INSTABILITY OF A SPHERICAL CONFIGURATION IN THE SIMPLEST MODEL OF A NUCLEUS

V. G. NOSOV

Submitted June 11, 1969

Zh. Eksp. Teor. Fiz. 57, 1765–1777 (November, 1969)

It is shown that small deformations of certain homogeneous bodies with a certain type of energy spectrum are accompanied by a macroscopic quantum effect. A complete analytical solution of the problem is given for a specific model of an ideal Fermi gas. In this case the change of the volume energy of the body associated with a deformation of the spherical surface turned out to be negative. The results are found to be in qualitative agreement with data on the static and dynamic deformations of atomic nuclei. The physical nature of the phase transition in nuclei, a thermodynamical theory of which was given earlier, is discussed.

1. INTRODUCTION

Volume Quantum Effect Associated with a Deformation of the Surface of a Body

THE accuracy of the well-known assumption of thermodynamics about the independence of the energy of a homogeneous and isotropic body on the shape of the volume occupied by it is determined by the nature of its proof. It is based on (see, for example, the monograph[1]) the concept of the additive nature of the total energy, which is comprised of the energies of the individual volume elements. Their displacement in empty space enables one to reproduce any arbitrary deformation which does not change the value of the total volume. An obvious limitation follows from the fact that the area of the body's surface nevertheless changes. This leads to the appearance of well-known effects of the type of surface attraction.[1)]

However, quantum effects compel one to also consider the possibility of volume effects in the dependence of a body's energy on deformation. Let us attempt to analyze this question for the case of a Fermi liquid.[4,1] The momentum p_f of the quasi-particles at the cutoff of the Fermi distribution is an important characteristic of its energy spectrum. The validity of a classical description of their motion is limited by the corresponding de Broglie wavelength k_f^{-1}, and the region of linear deformations which are small in comparison with it requires a more accurate and critical examination. Here the "quasi-completeness" of a separate region is limited by its volume interaction with the rest of the body, which may turn out to be more important than a simple surface effect. However, it is impossible to overlook the fact that in these considerations the difference between large and small deformations turned out to be almost completely arbitrary because it depended on an arbitrarily chosen initial "undistorted" configuration. Therefore, the actual possibility of the effect of interest to us appears only in the immediate vicinity of the chosen configurations of the body. For these configurations the second derivative of the energy with respect to a dimensionless deformation turns out, in general, to be a quantity of the order of the total volume energy (whereas for the surface energy, its second derivative with respect to the same dimensionless deformation is only proportional to the surface of the body).

The physical nature of this peculiar phenomenon consists in the fact that the geometry of the body determines the quantum numbers of the individual quasi-particles; this is also reflected in the energy of the body as a whole. Some of the quantum numbers exist only for sufficiently symmetric configurations, and the deformations eliminate their character of lasting integrals of the motion. The latter property has a decisive value. Macroscopically, the volume nature of the effect is due to the fact that, roughly speaking, it affects all of the particles.

Let us clarify what has been said for the specific example of a Fermi gas, for which the calculations can be carried to completion (see the following section). For a spherical configuration the individual fermions possess orbital angular momentum l. Even for such a simple system as a particle in a three-dimensional potential well, the existence of such an integral of the motion leads to a characteristic quantum-mechanical "repulsion" of levels possessing identical quantum numbers. Let us turn to the formulas for the free motion within the limits of a spherical region of radius R_0 (see, for example,[5]), and let us estimate the minimum distance between the energy levels of a particle in the case of interest, $kR_0 \gg 1$, where k denotes the wave number. For identical values of l the levels follow one after the other with intervals whose relative magnitude is $\gtrsim (kR_0)^{-1}$. However, if the orbital momenta are different, then beginning with $\Delta l = 2$, the analogous lower limit at once falls to a value of the order of $(kR_0)^{-2}$. We see that in connection with a calculation of the total energy, similar inhomogeneities of the spectra of the one-particle levels do not cancel at all. On the other hand, influencing them by means of disturbing the quantum number l, a distortion also changes the volume energy of the entire body. The "asymptotic" region of linear deformations which are large in comparison with the characteristic length k_f^{-1} corresponds to the classical

[1)] As applied to a nucleus similar surface terms, which depend on the deformation, in the energy were first considered in articles [2,3].

limit, where the orbital momentum of a fermion ceases to be even approximately conserved, and the corresponding inhomogeneities in its energy spectrum are finally smoothed out. Here the volume energy of the body tends to a limiting value which does not depend on the deformation, i.e., the quantum effect of interest to us decreases to zero (the surface effect, of course, remains). The phenomena described above should occur even in the more general case of a Fermi liquid, whose energy spectrum is basically constructed according to the model of a Fermi gas.[2)]

The quantum effect associated with deformation has an extremely important significance for nuclear physics. Experimental data accumulated during the last 15 years indicates that for not too light nuclei the characteristics of their shapes cease to be "dead" parameters and acquire all the properties of dynamical variables. Consequently, the correct posing of the question of the shape of a nucleus stipulates, in the first place, a minimization of its energy with respect to such collective variables. Surface attraction, of course, supports the result that the equilibrium configuration should not differ too strongly from a spherical configuration. However, it is precisely in its vicinity that the volume effect becomes dominant (see below). It is characteristic that in fact the linear deformations of nonspherical nuclei turn out to be of the order of the de Broglie wavelength k_f^{-1} of a nucleon (more precisely, of the quasiparticle corresponding to it inside a nucleus) at the Fermi surface. Taking into consideration the features of the quantum phenomena associated with deformation which were outlined above, it is difficult to verify that this coincidence could occur by chance.

Further, aside from the static configuration, the dynamical behavior of a deformation in the neighborhood of the new equilibrium position is of real interest (in particular, see article[6] which is devoted to this question). And here it is quite probable that the volume part of the dependence of a nucleus' energy (with respect to a collective variable, it plays the role of the potential energy) on the deformation is still dominant. Finally, the very existence of collective degrees of freedom of a nucleus sets in front of us a series of questions of a fundamental nature. It is clear that for such a system the selection of a collective variable, which is qualitatively different from "individual" or "internal" degrees of freedom, acquires a rigorous meaning only in the limit of a sufficiently large number of particles. It is important, therefore, in particular that for an arbitrary finite number of particles, taking into account the "individual" influences on the dynamical characteristics of a collective degree of freedom should not fall outside the framework of the accuracy of this concept itself. In order to obtain appropriate estimates it is desirable in the first place to trace the entire process of the "formulation" of the collective characteristics of the system as the number of particles in it increases for some primitive model of the type considered in the following Section. Intuitive arguments suggest that the most important features of such results retain their significance even in the real case of a nuclear Fermi liquid.

Up to this point the question has been the deformation of a body at zero temperature. However, the physical character of the picture indicates its validity for temperatures which are sufficiently small in comparison with the temperature of degeneracy. Apparently only this region is of interest in nuclear physics. If such a condition is violated, then the nucleus as such ceases to exist even if at the expense of an "instantaneous" escape of neutrons, bypassing the stage of the establishment of thermal equilibrium.

We shall touch upon the important question of the physical nature of the phase transition in nuclei in the final section.

2. MODEL EXAMPLE OF A FERMI GAS

Let us calculate the energy of a Fermi gas as a function of the shape of the surface confining it, where the equation of the surface has the form

$$R(\mu, \varphi) = R_0 + \xi(\mu, \varphi) = R_0 \left\{1 + \sum_{\lambda, \nu} \alpha_{\lambda\nu} y_{\lambda\nu}(\mu, \varphi)\right\}. \qquad (1)$$

Here $\mu = \cos\theta$, and φ denotes the azimuthal angle. It is convenient to normalize the spherical functions according to

$$y_{\lambda\nu} = (-1)^\nu \sqrt{\frac{(\lambda-\nu)!}{(\lambda+\nu)!}} P_\lambda^\nu(\mu) e^{i\nu\varphi}, \quad y_{\lambda 0} = P_\lambda(\mu). \qquad (2)$$

The wave function of an individual fermion satisfies the Schrödinger equation

$$\nabla^2\Psi + k^2\Psi = 0 \qquad (3)$$

and the boundary condition

$$\Psi|_{r=R(\mu,\varphi)} = 0. \qquad (4)$$

Since the deviation $\xi(\mu, \varphi)$ from a spherical shape will be assumed to be small, one can seek the eigenfunction Ψ and the eigenvalue k^2 in the form of corresponding series

$$\Psi = \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \ldots, \quad k^2 = k^{2(0)} + k^{2(1)} + k^{2(2)} + \ldots \qquad (5)$$

It is also convenient to expand the boundary condition (4) in powers of ξ. It reduces to a sequence of relations which the Ψ-function and its derivatives must satisfy at $r = R_0$:

$$\Psi^{(0)} = 0, \quad \frac{\partial\Psi^{(0)}}{\partial r}\xi + \Psi^{(1)} = 0,$$
$$\frac{1}{2}\frac{\partial^2\Psi^{(0)}}{\partial r^2}\xi^2 + \frac{\partial\Psi^{(1)}}{\partial r}\xi + \Psi^{(2)} = 0 \qquad (6)$$

and so forth.

The considerations following below enable one to simplify the subsequent calculations. Since a change of the volume is not of interest to us, and it is natural to regard the center of inertia of the body as fixed, the independent parameters of a deformation turn out to be $\alpha_{\lambda\nu}$ only with $\lambda \geq 2$. The energy of the body as a whole is expressed in terms of these parameters according to

$$E(\alpha_{\lambda\nu}) = E_0 + \frac{1}{2}\sum_{\lambda\geq 2} C_\lambda \sum_{\nu=-\lambda}^{\lambda} |\alpha_{\lambda\nu}|^2 \qquad (7)$$

for small deformations (only such an expression is

[2)] However, for the spectra of a Bose liquid [1] there are, in general, no quantum characteristics similar to the de Broglie wave-length k_f^{-1}. The question of the possible existence of a volume quantum effect associated with small deformations of similar bodies remains unclear for the present, and we shall not be concerned with this point.

invariant with respect to rotations and reflections of the coordinate axes). Thus, in the quadratic approximation with respect to α, the problem reduces to a determination of the spectrum of the "stiffness" C_λ. Therefore, without restricting the generality, we may set all of the $\alpha_{\lambda\nu}$ equal to zero except $\alpha_\lambda \equiv \alpha_{\lambda 0}$, i.e., the deformation

$$\xi(\mu) = R_0 \sum_\lambda \alpha_\lambda P_\lambda(\mu) \tag{8}$$

is assumed to be independent of φ. The axial symmetry which arises in this connection enables us to assign a strictly definite value of the quantum number m to each of the particles.

The initial zero-order approximation

$$\nabla^2\Psi^{(0)} + k^{2(0)}\Psi^{(0)} = 0 \tag{9}$$

corresponds to the elementary problem already mentioned above (see the Introduction); its well-known solution has the form

$$\Psi^{(0)} = j_l(k_{nl}r)\, y_{lm}(\mu, \varphi),$$
$$k_{nl}R_0 = \rho_{nl}, \quad k^{2(0)} = k_{nl}^2. \tag{10}$$

Here $j_l(\rho)$ are the spherical Bessel functions;[7] the ρ_{nl} are their zeros which, for a given value of l, are labeled by the integer subscript n = 1, 2, 3, 4, ... in order of increasing value. The equation of the first-order approximation,

$$\nabla^2\Psi^{(1)} + k_{nl}^2\Psi^{(1)} = -k^{2(1)}\Psi^{(0)} \tag{11}$$

differs from Eq. (9) by the presence of the right-hand side. The recursion relations for spherical Bessel functions[7] enable one to easily solve such inhomogeneous equations:

$$\Psi^{(1)} = -\frac{k^{2(1)}}{2k_{nl}^2} k_{nl} r j_{l+1}(k_{nl}r)\, y_{lm} + \sum_{l' \neq l} a_{l'}^{(1)} j_{l'}(k_{nl}r)\, y_{l'm}. \tag{12}$$

Substitution into the second of the conditions (6) gives

$$k^{2(1)} = -2k_{nl}^2 \sum_\lambda C_{l0\lambda 0}^{l0} C_{lm\lambda 0}^{lm} \alpha_\lambda,$$
$$a_{l'}^{(1)} = \frac{\rho_{nl} j_{l+1}(\rho_{nl})}{j_{l'}(\rho_{nl})} \sum_\lambda C_{l0\lambda 0}^{l'0} C_{lm\lambda 0}^{l'm} \alpha_\lambda, \tag{13}$$

where the $C^{jm}_{j_1 m_1 j_2 m_2}$ are Clebsch-Gordan coefficients.[5] The disturbance of the quantum number l associated with a deformation, which is expressed by the second of formulas (13), plays here a decisive role (see the Introduction). It is also reflected in the form of the quadratic correction to the energy, for which we obtain the following expression after analogous uncomplicated calculations ($\rho = \rho_{nl}$):

$$k^{2(2)} = k_{nl}^2 \sum_{\lambda,\lambda'} \Big\{ 3 C_{l0\lambda 0}^{l0} C_{l0\lambda' 0}^{l0} C_{lm\lambda 0}^{lm} C_{lm\lambda' 0}^{lm}$$
$$+ 2 \sum_{l' \neq l} C_{l0\lambda 0}^{l'0} C_{l'0\lambda' 0}^{l0} C_{l'm\lambda 0}^{lm} C_{lm\lambda' 0}^{l'm} \frac{1}{j_{l'}(\rho)} \frac{d}{d\rho}[\rho j_{l'}(\rho)] \Big\} \alpha_\lambda \alpha_{\lambda'}. \tag{14}$$

Now let us introduce the following abbreviated notation in order to sum over the particles in the gas:

$$\sum_{m=-l}^{l} k^2 = \overline{k^2}, \qquad \sum_{n,l}^{\rho_{nl} < \rho_f} \overline{k^2} = \overline{\overline{k^2}}; \tag{15}$$

$\rho_f = k_f R_0$ corresponds to the boundary of the Fermi distribution. The first summation is elementary to perform on the basis of well-known properties of the Clebsch-Gordan coefficients. Introducing the notation $l' = l + \Lambda$, we find ($\rho = \rho_{nl}$)

$$\overline{k^{2(0)}} = (2l+1)k_{nl}^2, \quad \overline{k^{2(1)}} = -2(2l+1)k_{nl}^2\alpha_0,$$
$$\overline{k^{2(2)}} = (2l+1)k_{nl}^2 \sum_\lambda \Big\{ 3|C_{l0\lambda 0}^{l0}|^2$$
$$+ 2 \sum_{\Lambda \neq 0} |C_{l0\lambda 0}^{l+\Lambda, 0}|^2 \frac{1}{j_{l+\Lambda}(\rho)} \frac{\partial}{\partial\rho}(\rho j_{l+\Lambda}) \Big\} \frac{\alpha_\lambda^2}{2\lambda+1}. \tag{16}$$

Keeping in mind that large values of the quantum numbers n and l play a dominant role, in what follows we replace the summation over the individual levels by an integration. In order to change to more convenient variables, we shall use an asymptotic representation of the functions j_l for large values of the order and the argument[7] (it corresponds to the quasiclassical approximation in connection with a description of the free motion of particles in spherical coordinates; see, for example,[5]):

$$\rho j_l(\rho) \cong (\sin\beta)^{-1/2} \sin\chi_l(\rho),$$
$$\chi_l(\rho) = \rho(\sin\beta - \beta\cos\beta) + \pi/4,$$
$$\beta = \arcsin[1 - (l+1/2)^2/\rho^2]^{1/2} = \arccos(l+1/2)/\rho. \tag{17}$$

Since the unperturbed eigenvalues ρ_{nl} are determined from the condition

$$\chi_l(\rho) = \pi n, \tag{18}$$

it is not difficult to verify that

$$dndl = \pi^{-1}\rho d\rho \sin^2\beta d\beta. \tag{19}$$

Now let us also change to the variables ρ and β in the expressions for the energy. It is important to note that in the quadratic approximation (16) the extraneous factor $\rho \sim \rho_f \gg 1$ appearing inside the derivative sign does not actually determine the order of magnitude of all of this correction to the energy. In fact, now after summing over the values of Λ which differ by sign, the corresponding terms cancel, and it is necessary to take into account the next term of the expansion in powers of the parameter $\rho_f^{-1} \ll 1$. The Clebsch-Gordan coefficient appearing in Eq. (16) can be expressed in terms of factorials[5] and, with the aid of Stirling's formula it can easily be expanded in inverse powers of the large quantity $l \sim \rho_f$. To the required approximation, we have

$$|C_{l0\lambda 0}^{l+\Lambda, 0}|^2 \cong 2^{-2\lambda} f(\lambda+\Lambda)(\lambda+\Lambda)!(\lambda-\Lambda)!$$
$$\times \Big[\Big(\frac{\lambda+\Lambda}{2}\Big)!\Big(\frac{\lambda-\Lambda}{2}\Big)!\Big]^{-2} \Big\{1 + \frac{\Lambda}{2\rho\cos\beta}\Big\}, \tag{20}$$

where

$$f(\lambda) = \begin{cases} 1 \text{ for even } \lambda, \\ 0 \text{ for odd } \lambda. \end{cases} \tag{21}$$

Then, in order to take the terms of (17) whose relative magnitude is of the order of ρ_f^{-1} into account in the wave functions, it is necessary to make a series expansion of the "action" $\chi_{l+\Lambda}(\rho)$ up to, inclusively, terms quadratic in the correction Λ to the index l. As to the pre-exponential factors which only depend on β, here it is sufficient to confine our attention to the linear approximation in Λ. Next, multiplying by the phase space element (19) and setting the limits of integration, the values of which are obvious, we arrive at the following expression for the quadratic correction:

$$\overline{\overline{k^{2(2)}}} = \frac{2}{\pi R_0^2} \int_0^{\rho_f} \rho^4 d\rho \sum_\lambda \frac{\alpha_\lambda^2}{2\lambda+1} \int_0^{\pi/2} \{3|C_{l0\lambda 0}^{l}|^2 \sin^2\beta\cos\beta$$

$$- 2 \sum_{\Lambda>0}^{'} |C_{l0\lambda 0}^{l,\Lambda,0}|^2 [\cos^3\beta + (\sin^3\beta - 2\sin\beta\cos^2\beta)\Lambda \operatorname{ctg}\Lambda\beta + \sin^2\beta\cos\beta\cdot\Lambda^2(1+\operatorname{ctg}^2\Lambda\beta)]\} d\beta. \quad (22)$$

The square of the Clebsch-Gordan coefficient is now to be understood as just the zero-order approximation given by the factor in front of the curly brackets on the right-hand side of Eq. (20).

The physical nature of the singularities which the integrand possesses in the interval $0 < \beta < \pi/2$ in general is required in certain explanations. According to Eq. (13) the imposition of a small deformation α_λ, as it were, leads to an interaction between the levels l and l' in the original spherical well. For very close levels $l \neq l'$ even a very small deformation might turn out to be a perturbation which is not small at all. And what is more, the transition to the continuous spectrum which is carried out in accordance with Eq. (19) now also formally admits the case of exact coincidence of their energies. With the aid of expressions (17) one can verify that the poles $\Lambda\beta = \pi\nu$ ($\nu = 1, 2, 3, 4 \ldots$) exactly correspond to such a "degeneracy." And nevertheless the applicability of the developed method is not disturbed. It is easy to see that the residues vanish at the poles $\beta_\nu = \pi\nu/\Lambda$, i.e., the integral does not depend on the method of going around the singular points. Physically this means that a small deformation of the well does not lead to a radical rearrangement of the initial state of the body, which might be expressed, let us suppose, in the appearance of an imaginary contribution to the energy. In other words, in regard to the approximation under consideration, expressions (10) still remain the correct wave functions for the zero-order approximation of the degenerate problem.

Let us take the integral (22) as indefinite, reducing it to the difference between the values of a certain single-valued function at the limits of integration. The normalization of the Clebsch-Gordan coefficients enables us to carry out the following summation over Λ. Finally we obtain

$$\overline{k^{2(0)}} = \frac{2}{15\pi}\frac{\rho_f^5}{R_0^2}, \quad \overline{k^{2(1)}} = -\frac{4}{15\pi}\frac{\rho_f^5}{R_0^2}\alpha_0,$$
$$\overline{k^{2(2)}} = \frac{2}{15\pi}\frac{\rho_f^5}{R_0^2}\sum_{\lambda}^{'}\{[8|C_{l0\lambda0}^{l0}|^2 - 3]f(\lambda) - 2\}\frac{\alpha_\lambda^2}{2\lambda+1}. \quad (23)$$

The condition for the conservation of the body's volume, which was mentioned above in connection with relation (7), gives

$$\alpha_0 = -\sum_{\lambda\geqslant 2}\frac{\alpha_\lambda^2}{2\lambda+1}. \quad (24)$$

Thus, since both corrections to the energy turned out to actually be of the same order of magnitude, let us combine them:[3)]

$$\overline{k^{2(1)}} + \overline{k^{2(2)}} = -\frac{2}{15\pi}\frac{\rho_f^5}{R_0^2}\sum_{\lambda\geqslant 2}^{'}\{3 - 8|C_{l0\lambda0}^{l0}|^2\}f(\lambda)\frac{\alpha_\lambda^2}{2\lambda+1}. \quad (25)$$

One can obtain the condition for smallness of the deformation, for example, by means of estimates of the different terms in (6). It turns out that

$$k_f R_0 \alpha \ll 1 \quad (26)$$

is required for the overwhelming majority of cases (in the Introduction this criterion is formulated more intuitively). Also a restriction holds on the value of the index λ. Analyzing the expansion of the right-hand side of (16) in powers of the correction Λ to the orbital momentum, we find that $\Lambda^2 \sim \lambda^2 \ll l \sim \rho_f$.

Hence

$$\lambda \ll \sqrt{k_f R_0}. \quad (27)$$

It is characteristic of the quasiclassical-type derivation given above that fermions with extremely small values of l are not taken into consideration. For example, in the case $\lambda = 2$ which is of practical importance, already only a few s-states will cancel the result (25), but an additional account of the p-states makes the correction to the energy positive. Formally, the "stiffness" of the body thus calculated would be determined by a negligible minority of its particles. The solution of the paradox consists in the fact that the motion of such levels is parabolic only in the physically uninteresting region $\alpha \ll (k_f R_0)^{-2}$. Further, the graph of a level changes its character, tending then to an asymptote which is linear in $|\alpha|$. It is clear that such an effect does not pertain to the collective stiffness C_2. An analogous "individual" nature also holds for the linear term in the energy which is due to partial occupation of the last level at the cutoff of the Fermi distribution (see the first of relations (13)). Characteristically both of these non-collective effects in general turn out to be of the same order of magnitude. Taking them into account would correspond to exceeding the accuracy of the very concept of a collective variable α (see also above, in the Introduction). Therefore, for real systems the question of what kind of singularities the curve $E(\alpha)$ has for $\alpha \lesssim (k_f R_0)^{-2}$ probably should not arise at all.

Another remark pertains to the calculation of the derivative on the right hand side of Eq. (16). Since upon differentiation of the slowly varying quasiclassical preexponential, a term arises with only a factor ρ in the denominator (see above in connection with the derivation of formula (22)), strictly speaking the analogous correction terms to the asymptotic representation (17) of the Bessel functions should also be taken into account in order to give the calculation a completely consistent character. However, Eq. (18) for ρ_{nl}, which must be substituted into (16), is then somewhat changed. One can convince oneself that a systematic account of all such corrections does not affect the result.

In regard to the method of its evaluation, expression (25) represents (after subtraction of the initial energy of the undistorted body) the sum of the energies of a certain set of levels with fixed quantum numbers l, n, and m.[4)] The state of the gas corresponding to these quantum numbers, as a whole, differs somewhat from the ground state in proportion to the deformation. In fact, by varying the dependence on the deformation α, some levels intersect the boundary of the Fermi distribution. Then a level being lifted above it must lose

[3)] In expressions (23) and (25) the index l enters only in a purely formal way. According to Eq. (20) the corresponding asymptotic limit for the square of a Clebsch-Gordan coefficient actually ceases to depend on l.

[4)] In particular, since the index l loses the meaning of orbital angular momentum after the deformation, this classification of the levels of a particle is to some extent of a formal nature.

particles; however, a level which is dropping down below the Fermi surface acquires a fermion. For our purpose it is sufficient to evaluate the correction to the body's energy corresponding to such a change of the quantum numbers, having taken the energy

$$\varepsilon(\alpha) = \varepsilon_0(1 + b\alpha) \tag{28}$$

of an individual particle in the linear approximation in α. The change in the energy caused by the departure of a particle from a level amounts to (see Fig. 1) $\delta\epsilon = \epsilon_f - \epsilon(\alpha)$ if $\tilde{\epsilon}_0 < \epsilon_0 < \epsilon_f$, where $\tilde{\epsilon}_0$ is determined from the equation $\tilde{\epsilon}(\alpha) = \epsilon_f$. Let us denote the number of similar levels in the interval $d\epsilon_0$ by $n(\epsilon_0)d\epsilon_0$. Integrating between the indicated limits, we find

$$\delta E = -\tfrac{1}{2} n(\varepsilon_f)(\varepsilon_f b\alpha)^2, \tag{29}$$

where ϵ_f denotes the Fermi energy. The same expression is also obtained for the levels intersecting the Fermi surface in the opposite direction ($b < 0$). Thus, the desired correction to the energy of the gas is expressed in terms of the square of the linear correction to the energy of a particle. The latter, of course, also depends on the other quantum numbers (in addition to the energy ϵ_0) over which the result (29) must be integrated in addition.

The generalization associated with the presence of many deformations α_λ is obvious. The energy density of the fermion states is determined by relation (19). Taking the first of formulas (13) into account and using the notation (15) we have

$$\delta\overline{\overline{k^2}} = -\frac{R_0^2}{4\pi}\int_0^{\pi/2} [\overline{k^{2(1)}}]^2 \sin^2\beta\, d\beta = -\frac{2}{3\pi}\frac{\rho_f^5}{R_0^2}\sum_{\lambda\geqslant 2}|C^{l_0}_{l_0\lambda_0}|^2\frac{\alpha_\lambda^2}{2\lambda+1} \tag{30}$$

as applied to the case of interest to us.

The change in the ground state energy of a Fermi gas associated with a deformation is now given by the sum of expressions (25) and (30). Also on the basis of the definition (7) and formulas (20), (21), and (23), we finally obtain the spectrum for the stiffness of the model under consideration in the form

$$C_\lambda = \begin{cases} -\dfrac{6E_0}{2\lambda+1}\left\{1 - \dfrac{(\lambda!)^2}{2^{2\lambda}[(\lambda/2)!]^4}\right\} & \text{for even } \lambda \\ 0 & \text{for odd } \lambda. \end{cases} \tag{31}$$

The proportionality of the result to the body's total energy E_0 indicates the volume nature of the effect[5] (see the Introduction). At the same time its quantum nature appears in the sharp dependence of the imposed distortion on the parity $(-1)^\lambda$. Having a classical origin, the well-known effects of surface attraction and also of the Coulomb repulsion of volume elements of nuclear matter from each other[2] are, of course, not revealed in such pronounced even-odd oscillations. The volume stiffness of a spherical configuration with respect to deformations with odd λ tended to zero for us. In connection with the contemporary status of the problem, it would probably be premature to discuss this derivation

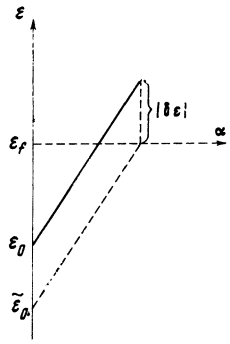

FIG. 1

in more detail. However, from experiment it follows that the equilibrium configuration of the nucleus apparently retains its center of symmetry, i.e., it does not contain odd harmonics (see, for example,[8]). This does not seem to be in contradiction with the second of relations (31), at least qualitatively.

The negative sign of the stiffness given by Eq. (31) for even values of λ indicates the instability of a spherical configuration: the body's energy falls with an increase of the corresponding deformation α. On the other hand, in the quasiclassical approximation the number of quantum states available for occupation by fermions is proportional to the phase volume required for them.[5] In general the latter remains unchanged if the number of particles and the volume of the gas are given. The apparent contradiction with formula (7) is explained by the narrowness of the domain of its applicability (26). As long as the deformation is so small, the effects caused by it are even further from its own classical limit and have little in common with it. It is characteristic that the total decrease in the body's energy here amounts to, in order of magnitude, only $C\alpha^2 \sim E_0/(k_f R_0)^2$. In the opposite limiting case

$$\alpha \gg (k_f R_0)^{-1}, \tag{32}$$

thanks to the constant number of "cells" in phase space, the change of a body's volume energy with deformation should cease.[6]

For the present it is necessary to regard the interesting question of whether the function $E(\alpha)$ monotonically approaches its classical limit as uninvestigated. In other words, at the present time there is still not enough solid foundation to finally yield a preference for one of the versions of the graph of this function which is schematically shown in Fig. 2. However, as a working hypothesis let us apply the second of the indicated possibilities (see Fig. 2b) and attempt to qualitatively analyze the physical consequences following from it. The estimate

$$\bar{\alpha} \sim (k_f R_0)^{-1} \tag{33}$$

for the equilibrium deformation of nonspherical nuclei is in satisfactory agreement with experiment. One can obtain an idea about the order of magnitude of the vibrational stiffness C which arises with respect to the new equilibrium position by comparing it to the absolute

[5] We did not yet take into consideration the fact that fermions possess spin. Spin degeneracy of the levels leads to a doubling of the expressions for the energy in any approximation with respect to the deformation. Therefore, when represented in the form (31), our result does not undergo any changes.

[6] This situation was already discussed above in the Introduction from a more general point of view which pertains not only to a gas.

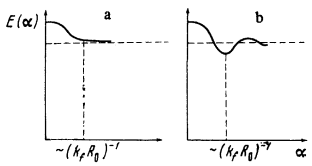

FIG. 2

value of the initial stiffness (31) of the spherical configuration. For a typical heavy nucleus with A ~ 200 this gives

$$C \sim |C_2| \sim 3000 \text{ meV} \qquad (34)$$

Unfortunately an investigation of the intensities of the electric quadrupole transitions accompanying a change of the vibrational state of nonspherical nuclei is still in an initial stage. However, the data cited, for example, in the monograph[8] concerning the excitation by multiply charged ions of such levels in thorium and uranium apparently does not contradict the rough estimate (34).

Speaking above about the experimental data, we primarily had in mind the quadrupole distortion, $\alpha = \alpha_2$, of a nucleus. However, even in this case, $\lambda = 2$, which is in practice the most important case, inequality (27) is not too strongly expressed. In order to eliminate the misgivings generated by such a situation, one can, without resorting to the integration according to formula (19), calculate, let us assume, the energy (25) by means of a direct summation of expressions (16) over the specific levels in the potential well. To be sure, the individual effect of very small orbital momenta which was mentioned above (see the text which immediately follows formula (27)) introduces some uncertainty in the estimates which are thus obtained. The general effect of these calculations is such that the corresponding collective degree of freedom is basically "formed" upon filling the 1d-state. Taking into account the quadrupling of the total number of fermions caused by the presence of the nucleon's spin and isotopic spin, we obtain A = 16 to 36. For heavier nuclei the characteristics of the quadrupole degree of freedom are apparently already sufficiently stable in this sense.

3. DISCUSSION. PHYSICAL NATURE OF THE PHASE TRANSITION

The unconstrained emergence, without some kind of complicating assumptions, of a distortion of the required order of magnitude already in the simplest conceivable model (see the preceding section) gives reason to doubt whether in general there exists a problem of "explaining" the nonspherical nature of nuclei as such. Preferably the actual explanation would be such as to explain why, sufficiently close to a magic number, the nuclei nevertheless turn out to be spherical. In regard to the accumulation of nuclear data, such a point of view apparently would correspond better to the observed facts than any other.

Roughly speaking the experimental data give the impression that the natural inherent nonsphericity of the nuclei can be disturbed only by the intrusion of a magic nucleus into their sequence. The unusual phenomena which accompany this breakdown cannot be fitted into the framework of the simplest ideas.[7] For example, immediately after the abrupt vanishing of the equilibrium distortion, the mass of the nucleus begins to systematically deviate from the predicted value given by the Weizsäcker semiempirical mass formula.[10] It is obvious that this energy effect is induced by a certain rearrangement of the ground state which should be reflected in some sort of region adjacent to it of the internal excitation spectrum of the nucleus. And, in fact, the spectra of the γ quanta for radiative capture of thermal neutrons[11] begin to deviate from the standard shape and to undergo qualitative changes exactly there where the equilibrium nonspherical nature is established, and the derivative of the nuclear mass with respect to the number of nucleons changes its value.

The connection between the strange phenomena described above and the proximity of a magic number nucleus seems to be definite. However, since the "magic number property" is interpreted as the result of a specific grouping of the levels of the individual nucleons for a spherically symmetric field, the distortion of the field α also plays the role of an independent variable with respect to the phase transition under consideration.[8] It is natural to assume that the change of the nucleus' internal state with distortion is continuous so that at the transition point $\alpha = \alpha_c$ both states coincide. It is extremely important that, according to generally accepted ideas about such continuous transformations (second-order phase transition[11]), both phases are not equivalent by far. An "ordered," so to speak, phase (in our case this will be the m-phase, $\alpha < \alpha_c$) is needed in addition to a certain phenomenological parameter in order to describe its properties, but for the normal "disordered" phase, $\alpha > \alpha_c$ (the n-phase in the terminology adopted in[9]), the corresponding quantity vanishes identically.

Thus, the "exotic" properties of the m-phase close to a magic nucleus, which the question raised above was about, are quite compatible with the spirit of the thermodynamical theory of a continuous transition[9] and they give a deeper meaning to it. On the other hand, in the case of an abrupt change of the internal state of the nucleus (a phase transition of the first kind) the question of which of the two phases is "more normal" would lose its physical content.

Just what kind of changes in the internal structure of nuclei near the magic numbers are those which give them the character of a distinctive "deviation from normal"? According to data on the masses,[10] the energy effect of a rearrangement of the ground state of a spherical nucleus (see above) in a recalculation per unit interval of the change in the number of nucleons only amounts to a quantity of the order of a megavolt. Therefore, the one-particle levels by themselves probably do not undergo any qualitative changes, but rather the phenomenon pertains to the residual interaction between

---

7) The limiting behavior of the collective characteristics of a nucleus at the phase transition point were considered in [6,9]. Here we discuss facts which more directly pertain to the internal state of the nucleus. Of course, in the final analysis it determines all collective properties of the system as well.

8) A thermodynamical theory of the phase transition in nuclei, based on these ideas, was developed in [9].

the nucleons. One can judge its strength, let us say, from the distance between the levels of a nucleus for which the states of the individual nucleons in the shell model are customarily assumed to be identical.

Information[8] pertaining to the region of heavy nuclei apparently gives an estimate in favor of ~0.02 MeV for the characteristic magnitude of such splitting. On the other hand, let us estimate the energy of the interaction of a nucleon with the surface of the nucleus (i.e., with the collective degree of freedom α). According to the results of the previous section, in order of magnitude it amounts to $\epsilon_f/(k_f R_0)^2 = \hbar^2/2MR_0^2$ ~ 0.4 meV (where M denotes the nucleon mass). The nearness of these values indicates that it is impossible to regard the "coupling scheme" for the indicated interactions inside a nucleus as completely predetermined beforehand. In principle two possibilities are conceivable here. If the residual interaction is, roughly speaking, more or less uniformly distributed between the different pairs of nucleons, then the interaction of an internal state of the nucleus with its surface will also primarily be realized by the individual nucleons. In this case, thanks to the well-known properties of the Fermi distribution, in particular particles possessing large values of the angular momenta will dominate in the interactions with the surface. The primitive model considered in the preceding Section gives a basis to assume that as a consequence a strongly expressed instability of the spherical configuration will appear.

However, near magic nuclei the dominance of the residual interaction precisely between nucleons located in one and the same shell may change the "coupling scheme," which is the topic of discussion. Here it is probable that the entire shell will turn out to be bound to the surface as a whole. At the present time we know very little about the mechanism for such a type of interaction of an internal state of the nucleus with a collective degree of freedom. However, the strange behavior of states with very small orbital momenta noted, for example, in Sec. 2 for the individual particle model in principle also admits the possibility of an increase of the energy upon deformation. Let us represent the naturally appearing case when the entire cluster of mutually interacting nucleons in a shell has some kind of small value for the total angular momentum J. It is quite possible that the interaction of such a cluster as a whole with the nucleus' surface imparts stability to its spherical configuration.

Thus, the phase transition is apparently due to a restructuring of the residual interaction in the nucleus. For the m-phase it is characteristic that, roughly speaking, the residual interaction in one and the same shell dominates to the detriment of the interactions between nucleons in different shells. Probably this is the explanation of the unusual properties of such nuclei mentioned in the present section. However, it would be too primitive to assume that in all other nuclei, which are farther away from a magic number, the normal phase of the residual interaction in general does not have any effect on the collective characteristics of the nucleus or reduces, let us assume, to only a replacement of the nucleons by the noninteracting quasiparticles corresponding to them. On the other hand, as one approaches the phase transition point the influence of the interaction between quasiparticles on the collective properties of the n-phase is felt more strongly. According to the results obtained in[9], this is especially clearly expressed by the relation

$$\frac{\partial \psi_n(\xi, \alpha)}{\partial \alpha}\Big|_{\alpha=\alpha_c} = 0 \qquad (35)$$

for the wave function of the nucleus at the transition point α_c (ξ is to be understood as the set of internal variables of the nucleus on which the wave function depends for a fixed deformation). Thus, if we confine our attention to an approximation that is linear with respect to the increase in the deformation, then here the residual interaction, as it were, completely disrupts the connection between the internal state of the nucleus and the collective degree of freedom. After this, a rearrangement of the energy spectrum obviously becomes unavoidable.

In conclusion we note that interest in a temperatureless phase transition of the second kind should not necessarily be limited to only the scope of nuclear physics. Since no physical principle is evident which would prohibit such transitions, perhaps in time they will be discovered for ordinary condensed matter. For such a phenomenon, varying as a function of the parameter α, after reaching the Curie point $\alpha = \alpha_c$ the ground state of the body begins to be continuously rearranged, taking on some kind of new quality. From this more general point of view the role of conditions of the type (35) still requires further investigation.

The author is sincerely grateful to E. M. Lifshitz, L. P. Pitaevskiĭ, and Ya. A. Smorodinskiĭ for a detailed discussion of the volume quantum effect associated with deformation of a body's surface which apparently is responsible for the nonspherical nature of many nuclei. The author also expresses his gratitude to A. M. Badalyan, A. I. Baz', D. P. Grechukhin, I. I. Gurevich, L. P. Kudrin, V. M. Kulakov, G. A. Pik-Pichak, Yu. A. Simonov, K. A. Ter-Martirosyan, E. L. Feĭnberg, and O. B. Firsov for discussions of the results of this work.

[1] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Fizmatgiz, 1964 (English Transl. of earlier ed., Addison-Wesley, 1958).

[2] N. Bohr and J. A. Wheeler, Phys. Rev. 56, 426 (1939).

[3] Ya. I. Frenkel', Zh. Eksp. Teor. Fiz. 9, 641 (1939).

[4] L. D. Landau, Zh. Eksp. Teor. Fiz. 30, 1058 (1956) [Sov. Phys.-JETP 3, 920 (1957)].

[5] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Fizmatgiz, 1963 (English Transl. of earlier ed., Addison-Wesley, 1958).

[6] V. G. Nosov, Yad. Fiz. 6, 44 (1967) [Sov. J. Nucl. Phys. 6, 32 (1968)].

[7] A. Sommerfeld, Partial Differential Equations in Physics, Academic Press Inc., New York, 1949 (Russ. Transl., IIL, 1950).

[8] Earl K. Hyde, The Nuclear Properties of the Heavy Elements, Vol. III, Fission Phenomena, Prentice Hall, Englewood Cliffs, N. J., 1964 (Russ. Transl., Atomizdat, 1968).

[9] V. G. Nosov, Zh. Eksp. Teor. Fiz. 53, 579 (1967) [Sov. Phys.-JETP 26, 375 (1968)].

[10]W. D. Myers and W. J. Swiatecki, Nucl. Phys. 81, 1 (1966); Lawrence Radiation Laboratory Preprint (1966).
[11]L. V. Groshev, A. M. Demidov, V. N. Lutsenko, and V. I. Pelekhov, Atlas spektrov γ-luchei radiatsionnogo zakhvata teplovykh neĭtronov (Charts of the Spectra of γ Rays of Radiative Capture of Thermal Neutrons), Atomizdat, 1958.

Translated by H. H. Nickle
209