0$). Then we have ([7], p. 970)

$$v_2|_{x>0} \approx -i\pi x^{1/2} H_1^{(1)}(2x^{1/2}). \tag{A.2}$$

b) $x < 0$. The exponential in (12) has a saddle point at $t = it_0 = i\beta^2|x|^{1/2}$, and under the condition (A.1) close to this point we apply the saddle-point method. For the calculation of v_2, we deform the contour C_2 so that it consists of the dotted line in Fig. 2 and the cut of the positive real axis from 0 to t_C. The integral over the cut of the real axis converges for $t \lesssim |x|^{-1}$ and therefore can be extended to infinity, simultaneously neglecting the term $t^3/3\beta^4$ in the exponent. We then have

$$v_2|_{x<0} \approx -\pi|x|^{1/2} H_1^{(1)}(2|x|^{1/2} e^{i\pi/2})$$

$$+ \pi^{1/2}\beta^{-3}|x|^{-5/4} \exp[-i(2/3\beta^2|x|^{3/2} + \pi/4)]. \tag{A.3}$$

2. **Calculation of v_1.** Expanding $e^{-t^3/3\beta^4}$ in a series, we can limit ourselves in (A.1) to the zero order term in this series. Then[7]

$$v_1 \approx x \int_{C_1} \frac{d\xi}{\xi^2} e^{\xi - x\xi^{-1}} = 2\pi i x^{1/2} J_1(2x^{1/2}). \tag{A.4}$$

3. **Calculation of v_4 for $x > 0$.** In this case, v_4 is calculated by the saddle point method, which gives

$$v_4 = \pi^{1/2}\beta^{-3}|x|^{-5/4} \exp(2/3\beta^2 x^{3/2})$$

For $x < 0$, v_4 is identical with the first term in (A.3), as follows from the procedure of calculation of (A.3).

It can be verified that (A.1)–(A.4) are superpositions of the WKB solutions at the "largest" of the branches and solutions of Eq. (10) for the condition $k_C^2 = -\alpha^3 z \ll q^2$. Equations (13) are the asymptotes of Eqs. (A.1)–(A.4) for $|x| \gg 1$.

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