RESONANT ACCELERATION OF A BEAM OF OSCILLATORS IN A MEDIUM WITH POPULATION INVERSION

V. B. KRASOVITSKIĬ

Submitted June 11, 1969

Zh. Eksp. Teor. Fiz. 57, 1760–1764 (November, 1969)

The resonant acceleration of a beam of oscillators generated by a relativistic electron beam moving along an external magnetic field in a medium with a population inversion is considered. It is shown that the beam acceleration time is of the same order as the emission time of the medium. The energy acquired by the beam during that time is determined. The acceleration method under consideration is sufficiently effective if the beam density is small in comparison with the density of the active molecules of the medium.

It is shown in [1] that the Cerenkov field of a particle moving in a nonequilibrium medium can stimulate coherent emission of the active molecules. Since the phase velocity of the oscillations excited by the particle is close to the velocity of the particle the reaction of the field changes the Cerenkov deceleration of the particle into acceleration.[1)]

The present paper deals with the acceleration of a beam of oscillators generated by a relativistic beam of charged particles moving along a dc magnetic field in a medium with population inversion. The beam is acted upon by a transverse wave field generated by the medium under normal Doppler-effect conditions, i.e., when the frequency of the accelerating field in the beam's coordinate system is equal to the gyrofrequency. Such an acceleration mechanism is of a particular interest because the synchronization between the accelerating field and the oscillator is not destroyed by the relativistic mass increase of the beam particles.[2,3] We show below that acceleration does occur and is sufficiently effective if a paramagnet is used as the active material. The solution of the nonlinear equations of motion of the beam and the equations of state of the medium determines the maximum energy acquired by the beam, and yields the accelerating time, which has the same order of magnitude as the emission time of the medium.

The initial system of equations consists of the equations of motion of the beam and the continuity equations, the equations of the magnetic moment $\mathbf{M}$ of the medium, and Maxwell's equations for the field:

$$\frac{\partial}{\partial t}\gamma\mathbf{v}+(\mathbf{v}\,\mathrm{grad})\gamma\mathbf{v}=\frac{e}{m}\mathbf{E}+\frac{e}{mc}[\mathbf{v},\mathbf{H}+\mathbf{H}_0],\quad \gamma=\left(1-\frac{|\mathbf{v}|^2}{c^2}\right)^{-1/2};$$
$$\frac{\partial n}{\partial t}+\mathrm{div}\,n\mathbf{v}=0;$$
$$\frac{\partial\mathbf{M}}{\partial t}=-\frac{g\mu}{\hbar}[\mathbf{M},\mathbf{H}+\mathbf{H}_0]; \tag{1*}$$
$$\mathrm{rot}\,\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t};\quad \mathrm{rot}\,\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+4\pi\,\mathrm{rot}\,\mathbf{M}+\frac{4\pi}{c}en\mathbf{v};$$

where $\mathbf{v}$ is the velocity and n the density of the beam; $\mathbf{M}$ is the magnetic moment per unit volume; $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic components of the self-consistent field; $\mathbf{H}_0$ is the dc magnetic field and g is the Landé factor; μ is the elementary magnetic moment of an atom (Bohr magneton), and $\hbar$ is Planck's constant.

We consider a one-dimensional problem, assuming that all quantities depend only on the z coordinate parallel to the magnetic field $\mathbf{H}_0$ and seek the solution of (1) in the form of plane waves with circular polarization:

$$\begin{aligned}&E_x+iE_y=H_y-iH_x=E(t)\exp[i\Phi+i\psi(t)];\\&M_x+iM_y=b(t)\exp[i\Phi+i\eta(t)];\\&v_x+iv_y=v_\perp(t)\exp[i\Phi+i\vartheta(t)],\end{aligned} \tag{2}$$

where $\Phi=\omega(z/c-t)$. Furthermore, assuming that $v_z(t,z)=v_\parallel(t)$ and $n(t,z)=n_0$ (the continuity equation is satisfied automatically[4]), and substituting (2) into (1), we obtain a system of ordinary differential equations

$$\frac{d}{dt}\gamma v_\perp=\frac{e}{m}\left(1-\frac{v_\parallel}{c}\right)E\cos(\vartheta-\psi); \tag{3}$$
$$\frac{d}{dt}\gamma v_\parallel=\frac{e}{mc}v_\perp E\cos(\vartheta-\psi); \tag{4}$$
$$\frac{db}{dt}=-\frac{g\mu}{\hbar}(N^2\mu^2-b^2)^{1/2}E\cos(\eta-\psi); \tag{5}$$
$$\frac{dE}{dt}=-2\pi\Omega b\cos(\eta-\psi)-2\pi en_0v_\perp\cos(\vartheta-\psi); \tag{6}$$
$$\gamma\frac{d\theta}{dt}=\Omega\left(1-\frac{v_\parallel}{c}\right)\gamma-\omega_H-\frac{e}{m}\left(1-\frac{v_\parallel}{c}\right)\frac{E}{v_\perp}\sin(\vartheta-\psi); \tag{7}$$
$$\frac{d\eta}{dt}=-\frac{g\mu}{\hbar}(N^2\mu^2-b^2)^{1/2}\frac{E}{b}\sin(\eta-\psi); \tag{8}$$
$$\frac{d\psi}{dt}=2\pi\Omega\frac{b}{E}\sin(\eta-\psi)-2\pi en_0\frac{v_\perp}{E}\sin(\vartheta-\psi). \tag{9}$$

In the derivation of (3)–(9) we assumed that the amplitudes and phases of the velocity, magnetic moment, and electromagnetic field are slowly varying functions of time. In addition, putting $\omega=\Omega\equiv g\mu H_0/\hbar$ (where Ω is the resonance frequency of the medium, equal to the angular precession frequency of the magnetic moment), and using the law of conservation of total momentum we eliminate the z-component of the vector $\mathbf{M}$, namely $M_z=-(N^2\mu^2-M_x^2-M_y^2)^{1/2}$ (where N is the density of active molecules and the minus sign in front of the square root means that the magnetic-moment vector is

---

[1)] The possibility of direct conversion of energy stored in the active medium into kinetic energy of an accelerated particle beam was noted by Ya. B. Faĭnberg, and the use of an inverted paramagnet for acceldeation was suggested by E. K. Zavoĭskiĭ.

*$[\mathbf{v},\mathbf{H}+\mathbf{H}_0]\equiv\mathbf{v}\times(\mathbf{H}+\mathbf{H}_0)$.

directed against the magnetic field at the beginning of acceleration, i.e., the paramagnet is inverted[5]).

A formula determining the beam energy can be obtained from (3) and (4):

$$\frac{d}{dt} mc^2\gamma = eEv_\perp \cos(\vartheta - \psi). \tag{10}$$

Subtracting (4) and (10) term by term we find that the quantity

$$\left(1 - \frac{v_\parallel}{c}\right)\gamma = \left(1 - \frac{v_\parallel}{c}\right)\left(1 - \frac{v_\parallel}{c^2} - \frac{v_\perp^2}{c^2}\right)^{-1/2} \equiv q, \tag{11}$$

representing the difference between the energy and the longitudinal momentum of the beam is conserved and is an integral of motion.[2,3] The system (3)–(9) can be simplified by using (11) and assuming that $q = \omega_H/\Omega$ ($\omega_H = eH_0/mc$ is the gyrofrequency). It can be readily seen that a resonant solution exists and yields $\vartheta = \psi = \eta = 0$ satisfying (3)–(9) for any values of the functions $v_\perp(t)$, $v_\parallel(t)$, $b(t)$ and $E(t)$.

Expressing $v_\parallel(t)$ in terms of $v_\perp(t)$ from (11):

$$\frac{v_\parallel}{c} = \frac{1}{1+q^2}\left[1 - q^2\left(1 - \frac{1+q^2}{q^2}\frac{v_\perp^2}{c^2}\right)^{1/2}\right], \tag{12}$$

and substituting the variables

$$\frac{v_\perp}{c} = \frac{q}{\sqrt{1+q^2}}\sin x; \quad b = N\mu \sin y, \tag{13}$$

we can reduce (3)–(9) to the form

$$\begin{gathered} \frac{dx}{dt} = \frac{e}{mc}\frac{q}{(1+q^2)^{3/2}}(1+\cos x)^2 E; \\ \frac{dy}{dt} = -\frac{g\mu}{\hbar}E; \\ \frac{dE}{dt} = -2\pi\Omega N\mu \sin y - 2\pi e n_0 c \frac{q}{\sqrt{1+q^2}}\sin x. \end{gathered} \tag{14}$$

We consider the solution of (14) assuming that at the initial time $t = 0$ the transverse beam velocity is zero ($x(0) = 0$) and the magnetic moment is directed against the magnetic field ($b(0) = 0$). For these initial conditions, the integrals of motion of (14) can be represented in the following form:

$$\begin{gathered} \operatorname{tg}\frac{x}{2} + \frac{1}{3}\operatorname{tg}^3\frac{x}{2} = -\frac{2e\hbar}{mcg\mu}\frac{q}{(1+q^2)^{3/2}}y; \\ E^2 - 8\pi N\hbar\Omega \sin^2\frac{y}{2} + 2\pi n_0 mc^2(1+q^2)\operatorname{tg}^2\frac{x}{2} = 0. \end{gathered} \tag{15}$$

Eliminating E and y from the second equation in (15) by means of the first equation in (14) and the first equation in (15) we obtain an equation for the motion of the beam:

$$\left(\frac{dw}{dt}\right)^2 = 8\pi\frac{e^2}{m^2c^2}\frac{q^2}{(1+q^2)^3}\frac{1}{(1+w^2)^2}\left\{\frac{4}{g}N\hbar\Omega\sin^2\left[\frac{mcg\mu}{4e\hbar}\frac{(1+q^2)^{3/2}}{q} \times \left(w + \frac{1}{3}w^3\right)\right] - n_0 mc^2(1+q^2)w^2\right\}, \tag{16}$$

where $w = \tan \frac{1}{2}x$.

We assume below that the total angular momentum of the active atoms is determined by the spin angular momentum ($g = 2$, $\Omega = \omega_H$) and the beam velocity at the initial time is zero: $q = 1$. In this case (16) assumes the form

$$\frac{dw}{dt} = 2\sqrt{\frac{2N\mu^2\omega_H}{\hbar}}\frac{1}{1+w^2}\left\{\sin^2\left[\frac{1}{\sqrt{2}}\left(w + \frac{1}{3}w^3\right)\right] - \frac{n_0 mc^2}{N\hbar\omega_H}w^2\right\}^{1/2} \tag{17}$$

In the absence of the beam ($n_0 = 0$), Eq. (17) describes the natural oscillations of a paramagnet that is in inverted state at the initial time. The transverse amplitude of the magnetic moment y(t) varies like a nonlinear pendulum with the equilibrium position at the upper point and a period of oscillations $T = \frac{1}{2}(\hbar/N\mu^2\omega_H)^{1/2}$,[6] while the magnetic-moment vector that was directed against H_0 at the initial time rotates and becomes parallel to the magnetic field in one-half of a period. At the same time, the entire energy stored in the medium is transformed into the electromagnetic field energy. This is followed by the process of field absorption by the medium, returning the system to the initial state ($y = 2\pi$). The process is then repeated, although the value of y(t) decreases to zero and the field phase is shifted by π. If the beam density is different from zero the absorption of part of the energy by the beam gives rise to a maximum oscillation amplitude y_m different from 2π. We obtain the equation for the function w_m by setting $\dot{w}(w_m) = 0$ in (17):

$$\sin^2\frac{1}{\sqrt{2}}\left(w_m + \frac{1}{3}w_m^3\right) = \Delta w_m^2, \quad \Delta \equiv \frac{n_0 mc^2}{N\hbar\omega_H}. \tag{18}$$

In a general case this equation can be solved numerically, although when $\Delta \ll 1$ the solution can be obtained by the method of successive approximations:

$$w_m = w_0 - \sqrt{2\Delta}\frac{\omega_0}{1+w_0^2}; \quad w_0 + \frac{1}{3}w_0^3 = \sqrt{2}\pi. \tag{19}$$

In the first approximation we find: $w_m \approx w_0 \approx 2.4$.

Using (19) we can determine the maximum energy acquired by the beam:

$$mc^2\gamma_m = mc^2(1 + w_0^2) \lesssim 7mc^2. \tag{20}$$

Thus a low-density beam initially at rest, $n_0 \ll N\hbar\omega_H/mc^2$, can be accelerated up to the energy of $\lesssim 3.5$ MeV in time $T \sim \frac{1}{2}(\hbar/N\mu^2\omega_H)^{1/2}$ and the energy acquired by the beam is weakly dependent on the beam density. As beam density increases the maximum energy decreases and when $\Delta > 1$ acceleration does not occur.[2)]

The above theory does not take into account the longitudinal field $E_\parallel(t)$ generated by the z-component of the accelerated beam; it is therefore necessary to determine the conditions under which this approximation is valid. As shown in [4] the main effect obtained by including $E_\parallel$ is a phase shift of the beam relative to the accelerating field, which stops the acceleration.

Assuming that $v_\parallel = c$, we find the longitudinal field generated by current en_0c: $E_\parallel(t) = -4\pi en_0ct$. The quantity q, defined by (11), varies in time according to

$$q = 1 + {}^1/_{14}\omega_0^2 t^2, \quad \omega_0^2 = 4\pi e^2 n_0/m. \tag{21}$$

Substituting (21) into (7) we determine the time dependence of the beam phase shift

$$\vartheta(t) \approx \frac{1}{294}\omega_0^2\omega_H t^3. \tag{22}$$

The acceleration process obviously terminates at $\vartheta \sim 1$.

2) The interaction of a high-density charged-particle beam with an active dielectric in the absence of magnetic field was considered in [7]. In this case the energy stored in the medium can be converted into the energy of a longitudinal wave and the beam acts as a slow-wave waveguide.

The time τ determined by this condition turns out to be equal to $\tau \sim 7\omega_0^{-2/3}\omega_H^{-1/3}$. Comparing τ with the characteristic acceleration time T we find that the longitudinal field can be neglected with $T \ll \tau$:

$$\left(\frac{\hbar}{N\mu^2\omega_H}\right)^{1/2} \ll \frac{14}{\omega_0^{2/3}\omega_H^{1/3}} \tag{23}$$

In conclusion we evaluate the maximum density in the accelerated beam for a given value of the magnetic field. For $\omega_H \sim 2\times 10^{11}$ (wavelength of the order of a cm) and $N \sim 10^{20}$ we find $n_0 \lesssim 10^{10}$ and $T \sim 5\times 10^{-5}$. With increasing emission frequency the particle density n_0 can be increased: $\omega_H \sim 10^{15}$ (optical frequency range), $n_0 \lesssim 10^{14}$ and $T \sim 5\times 10^{-12}$. Consequently the above effect can be used in the design of heavy-current accelerators with moderate energy.

[1] V. B. Krasovitskiĭ and V. I. Kurilko, Zh. Eksp. Teor. Fiz. **57**, 864 (1969) [Sov. Phys.-JETP **30**, 473 (1970)].

[2] A. A. Kolomenskiĭ and A. N. Lebedev, Dokl. Akad. Nauk SSSR **145**, 1259 (1962) [Sov. Phys.-Dokl. **7**, 745 (1963)]; Zh. Eksp. Teor. Fiz. **44**, 261 (1963) [Sov. Phys.-JETP **17**, 179 (1963)].

[3] V. Ya. Davydovskiĭ, Zh. Eksp. Teor. Fiz. **43**, 886 (1962) [Sov. Phys.-JETP **16**, 629 (1963)].

[4] V. B. Krasovitskiĭ and V. I. Kurilko, Zh. Eksp. Teor. Fiz. **49**, 1831 (1965) [Sov. Phys.-JETP **22**, 1252 (1966)].

[5] A. E. Siegman, Microwave Solid State Masers. McGraw-Hill (Russ. Transl., Mir, 1966, p. 170).

[6] V. B. Krasovitskiĭ and V. I. Kurilko, Zh. Eksp. Teor. Fiz. **48**, 353 (1965) [Sov. Phys.-JETP **21**, 232 (1965)].

[7] V. B. Krasovitskiĭ, Zh. Eksp. Teor. Fiz. **53**, 573 (1967) [Sov. Phys.-JETP **26**, 371 (1968)].

Translated by S. Kassel
208