PARTICLE DISTRIBUTION AND ELECTRIC FIELD IN THE VICINITY OF A BODY MOVING SLOWLY IN A RAREFIED PLASMA

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We obtain and analyze expressions for the particle concentration and current densities in the vicinity of a charged spherical body moving in a rarefied plasma. We also write down equations for the electric field. We assume that the dimensions of the body are small compared with the particle mean free path and that its speed is small compared with the thermal velocity of the ions.

We consider a spherical body of radius R_0 moving with a velocity V_0 in a rarefied plasma. We assume the dimensions of the body to be small compared to the particle mean free path. Let the surface of the sphere have a potential φ_0. The plasma close to the sphere is then, of course, perturbed. We must determine the total field in the plasma produced by the body and the space charge and also the particle distribution in the perturbed region.

We perform all our calculations in a spherical system of coordinates with the origin at the center of the sphere and the z axis in the direction of its velocity. The sphere is, of course, not moving in this system of coordinates and a particle current with average velocity $-V_0$ is incident upon it.

The distribution functions of the ions, f_i, and the electrons, f_e, and the potential of the electric field φ are determined by stationary kinetic equations and the Poisson equation:

$$\mathbf{v}\frac{\partial f_i}{\partial \mathbf{r}} - \frac{e}{m_i}\nabla\varphi\frac{\partial f_i}{\partial \mathbf{v}} = 0, \qquad \mathbf{v}\frac{\partial f_e}{\partial \mathbf{r}} + \frac{e}{m_e}\nabla\varphi\frac{\partial f_e}{\partial \mathbf{v}} = 0, \tag{1}$$

$$\nabla^2\varphi = -4\pi e\left\{\int f_i d^3v - \int f_e d^3v\right\}, \tag{2}$$

where e is the charge of the ions (the electron charge is equal to $-e$), m_i the ion mass, m_e the electron mass, v the velocity, and r the radius vector.

We must solve Eqs. (1) and (2) taking into account the boundary conditions at the surface of the body ($r = R_0$) and at infinity:

$$\varphi(R_0, \vartheta) = \varphi_0, \quad \varphi(r, \vartheta)_{r\to\infty} = 0 \tag{3}$$

(ϑ is the polar angle). We have here taken into account that the field possesses axial symmetry around the direction of motion.

The distribution functions at infinity are Maxwellian:

$$(f_{i,e})_{r\to\infty} = N_0\left(\frac{m_{i,e}}{2\pi T}\right)^{3/2}\exp\left\{-\frac{m_{i,e}}{2T}(\mathbf{v}+\mathbf{V}_0)^2\right\}, \tag{4}$$

where N_0 is the unperturbed ion (electron) concentration and T the plasma temperature in energy units.

If all ions are neutralized and the electrons are absorbed when colliding with the sphere, we have at the surface of the sphere for the functions

$$(f_{i,e})_{r=R_0} = 0. \tag{5}$$

As we consider particles which can have infinite speeds, Eq. (5) need be satisfied only for $v_r > 0$ where v_r is the velocity component along the radius vector.

A. V. Gurevich (see, for instance,[1]) has solved the set of Eqs. (1) and (2) with the boundary conditions (3) to (5) for a sphere at rest ($V_0 = 0$).

Let now $V_0 \ll v_i$, where $v_i = (2T/m_i)^{1/2}$ is the ion thermal velocity. We expand the particle distribution functions and the electrical field potential in a power series in the small parameter V_0/v_i, restricting ourselves to the first order terms. Then

$$f_{i,e} = f_{i,e}^{(0)}(r; \mathbf{v}) + \frac{V_0}{v_i}f_{i,e}^{(1)}(r, \vartheta; \mathbf{v}), \tag{6}$$

$$\varphi = \varphi^{(0)}(r) + \frac{V_0}{v_i}\varphi^{(1)}(r, \vartheta). \tag{7}$$

When writing down Eqs. (6) and (7) we took into account that the field of the body at rest is spherically symmetric and that $f_i^{(0)}$ and $f_e^{(0)}$ depend only on r and v because of the spherical symmetry.

Substituting these expressions into Eq. (1) we get for the zeroth and first approximations, respectively,

$$\mathbf{v}\frac{\partial f_{i,e}^{(0)}}{\partial \mathbf{r}} \mp \frac{e}{m_{i,e}}\nabla\varphi^{(0)}\frac{\partial f_{i,e}^{(0)}}{\partial \mathbf{v}} = 0 \tag{8}$$

and

$$\mathbf{v}\frac{\partial f_{i,e}^{(1)}}{\partial \mathbf{r}} \mp \frac{e}{m_{i,e}}\nabla\varphi^{(0)}\frac{\partial f_{i,e}^{(1)}}{\partial \mathbf{v}} = \pm\frac{e}{m_{i,e}}\nabla\varphi^{(1)}\frac{\partial f_{i,e}^{(0)}}{\partial \mathbf{v}}, \tag{9}$$

where the upper (lower) sign refers to ions (electrons).

We note that the field of the sphere at rest, φ_0, occurs in the left-hand sides of Eqs. (8) and (9). This means that when we change from a sphere at rest to one which is moving slowly the particle trajectories in the (r, v)-phase space are unchanged. Hence, in the method used by us to solve the problem the phase volume occupied by the particles remains unchanged and only the particle density in phase space is changed. We shall use this fact in what follows.

We now shall consider the Eqs. (8) and (9) for the ions. Instead of the ion velocity components we introduce new variables E, M, and ψ, where

$$E = m_i v^2/2 + e\varphi^{(0)}, \qquad M = m_i v_\perp r \tag{10}$$

(here $v_\perp$ is the ion velocity component at right angles to its radius vector), while ψ is the angle between the projections onto a plane through the origin and at right angles to the ion radius vector of the ion velocity vector and the vector $-\mathbf{V}_0$. It is reckoned from the projection of the vector $-\mathbf{V}_0$ counterclockwise (the angle φ in[2]). We note that for an ion in the field of a sphere at rest E, M, and ψ are integrals of motion.

The solution of the homogeneous Eq. (9) satisfying condition (4) was obtained in[2]. Expanding this solution in a power series in V_0/v_i, we get

$$f^{(1)}_{i\,\mathrm{hom}} = f^{(1)-}_{i\,\mathrm{hom}} + f^{(1)+}_{i\,\mathrm{hom}},$$

$$f^{(1)\mp}_{i\,\mathrm{hom}} = 2N_0\left(\frac{m_i}{2\pi T}\right)^{3/2} e^{-E/T}\left(\frac{E}{T}\right)^{1/2}(\cos C^{\mp}\cos\vartheta + \sin C^{\mp}\sin\vartheta\cos\psi). \tag{11}$$

Here f_i^- (f_i^+) corresponds to values $v_r < 0$ ($v_r > 0$). The angles C^- and C^+ are defined as follows:

$$C^- = \int_r^\infty \frac{M}{x^2F(x)}dx, \tag{12}$$

$$C^+ = \int_{r_{min}}^r \frac{M}{x^2F(x)}dx + \int_{r_{min}}^\infty \frac{M}{x^2F(x)}dx, \tag{13}$$

where

$$F(x) \equiv \{2m_i[E - e\varphi^{(0)}(x)] - (M/x)^2\}^{1/2}, \tag{14}$$

while r_{min} is the positive root of the equation $F(r) = 0$.

If the ion distribution function in the field $\varphi^{(0)}$ depends only on the ion energy, i.e., $f_i^{(0)} = f_i^{(0)}(E)$ we can transform in the stationary case the right-hand side of Eq. (9) for ions to the form

$$\frac{e}{m_i}\nabla\varphi^{(1)}\frac{\partial f_i^{(0)}}{\partial \mathbf{v}} = e\frac{\partial f_i^{(0)}}{\partial E}\mathbf{v}\nabla\varphi^{(1)} = e\frac{\partial f_i^{(0)}}{\partial E}\frac{d}{dt}\varphi^{(1)}.$$

As the quantity $\partial f_i^{(0)}/\partial E$ is an integral of motion along the trajectory in the field $\varphi^{(0)}$ of the sphere at rest, this expression can be written in the form

$$\frac{e}{m_i}\nabla\varphi^{(1)}\frac{\partial f_i^{(0)}}{\partial \mathbf{v}} = \frac{d'}{dt}\left(\frac{\partial f_i^{(0)}}{\partial E}e\varphi^{(1)}\right),$$

where now already the time derivative is taken along the trajectory in the field $\varphi^{(0)}$ (which is indicated by a prime). Taking this into consideration we get from Eq. (9)

$$\frac{d'}{dt}f_i^{(1)} = \frac{d'}{dt}\left(\frac{\partial f_i^{(0)}}{\partial E}e\varphi^{(1)}\right). \tag{15}$$

The time derivatives are here taken along the phase trajectory in the field $\varphi^{(0)}$.

It is clear from Eq. (15) that the functions $f_i^{(1)}$ and $e\varphi^{(1)}\partial f^{(0)}/\partial E$ can differ only by an arbitrary function of the integrals of motion of the ion in the field $\varphi^{(0)}$. Hence we have

$$f_i^{(1)} = f_i^{(1)-} + f_i^{(1)+}, \quad f_i^{(1)\mp} = \frac{\partial f_i^{(0)\mp}}{\partial E}e\varphi^{(1)} + f^{(1)\mp}_{i\,\mathrm{hom}}. \tag{16}$$

As the ion distribution functions $f_i^{(0)\mp}$ are Maxwellian, i.e., $f_i^{(0)\mp} = N_0(m_i/2\pi T)^{3/2}e^{-E/T}$, the functions $f_i^{(1)\mp}$ are equal to

$$f_i^{(1)\mp} = N_0\left(\frac{m_i}{2\pi T}\right)^{3/2}e^{-E/T}\left[-\frac{e\varphi^{(1)}}{T} + 2\left(\frac{E}{T}\right)^{1/2}\right.$$
$$\left.\times(\cos C^{\mp}\cos\vartheta + \sin C^{\mp}\sin\vartheta\cos\psi)\right]. \tag{17}$$

The functions $f_e^{(1)\mp}$ are determined in a similar way:

$$f_e^{(1)\mp} = -\frac{\partial f_e^{(0)\mp}}{\partial E}e\varphi^{(1)}. \tag{18}$$

If the electron distribution functions $f_e^{(0)\mp}$ are Maxwellian, i.e., $f_e^{(0)\mp} = N_0(m_e/2\pi T)^{3/2}\ E/T$, we have

$$f_e^{(1)\mp} = N_0\left(\frac{m_e}{2\pi T}\right)^{3/2}e^{-E/T}\frac{e\varphi^{(1)}}{T}. \tag{19}$$

In the same approximation in V_0/v_i we have for the particle concentrations

$$N_{i,e} = N^{(0)}_{i,e}(r) + \frac{V_0}{v_i}N^{(1)}_{i,e}(r,\vartheta). \tag{20}$$

Here $N_i^{(0)}$ and $N_e^{(0)}$ are, respectively, the ion and electron concentrations close to the body at rest.

When integrating the ion and electron distribution functions it is convenient as in[3,4] to change to the variables E, M^2, and ψ. For instance, in the case of ions

$$d^3v = \frac{d\psi dE dM^2}{2m_i^2r^2F(r)}, \tag{21}$$

where F(r) was defined in (14).

As the field $\varphi^{(0)}$ occurs on the left-hand sides of Eqs. (8) and (9) the region of integration when we wish to find the moments of the functions f_i and f_e is determined by the field in the vicinity of the body at rest. After integrating the functions (17) and (19) we get thus

$$N_i^{(1)} = -\frac{e\varphi^{(1)}}{T}N_i^{(0)} + N_0\frac{\cos\vartheta}{r^2(2\pi m_iT^3)^{1/2}}\left[\iint_{S_1}\frac{e^{-E/T}(E/T)^{1/2}\cos C^-}{F(r)}dEdM^2\right.$$
$$\left.+\iint_{S_2}\frac{e^{-E/T}(E/T)^{1/2}\cos C^+}{F(r)}dE\,dM^2\right], \tag{22}$$

$$N_e^{(1)} = \frac{e\varphi^{(1)}}{T}N_e^{(0)}. \tag{23}$$

The regions of integration S_1 and S_2 can be found, for instance, in the book[1], but only in (v_r, $v_\perp$)-space. In the particular case of a sphere with $R_0 \gg D$ and $|\varphi_0| \ll (T/e)(R_0/D)^{4/3}$, where $D = (T/4\pi e^2N_0)^{1/2}$ is the Debye radius of the unperturbed plasma the integration domains in (E, M^2)-space were determined in ref.[3]. Some information about integration domains is given in[5].

The expressions for the components of the ion and electron current densities along the inward normal to the spherical surface of radius r have the following form in the approximation which is linear in V_0/v_i:

$$j_{i,e} = j^{(0)}_{i,e}(r) + \frac{V_0}{v_i}j^{(1)}_{i,e}(r,\vartheta). \tag{24}$$

Here $j_i^{(0)}$ and $j_e^{(0)}$ are, respectively, the components of the ion and electron current densities along the inward normal to the spherical surface of radius r in the vicinity of the sphere at rest while the complete expressions for $j_i^{(1)}$ and $j_e^{(1)}$ based upon (17), (19), and (21) are equal to

$$j_i^{(1)} = -\frac{e\varphi^{(1)}}{T}j_i^{(0)} + N_0\frac{\cos\vartheta}{r^2(2\pi m_i{}^3T^3)^{1/2}}\iint_{S_1}e^{-E/T}\left(\frac{E}{T}\right)^{1/2}\cos C^-\,dE\,dM^2, \tag{25}$$

$$j_e^{(1)} = \frac{e\varphi^{(1)}}{T}j_e^{(0)}. \tag{26}$$

After the ion and electron concentrations are determined the problem is reduced to solving Eq. (2) for the potential of the field. Substituting expansions (7) and (20) into this equation we have

$$\nabla^2\varphi^{(0)} = -4\pi e(N_i^{(0)} - N_e^{(0)}), \tag{27}$$

$$\nabla^2\varphi^{(1)} = 4\pi e(N_i^{(1)} - N_e^{(1)}). \tag{28}$$

We get the boundary conditions for the last equation from Eq. (3):

$$\varphi^{(1)}(R_0, \vartheta) = 0, \quad \varphi^{(1)}(r, \vartheta)_{r\to\infty} = 0. \tag{29}$$

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