SPECTRAL REPRESENTATIONS IN A RENORMALIZABLE THEORY OF A MASSIVE VECTOR FIELD

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Spectral representations for the commutator and Green-like functions of vector currents and fields of a renormalizable theory are derived in the framework of dispersion theory. The contribution of one-particle intermediate states to the spectral density is taken into account exactly. The mass and wave function renormalization constants are expressed in terms of spectral integrals. Canonical commutators are constructed for the Heisenberg fields and an asymptotic relation is derived for the quantization scheme involving a supplementary condition.

1. INTRODUCTION

IN connection with the problems of current algebra, several authors[1] have pointed out formal contradictions between traditional dynamical calculations of equal-time commutators and the results which seem to correspond to them, based on spectral representations. In evaluating these contradictions, one should clearly keep in mind which is the correspondence between the quantities which are being compared in these two approaches. Usually, in the dynamical calculation one starts from a definite Lagrangian scheme, and relates the contradictions which arise with an incorrect definition of the current in the framework of that scheme. This mode of operation is not unique.

In[2-4] an improved version of the Lagrangian formalism has been developed, closely related to the dispersion relations approach to quantum field theory. It turned out that such a formalism, allowing to take into account more consistently the effects of renormalization, does not remove the above mentioned contradictions. But we hope that at least the renormalizable theories are in principle self-consistent. Therefore, in order to remove the contradictions it is necessary to pay special attention to both the definition of the S-matrix satisfying the usual axioms of field theory in the Lagrangian scheme, and to making more precise the form of the spectral representations in renormalizable theories. The spectral representations for the commutators of vector fields are usually written down without difficulties[1]. The difficulties appear when one considers the representations of Green's functions where subtractions become necessary, since the structure of the counterterms can in principle be quite arbitrary. The usual reasonings do not answer the question of the order of the differential operators, or the tensor structure of the counterterms. In Sec. 3 it is shown that a unique answer can be obtained in a special model, in which the whole interaction reduces to wave function renormalization. In addition, as has been made clear on the example of the scalar field theory[5], the transition to the Green's functions requires a consistent consideration of the contribution of one-particle terms to the spectral density. As a result we obtain the correct spectral representations for a renormalizable theory of a massive vector field, automatically taking into account the effects of renormalization.

As usual in the dispersion relations approach[6], we shall operate with a vector source-current defined by

$$J_\mu(x) = i\frac{\delta S}{\delta u_\mu(x)}S^+, \tag{1}$$

where $u_\mu(x)$ is the out-field of a massive neutral vector boson, satisfying the commutation relations

$$[u_\mu(x), u_\nu(y)] = -iD_{\mu\nu}(x-y) = -i\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\right)D(x-y), \tag{2}$$

corresponding to quantization with the supplementary condition $\partial_\mu u_\mu(x) = 0$, whereas the free Green's functions satisfy the transversality condition: $\partial_\mu D^a_{\mu\nu}(x) = m^{-2}\partial_\nu\delta(x)$. In addition we have to introduce into consideration current-like operators satisfying the "equations of motion"[7] of the type

$$\Lambda_{\mu\nu}(x,y) = \frac{\delta J_\mu(x)}{\delta u_\nu(y)} - i\vartheta(y_0-x_0)[J_\mu(x), J_\nu(y)]. \tag{3}$$

Below we shall not deal directly with the operators (1) and (3), but rather with their vacuum expectation values, denoting, as usual,

$$-i\langle[J_\mu(x), J_\nu(y)]\rangle_0 = f_{\mu\nu}(x-y), \tag{4}$$

$$-\left\langle\frac{\delta J_\mu(x)}{\delta u_\nu(y)}\right\rangle_0 = f_{\mu\nu}{}^a(x-y), \tag{5}$$

$$-i\vartheta(y_0-x_0)\langle[J_\mu(x), J_\nu(y)]\rangle_0 = f^{a(D)}_{\mu\nu}(x-y), \tag{6}$$

$$-\langle\Lambda_{\mu\nu}(x,y)\rangle_0 = \lambda_{\mu\nu}(x-y). \tag{7}$$

The multiplication with the theta-function in (6) is to be understood in the sense of Dyson, defined unambiguously in[6].

2. DERIVATION OF THE SPECTRAL REPRESENTATIONS IN THE DISPERSION RELATIONS APPROACH

The conditions of locality and relativistic invariance allow us to assert that $f_{\mu\nu}(x-y)$ is an antisymmetric Lorentz-covariant function vanishing outside the cone $(x-y)^2 \geq 0$. It can therefore be written in the form

$$f_{\mu\nu}(x-y) = \int ds\, d_{\mu\nu}(s) D_s(x-y), \tag{8}$$

where $D_s(x-y)$ is the Pauli-Jordan commutator function for mass $s^{1/2}$, and

$$d_{\mu\nu}(s) = g_{\mu\nu}I_1(s) + I_2(s)\partial_\mu\partial_\nu$$

(since D_s satisfies the Klein-Gordon equation for the free field, one may consider $d_{\mu\nu}(s)$ to be a differential operator of at most second order). In (2) we have assumed that the complete set of out-states contains only transverse (in the four-dimensional sense) vector particles of mass m. Therefore the integration in (8) starts from a threshold $s_0 > 0$.

The renormalizability condition for vector theories is the conservation of the vector source-current $\langle n | \partial_\mu J_\mu(x) | m \rangle = 0$ [9,10]. This implies the transversality of the function $f_{\mu\nu}$: $\partial_\mu f_{\mu\nu}(x) = 0$, leading for the spectral functions $I_1(s)$ and $I_2(s)$ to the relation $I_1(s) = s\,I_2(s)$. Since the point $s = 0$ does not belong to the spectrum, the representation (8) takes the form (we omit below the subscript of the function $I_1(s)$):

$$f_{\mu\nu}(x-y) = \int_{s_0}^{\infty} ds\left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{s}\right)D_s(x-y)I(s), \tag{9}$$

where the spectral function $I(s)$ has the well known expression in terms of the current matrix elements:

$$I(k^2) = \frac{(2\pi)^3}{3}\sum_n \langle 0|J_\lambda(0)|n\rangle\langle n|J_\lambda(0)|0\rangle\delta(P_n - k). \tag{10}$$

Since current conservation implies $P_\mu J_\mu(0) = 0$, where P_μ is the timelike energy-momentum vector of the intermediate state $|n\rangle$, the 4-vector $\langle 0 | J_\lambda(0) | n\rangle$ is spacelike; therefore the spectral function is positive definite.

Usually one does not include the one-particle state into the sum over intermediate states (10), referring to the stability condition $\langle 0 | J_\lambda(0) | 1\rangle = 0$, and picking the lower integration limit in (9) to correspond to the two-particle state: $s_0 > m^2$. In fact, as has been shown by the example of the scalar field theory[5], the one-particle states, although not contributing to the spectral representation for the currents, become essential for the corresponding representations of the field operators. It is still more convenient not to include them in the sum and to consider them separately.

In order to obtain the spectral representations of the advanced functions $f^a_{\mu\nu}$ we form the vacuum expectation values of the equations of motion (3), in the same manner as in[5]:

$$f_{\mu\nu}{}^a(x) = f_{\mu\nu}^{a(D)}(x) + \lambda_{\mu\nu}(x) = \vartheta(-x_0)f_{\mu\nu}(x) + \lambda_{\mu\nu}(x). \tag{11}$$

We now make use of the fact that the multiplication by the theta-function is here fixed by our method of extending the Fourier transform of the advanced function into the complex plane in such a manner that the contribution from the integral along a large circumference tends to zero. Formally this reduces to using the usual integral representation for the theta-function, and in particular the condition:

$$\vartheta(x_0 - y_0)\vartheta(y_0 - z_0)\ldots\vartheta(v_0 - u_0) = 0, \text{ if } x_0 = u_0.$$

Then

$$f_{\mu\nu}^{a(D)}(x) = \int_{s_0}^{\infty} ds I(s) D^{a(D)}_{\mu\nu,\,s}(x), \tag{12}$$

where, by definition

$$D^{a(D)}_{\mu\nu,\,s}(x) = -\vartheta(-x_0)\left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{s}\right)D_s(x)$$
$$= \left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{s}\right)D_s{}^a(x) - \delta_{\mu 0}\,\delta_{\nu 0}\frac{\delta(x)}{s}.$$

It is usually assumed that the integral with respect to s in the spectral representation (12) for $f^{a(D)}_{\mu\nu}$ diverges. Therefore it is convenient to effect in (12) a identity transformation, separating explicitly the convergent part. As a result, one obtains for $f^{a(D)}_{\mu\nu}$ the representation:

$$f^{a(D)}_{\mu\nu}(x-y) = (-Q_{\mu\rho})_x \int_{s_0}^{\infty} ds \frac{I(s)}{(s-m^2)^2} D^a_{\rho\sigma,\,s}(x-y)(-Q_{\sigma\nu})_y$$
$$+ c_{1D}(-Q_{\mu\nu})\delta(x-y) + c_{0D}g_{\mu\nu}\delta(x-y) + c_{0D}'\delta_{\mu 0}\delta_{\nu 0}\delta(x-y), \tag{13}$$

where

$$c_{1D} = -\int_{s_0} \frac{I(s)\,ds}{(s-m^2)^2}, \quad c_{0D} = \int_{s_0}^{\infty} \frac{I(s)\,ds}{s-m^2}, \quad c_{0D}' = -\int_{s_0} \frac{I(s)\,ds}{s}, \tag{14}$$

and the notation $-Q_{\mu\nu} = g_{\mu\nu}(m^2 - \Box) - \partial_\mu\partial_\nu$ has been introduced. We note that the operator $-Q_{\mu\nu}$ plays with respect to the function $D^a_{\mu\nu}(x)$ the same role as the operator $m^2 - \Box$ plays in the scalar theory with respect to the function $D^a(x)$: $-Q_{\mu\nu}D^a_{-\nu\lambda}(x) = g_{\mu\lambda}\delta(x)$.

In renormalizable theories the degree of growth of the Fourier transform $f^a_{\mu\nu}(x)$ cannot be larger than two. Therefore the function $\lambda_{\mu\nu}(x)$, corresponding to the contribution of the large circle to $f^a_{\mu\nu}$, must be a polynomial of degree at most two in the derivatives of the delta function. In general, this polynomial may have a complicated, noncovariant form. However, if the whole function $f^a_{\mu\nu}$ is covariant, the noncovariant parts in $f^{a(D)}_{\mu\nu}$ and $\lambda_{\mu\nu}$ must compensate each other. Taking into account the structure of the quasilocal terms in (13), it is convenient to select $\lambda_{\mu\nu}$ in the form

$$\lambda_{\mu\nu}(x-y) = \lambda_1(-Q_{\mu\nu})\delta(x-y) + \lambda_0 g_{\mu\nu}\delta(x-y) + \lambda_0'\delta_{\mu 0}\delta_{\nu 0}\delta(x-y). \tag{15}$$

Summarizing, we obtain for the total function $f^a_{\mu\nu}(x)$ the expression

$$f^a_{\mu\nu}(x-y) = (-Q_{\mu\rho})_x \int_{s_0}^{\infty} \frac{I(s)\,ds}{(s-m^2)^2} D^a_{\rho\sigma,\,s}(x-y)(-Q_{\sigma\nu})_y$$
$$+ c_1(-Q_{\mu\nu})\delta(x-y) + c_0 g_{\mu\nu}\delta(x-y) + c_0'\delta_{\mu 0}\delta_{\nu 0}\delta(x-y), \tag{16}$$

where

$$c_1 = c_{1D} + \lambda_1, \quad c_0 = c_{0D} + \lambda_0, \quad c_0' = c_{0D}' + \lambda_0'.$$

Equation (16) determines the spectral representation for the advanced Green's function up to a possible contribution of the one-particle terms, which still needs to be investigated. The coefficients c_1, c_0, and c_0' must be finite. At the same time, their components c_{iD} and λ_i may actually be infinite, after removing the intermediate regularization. Appeal to additional physical requirements helps to make more concrete the values of these coefficients.

3. STABILITY AND UNITARITY CONDITIONS

Imposing on the Green's functions the condition of stability of the physical one-particle state, we obtain two conditions

$$\lambda_0 = -c_0, \quad \lambda_0' = -c_0'. \tag{17}$$

Before discussing the selection of the coefficient c_1, we consider the role of the one-particle states in the spectral representations.

The contribution of the one-particle intermediate state to the spectral density is proportional to $(s - m^2)^2\delta(s - m^2)$. It is obvious that the inclusion of such a term will not manifest itself in the functions $f_{\mu\nu}$. For $f^a_{\mu\nu}$ it should reduce to mutually compensating additions to the integral term and the constant c_iD. However the question arises as to the direction of operation of $-Q_{\mu\nu}$, or equivalently, the order in which multiplications are carried out in the nonassociative product of generalized functions $(p^2 - m^2 - i\epsilon p_0)^{-1} \times (p^2 - m^2)\delta(p^2 - m^2)$ in the p-representation. This problem does not arise in the integral term, for which the positions of all the singularities of the delta-functions and the fractions do not coincide, and the order of action of $-Q_{\mu\nu}$ is irrelevant. Therefore, following the procedure developed for the scalar case, it is more convenient to consider that the integration in the spectral integral is carried out for $s_0 > m^2$, and the one-particle term is considered separately, determining it from the solvability and unitarity conditions.

On the basis of the solubility condition

$$f_{\mu\nu}(x-y) = f_{\mu\nu}{}^r(x-y) - f_{\mu\nu}{}^a(x-y).$$

On the other hand, substituting a complete set of intermediate states $|n\rangle\langle n|$ into the current commutator, and separating the contribution of the one-particle state, we have

$$f_{\mu\nu}(x-y) = f^{(1)}_{\mu\nu}(x-y) + \text{contribution with } c\ n \geqslant 2,$$

where

$$f^{(1)}_{\mu\nu}(x-y) = \int dz\, du\, f_{\mu\rho}{}^a(x-z) D_{\rho\sigma}(z-u) f_{\sigma\nu}{}^r(u-y). \quad (18)$$

The integral terms in $f^a_{\mu\rho}$, $f^r_{\mu\rho}$ do not contribute to (18), since $D_{\rho\sigma}$ satisfies the free Proca equation. Consequently, one might try to find a nontrivial expression for $f^{(1)}$, setting from the beginning $I(s) = 0$. This is done in the next section.

4. A MODEL FIELD THEORY BASED ON THE UNITARITY CONDITION

Thus, we consider a model in which the one-particle state unitarity condition is exactly fulfilled, i.e., all current matrix elements $\langle 0 | J_\lambda(0) | n\rangle$ vanish for $n \neq 1$. In such a model the equation $f_{\mu\nu} = f^{(1)}_{\mu\nu}$ is exactly satisfied, and the model is determined by the equations

$$f_{\mu\nu}(x-y) = \int dz\, du\, f_{\mu\rho}{}^a(x-z) D_{\rho\sigma}(z-u) f_{\sigma\nu}{}^r(u-y), \quad (19)$$

$$f_{\mu\nu}{}^a(x-y) = f_{\nu\mu}{}^r(y-x), \quad f_{\mu\nu}(x-y) = f_{\mu\nu}{}^r(x-y) - f_{\mu\nu}{}^a(x-y). \quad (20)$$

Taking into account the analogy of the form of the spectral representations for the vector and scalar[5,11] cases, we obtain directly the following solution of the system (19) and (20):

$$f^{a,r}_{\mu\nu}(x-y) = \underleftarrow{(-Q_{\mu\rho})}\,\hat\rho D^{a,r}_{\rho\sigma}(x-y)\hat\rho\underrightarrow{(-Q_{\sigma\nu})}$$
$$- \underleftarrow{(-Q_{\mu\nu})}\hat\rho\delta(x-y) - \delta(x-y)\hat\rho\underrightarrow{(-Q_{\mu\nu})},$$
$$f_{\mu\nu}(x-y) = \underleftarrow{(-Q_{\mu\rho})}\hat\rho D_{\rho\sigma}(x-y)\hat\rho\underrightarrow{(-Q_{\sigma\nu})}, \quad (21)$$

where $\hat\rho$ is an integro differential operator of the type of $\hat{I}$ or $\hat{N}$ (cf.[12]). It is imperative to indicate which way the operator $-Q_{\mu\nu}$ acts, since, e.g., by reversing these directions we would get for $f_{\mu\nu}$ an expression which vanishes identically.

In this case it is of particular interest that the solution (21) allows one to reconstruct the dynamics of the model. Indeed, since only the matrix element $\langle 0 | J_\lambda | 1\rangle$ is different from zero, the Heisenberg current J_λ is linear in the out-field, so that the relation (5) can be explicitly "integrated," yielding

$$J_\mu(x) = -\int dy\, f_{\mu\nu}{}^a(x-y) u_\nu(y).$$

After this the Heisenberg field can be determined from its expression in terms of the Yang-Feldman equation. Utilizing the concrete form (21) of the solution for $f^a_{\mu\nu}$, we obtain

$$J_\mu(x) = \underleftarrow{(-Q_{\mu\rho})}\hat\rho\int dy\,\{\delta(x-y)g_{\rho\nu} - D^a_{\rho\sigma}(x-y)\hat\rho\underrightarrow{(-Q_{\sigma\nu})}\}$$
$$\times u_\nu(y) + \hat\rho\underrightarrow{(-Q_{\mu\nu})}u_\nu(x), \quad (22)$$

$$U_\mu(x) = (1 - \hat{I}\hat\rho)\int dy\,\{\delta(x-y)g_{\mu\nu} - D^a_{\mu\rho}(x-y)\hat\rho\underrightarrow{(-Q_{\rho\nu})}\}\, u_\nu(y). \quad (23)$$

One can now recognize how the Heisenberg current is expressed in terms of the Heisenberg field, i.e., reestablish the form of the dynamical law that would define our model in a Lagrangian approach. For this purpose it suffices to express the integral in (22) in terms of (23), and to determine the term $\underrightarrow{(-Q_{\mu\nu})}u_\nu$ by applying the operator $-Q_{\mu\nu}$ to the Yang-Feldman equation:

$$\underrightarrow{(-Q_{\mu\nu})}u_\nu(x) = \underrightarrow{(-Q_{\mu\nu})}U_\nu(x) + \hat{I}J_\mu(x). \quad (24)$$

As a result we obtain

$$J_\mu(x) = \frac{1}{1-\hat{I}\hat\rho}\left\{\underleftarrow{(-Q_{\mu\nu})}\frac{\hat\rho}{1-\hat{I}\hat\rho} + \hat\rho\underrightarrow{(-Q_{\mu\nu})}\right\}U_\nu(x). \quad (25)$$

Comparison of (25) with (22) shows that no matter how we select the operator ρ we cannot simultaneously get rid in (22) of the integral kernels containing $\hat{I}$. The largest interest is presented by two special cases, when the operators $\hat{I}$ are absent in either of the two equations.

The first possibility corresponds to the "pure renormalization model," considered in[12]. In this case $\hat\rho = -(1 - Z^{1/2})\hat{N} = -(1 - Z^{1/2})/[1 - (1 - Z^{1/2})\hat{I}]$, and the simplest form is taken by the dynamical law in the Lagrangian formulation:

$$J_\mu(x) = -(1-Z)\int dy\,(-Q_{\mu\nu})\delta(x-y)U_\nu(y), \quad (26)$$

whereas the expressions (22) and (23) of J_μ and U_μ turn out to be fairly complicated (the operators $\hat{I}$ in these expressions, as in that for $\hat\rho$ show up in the denominators).

The second possibility, $\rho = -(1 - Z^{1/2})/Z^{1/2}$, corresponds to a simpler form (without the operator $\hat{I}$) of all the expressions characteristic for the axiomatic S-matrix theory:

$$J^R_\mu(x) = -\int dy\, f^{a(R)}_{\mu\nu}(x-y)u_\nu(y) =$$
$$= -\frac{1-\sqrt{Z}}{\sqrt{Z}}\{\underleftarrow{(-Q_{\mu\nu})} + \underrightarrow{(-Q_{\sigma\nu})}\}u_\nu(x) -$$
$$-\left(\frac{1-\sqrt{Z}}{\sqrt{Z}}\right)^2\underleftarrow{(-Q_{\mu\rho})}\int dy\, D^a_{\rho\sigma}(x-y)\underrightarrow{(-Q_{\sigma\nu})}\,u_\nu(y),$$
$$\lambda_{\mu\nu}{}^R(x-y) = \frac{1-Z}{Z}(-Q_{\mu\nu})\delta(x-y) \quad (27)$$

(the superscript R refers in the sequel to quantities in the pure renormalization model). However, if we wish to have in the theory a dynamical law of the form $J^R = f(U^R)$, for which it is necessary to introduce first the Heisenberg field U^R, the formalism of the theory will involve the operators $\hat{I}$. The relation between J^R and U^R, in place of (26), will have the form

$$J_\mu^R(x) = -\frac{1-Z}{Z}\left(1+\frac{1-\sqrt{Z}}{\sqrt{Z}}\hat{I}\right)^{-1}\int dy(-Q_{\mu\nu})\delta(x-y) \times\left(1+\frac{1-\sqrt{Z}}{\sqrt{Z}}\hat{I}\right)^{-1}U_\nu^R(y),$$

i.e., contains "infinite polynomials" in $\hat{I}$. Since in the present paper we deal only with the spectral representations, i.e., essentially with S-matrix theory, it is more convenient to select the second version.

In the framework of this version of the theory it is fairly simple to consider a more general case of soluble model, which is of independent interest. Indeed, the system (19)—(20) is satisfied not only by (27), but also by the more general solution:

$$f^a_{\mu\nu}(x-y) = A^2\underleftarrow{(-Q_{\mu\rho})}D^a_{\rho\sigma}(x-y)\underrightarrow{(-Q_{\sigma\nu})} - 2A(-Q_{\mu\nu})\delta(x-y) + B\partial_\mu\partial_\nu\delta(x-y),$$
$$f_{\mu\nu}(x-y) = A^2\underleftarrow{(-Q_{\mu\rho})}D_{\rho\sigma}(x-y)\underrightarrow{(-Q_{\sigma\nu})}, \tag{28}$$

where A and B are ordinary numbers: the solution (27) corresponds to $A = -(1 - Z^{1/2})/Z^{1/2}$; $B = 0$. In this solution, in addition to the operators $-Q_{\mu\nu}$, which together with D^a, and I would generate the "algebra" of the pure renormalization model, there appear the new differential operators ∂_μ. The rules of operation with these should be defined independently, and we adopt the simplest possibility, when the direction in which the operator ∂_μ acts is indifferent (this does not lead to any ambiguities).

The expressions for the current, the operator $\Lambda_{\mu\nu}$, and the Heisenberg field in the model (28) are obtained from the pure renormalization model (27) by means of the substitution:

$$J_\mu(x) = J_\mu^R(x) + B\partial_\mu\partial_\nu u_\nu(x),$$
$$\Lambda_{\mu\nu}(x, y) = \Lambda_{\mu\nu}^R(x, y) + B\partial_\mu\partial_\nu\delta(x-y),$$
$$U_\mu(x) = U_\mu^R(x) - Bm^{-2}\partial_\mu\partial_\nu u_\nu(x). \tag{29}$$

The most interesting feature of this new model becomes manifest if we try to find the relation between the current and the Heisenberg field, for which it is necessary to use, in addition to (24), the relation

$$\partial_\mu u_\mu(x) = \partial_\mu U_\mu(x) + m^{-2}\partial_\mu J_\mu(x).$$

It turns out that

$$J_\mu(x) = \frac{A(2-A)}{1-A\hat{I}}\int dy(-Q_{\mu\nu})\delta(x-y)\frac{1}{1-A\hat{I}}U_\nu(y) + B(1-A\hat{I})^{-4}(1+Bm^{-2}\Box(1-A\hat{I})^{-2})^{-1}\partial_\mu\partial_\nu U_\nu(x). \tag{30}$$

Thus, we encounter an interesting situation. The Heisenberg current is Hermitian, locally related to the out-fields and the causality condition is verified. At the same time the relation between J_μ and U_μ which was found here corresponds to a nonlocal interaction Lagrangian of the form¹⁾:

$$\mathscr{L}_I(x) = -\frac{A(A-2)}{2}:u_\mu(x)(1-A\hat{I})^{-1}\int dy(-Q_{\mu\nu})\delta(x-y)(1-A\hat{I}) \times u_\nu(y):-\frac{1}{2}B(1-A\hat{I})^4:u_\mu(x)\frac{\partial_\mu\partial_\nu}{1+Bm^{-2}\Box(1-A\hat{I})^{-2}}u_\nu(x):. \tag{31}$$

We have arrived at a model in which the S-matrix and the current satisfy all the axioms of local field theory, whereas the Lagrangian formulation formally corresponds to a nonlocal (or formally unrenormalizable) theory.

This investigation of the model allows us to take into account exactly only the contribution of one-particle terms in the spectral representation.

5. COMPLETE SPECTRAL REPRESENTATIONS FOR CURRENTS AND FIELDS

We now write out the complete spectral representations for the currents, taking into account the conditions of stability (17), and adding to (16) the contributions of the one-particle terms of the type (27). Here we set B = 0, since we always include in $\lambda_{\mu\nu}$ only the terms which coincide in structure with the quasilocal terms in the spectral representation for $f^{a(D)}_{\mu\nu}$. In other words, we write

$$f^a_{\mu\nu}(x-y) = \underleftarrow{(-Q_{\mu\rho})}\left\{\int_{s_0}^{\infty}\frac{I(s)\,ds}{(s-m^2)^2}D^a_{\rho\sigma,s}(x-y) - A^2D^a_{\rho\sigma}(x-y)\right\}\underrightarrow{(-Q_{\sigma\nu})} + (c_{1D}+\lambda_1-A^2)(-Q_{\mu\nu})\delta(x-y) \tag{32}$$

and accordingly

$$f_{\mu\nu}(x-y) = \underleftarrow{(-Q_{\mu\rho})}\left\{\int_{s_0}^{\infty}\frac{I(s)\,ds}{(s-m^2)^2}D_{\rho\sigma,s}(x-y) - A^2D^a_{\rho\sigma}(x-y)\right\}\underrightarrow{(-Q_{\sigma\nu})}. \tag{33}$$

The representations (32) and (33) are structurally analogous to the appropriate representations of the scalar case[5], and the unitarity condition yields exactly the same relation $-2A = c_{1D} + \lambda_1 - A^2$, whence $A = 1 - (1 - c_{1D} + \lambda_1)^{1/2}$.

Naturally, the identification of the constants A, c_{1D} and λ_1 in the renormalized (R) and unrenormalized (U) theories will be the same:

a) R-theory $\lambda_1 = \frac{1-Z_3}{Z_3}$, $c_{1D} = -\lambda_1$, $A = 0$,

b) U-theory $\lambda_1 = 0$, $c_{1D} = Z_3 - 1$, $A = 1 - \sqrt{Z_3}$.

Going over to the representations of the fields we obtain for the cummutator function:

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¹⁾The denominator involving the d'Alembertian in the second term of (31) should be interpreted in such a manner as to verify the solvability and unitarity conditions for the S-matrix. It is easy to see that for this the second terms has to be interpreted in the form (for simplicity we set now $\hat{I} = 1$)

$$-\frac{1}{2}m^{-2}(1-A)^{-2}:u_\mu(x)\int dy\bar{D}(x-y)\partial_\mu\partial_\nu u_\nu(y):,$$

where $\bar{D}(x-y)$ is the Green's function with symmetrical evasion of the pole, corresponding to an effective mass $M^2 = -m^2(1-A)^2B^{-1}$.

A second possibility consists of representing the second term as the formal sum of an infinite series in the powers of the d'Alembert operator. Such a treatment corresponds to a formally unrenormalizable theory, in which the degrees of growth of the matrix elements increases without bound as the order of perturbation theory is increased. However, the summation of the series yields, e.g., for the current Green's function growth of second order, agreeing completely with the "nonlocal" interpretation.

$$G_{\mu\nu}(x-y)=i\langle[U_\mu(x),U_\nu(y)]\rangle_0=(1-A)^2D_{\mu\nu}(x-y)$$
$$+\int_{s_0}^{\infty}\frac{I(s)\,ds}{(s-m^2)^2}D_{\mu\nu,\,s}(x-y).$$

In addition, it is interesting to find the expression of the matrix element of the Heisenberg field between the vacuum and the one-particle state. Starting from the spectral representation for $f^a_{\mu\nu}$ it is easy to obtain that

$$\langle 0|U_\mu(x)|1\rangle=(1-A)\langle 0|u_\mu(x)|1\rangle. \qquad (34)$$

As regards the vacuum expectation value of the equal-time commutator of the currents could be computed, of course, by multiplying $\delta(x_0-y_0)$ by $f_{\mu\nu}$; but such a procedure would be very clumsy for $A\neq 0$. Therefore it is convenient to use a roundabout way, used earlier[13,14] for the Heisenberg field operators themselves (rather than their matrix elements). Owing to the current conservation, $\partial_\mu f_{\mu\nu}=0$ and therefore, taking the divergence of the Dyson part $f^{a(D)}_{\mu\nu}$, yields exactly the necessary combination $\delta(x_0-y_0)f_0$. Equation (11) implies the relation

$$\delta(x_0-y_0)f_{0\nu}(x-y)=\partial_\mu f_{\mu\nu}{}^a(x-y)-\partial_\mu\lambda_{\mu\nu}(x-y).$$

Substituting here (32) for $f^a_{\mu\nu}$ and (15) for $\lambda_{\mu\nu}$ and taking into account the interpretation of the coefficients A and χ_i we obtain for the R-theory and for the U-theory

$$\delta(x_0-y_0)\langle[J_0(x),J_\nu(y)]\rangle_0=\delta_{\nu k}\int_{s_0}^{\infty}\frac{I(s)\,ds}{s}\partial_k\delta(x-y). \qquad (35)$$

We have already related the divergent integral c_1D with the renormalization constant Z_3 for the field operators. One could try to relate also the other two integrals c_0D and $c_0'D$ (cf. (14)) with the renormalization constants which appear in the usual Lagrangian scheme. First of all, the argument developed in the scalar case[15] lead here also to the identification

$$-c_{0D}=\delta m^2=\int_{s_0}^{\infty}\frac{I_U(s)\,ds}{s-m^2},\qquad I_U(s)=Z_3I_R(s),\qquad m_0^2=m^2+\delta m^2.$$

The last integral $c_0'D$ is not expressible only in terms of the renormalization constants Z_3 and m^2; after separating the standard divergent constants, there remains a presumably convergent integral:

$$c_{0D}'=\int_{s_0}^{\infty}\frac{I_R(s)\,ds}{s}=\frac{m^2+\delta m^2}{Z_3}-m^2+\int_{s_0}\frac{I(s)\,ds}{s(s-m^2)^2}m^4.$$

6. CANONICAL COMMUTATORS

One could also write out the equal-time commutation relations for the fields U_μ, but they are not very instructive, since they involve all four components of the field U_μ, of which only three correspond to independent physical quanta. Therefore, in order to verify the correctness of our adopted identification of the integrals, it is desirable to consider the canonical commutation relations, which are most conveniently sought in the form of the commutators between the creation and annihilation operators of physical quanta. The analogs of these operators for the Heisenberg fields are the amplitudes introduced by Lehmann, Symanzik and Zimmermann[16] for the formulation of the asymptotic condition.[17]

For the scalar field such amplitudes are the (time-dependent) coefficients of expansion of the Heisenberg field B(x) with respect to an orthonormal basis of positive- and negative-frequency solutions of the free Klein-Gordon equation,

$$\varphi_{\mathbf{k}}{}^{\pm}(x)=(2\pi)^{-3/2}(2k_0)^{-1/2}e^{\pm ikx}\,(k_0=\sqrt{\mathbf{k}^2+m^2}):$$
$$B(x)=\int d\mathbf{k}\,\{\varphi_{\mathbf{k}}{}^+(x)B^+(\mathbf{k},x_0)+\varphi_{\mathbf{k}}{}^-(x)B^-(\mathbf{k},x_0)\},$$

where the Lorentz-invariant inner product on the hyperboloid has the x-representation:

$$(\varphi_1(x),\varphi_2(x))=-i\int d\mathbf{x}\,(\partial_0\varphi_1(x)\varphi_2{}^*(x)-\varphi_1(x)\partial_0\varphi_2{}^*(x)).$$

The vector Heisenberg field off the energy shell has both transverse and longitudinal components:

$$U_\mu(x)=V_\mu(x)+\Lambda_\mu(x),$$

where $\partial_\mu V_\mu(x)=0$, $\partial_\mu\Lambda_\nu(x)-\partial_\nu\Lambda_\mu(x)=0$. We accordingly introduce two sets of solutions of the Klein-Gordon equation for the vector field: the transverse ones $v_{\mu;\mathbf{k},a}(x)=(2\pi)^{-3/2}(2k_0)^{-1/2}e^{\pm ikx}(\delta_{\mu 0}\frac{k_i}{k_0}-\delta_{\mu i})e^a_i(\mathbf{k})$ and the longitudinal ones $\lambda^{\pm}_{\mu\mathbf{k}}(x)=(2\pi)^{-3/2}(2k_0)^{-1/2}\times e^{\pm ikx}k_\mu/m$. Their norm is defined through the inner product

$$\langle\varphi_\mu(x),\psi_\mu(x)\rangle=-i\int d\mathbf{x}\,(\partial_0\varphi_\mu(x)\psi_\mu{}^*(x)-\varphi_\mu(x)\partial_0\psi_\mu{}^*(x)),$$

and the orthonormality conditions take the form

$$(v^{\pm}_{\mu;\,\mathbf{k},a},v^{\pm}_{\mu;\,\mathbf{k}',a'})=\delta_{aa'}\delta(\mathbf{k}-\mathbf{k}'),$$
$$(\lambda^{\pm}_{\mu;\,\mathbf{k}},\lambda^{\pm}_{\mu;\,\mathbf{k}'})=\delta(\mathbf{k}-\mathbf{k}').$$

We now expand the field $U_\mu(x)$ with respect to these solutions:

$$U_\mu{}^{\pm}(x)=\int d\mathbf{k}\,\{v^{\pm}_{\mu;\,\mathbf{k},a}(x)V_a{}^{\pm}(\mathbf{k},x_0)+\lambda^{\pm}_{\mu;\,\mathbf{k}}(x)\Lambda^{\pm}(\mathbf{k},x_0)\},$$
$$U_\mu(x)=U_\mu{}^+(x)+U_\mu{}^-(x)$$

where

$$V_a{}^{\pm}(\mathbf{k},x_0)=(U_\mu(x),v^{\pm}_{\mu;\,\mathbf{k},a}(x)),\qquad\Lambda^{\pm}(\mathbf{k},x_0)=(U_\mu(x),\lambda^{\pm}_{\mu;\,\mathbf{k}}(x)).$$

For the amplitudes $V^{\pm}_a(\mathbf{k},x_0)$, $\Lambda^{\pm}(\mathbf{k},x_0)$ we construct the (vacuum-averaged) canonical commutators, which are expressed in terms of the Heisenberg commutation function $G_{\mu\nu}$:

$$\langle[V_a{}^-(\mathbf{k},x_0),V_{a'}{}^+(\mathbf{k}',x_0)]\rangle_0=\left\{(1-A)^2+\int_{s_0}^{\infty}\frac{I(s)\,ds}{(s-m^2)^2}\right\}\delta_{aa'}\delta(\mathbf{k}-\mathbf{k}'),$$
$$\langle[V_a{}^-(\mathbf{k},x_0),\Lambda^+(\mathbf{k}',x_0)]\rangle_0=\langle[\Lambda^-(\mathbf{k},x_0),\Lambda^+(\mathbf{k}',x_0)]\rangle_0=0.$$

For the same amplitudes one can formulate the asymptotic condition:

$$\lim_{x_0\to\infty}\langle 0|V_a{}^-(\mathbf{k},x_0)|n\rangle=(1-A)\langle 0|b_a{}^-(\mathbf{k})|n\rangle\,\delta_{n1}$$

((the operators $b^{\pm}_a(\mathbf{k})$ referring to the physical quanta are introduced in[10])

$$\lim_{x_0\to\infty}\langle 0|\Lambda^-(\mathbf{k},x_0)|n\rangle=0.$$

Thus, in complete agreement with the fact that there are no longitudinal quanta in a complete set of asymptotic states, it turns out that the correctly selected longitudinal components of the Heisenberg field $U_\mu(x)$ vanish in the limit $x_0\to\infty$. On the other hand, the vanishing of the second commutator tells us about the dynamical independence (in a weak sense) of the longitudinal and transverse components for the field $U_\mu(x)$. Finally, the canonical commutator of the longitudinal components themselves also vanishes.

Thus, it turns out that not only in the case of simple extensions off the mass shell, but also in the transition to interpolating fields, the longitudinal component of the vector field is asymptotically bounded by a vanishing c-number. As regards the transverse component, it naturally acquires a operator addition when the interaction is switched on, which in the canonical commutator leads to the substitution

$$1 \to (1-A)^2 + \int_{s_0}^{\infty} \frac{I(s)\,ds}{(s-m^2)^2} = \begin{cases} Z_3^{-1} & \text{in the R-theory} \\ 1 & \text{in the U-theory,} \end{cases}$$

justifying the above-mentioned interpretation of the spectral integral $c_1 D$.

The last integral $c_1' D$ which is specific for the vector theory does not enter anywhere, except into the Schwinger term of the current commutator (35).

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