FIRST ORDER PHASE TRANSITION IN A MODEL OF A BOSE GAS WITH ATTRACTION

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The thermodynamic functions are calculated for a weakly nonideal Bose gas. In Sec. 1, general formulas are obtained for the thermodynamic potential, allowing us to investigate the thermodynamics of a Bose gas in the region of arbitrary temperatures, except the immediate neighborhood of a phase transition. The formulas include terms of second order in the expansion parameter. It is shown that the expansion parameter (gas parameter) changes during a transition from the low temperature region to the high temperature region. A model with attraction at large distances is investigated. The presence of a first-order phase transition in such a system is shown, and all thermodynamic functions of the transition are evaluated, and also the diagrams of phase stratification in the T, n plane are constructed. The existence of a first-order phase transition is shown for the model not only with an effective attraction, but also with an effective repulsion.

THE goal of the present article is a study of the possibility of a first-order phase transition at finite temperatures in a model of a rarefied Bose gas with long-range attraction and short-range repulsion between the particles. A consistent model of this kind was proposed by Iordanskiĭ[1] who showed that at absolute zero temperature stable homogeneous states in this case exist only for sufficiently large densities. One can therefore surmise that the system undergoes a first-order phase transition at finite temperatures. This problem is of well-known interest. We note, incidentally, that in real helium a first-order phase transition of the first kind occurs at sufficiently low pressures and temperatures.

Just as in[1] we shall assume that the potential energy $V(\mathbf{r}) = V_1(\mathbf{r}) + V_2(\mathbf{r})$ of the pair interaction of the Bose particles consists of a repulsive core $V_1(\mathbf{r}) > 0$ with a radius of action a and a long-range attraction $V_2(\mathbf{r}) < 0$ with a radius of action $R \gg a$. Let the relation between these quantities be such that in the essential region $V_1a^3 \sim V_2R^3$, so that the corresponding Fourier components $U_1(\mathbf{p})$ and $U_2(\mathbf{p})$ are of the same order of magnitude for $p < 1/R$. For simplicity we shall assume the interaction to be weak, i.e., the Born parameters $\xi_1 = U_1/a \ll 1$ and $\xi_2 = U_2/R \ll 1$ (the mass m = 1 everywhere). Such an approach saves us from the necessity to sum the Born series for the scattering amplitude. One can show that the results do not depend on this (see[2]).

The amplitude $f(\mathbf{p}, \mathbf{p}')$ for the scattering of two particles which interact by means of the potential V is related to U by the equation[1-3]

$$4\pi f(\mathbf{p}, \mathbf{p}') = U(\mathbf{p}' - \mathbf{p}) + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{U(\mathbf{p}' - \mathbf{q}) \cdot 4\pi f(\mathbf{p}, \mathbf{q})}{\mathbf{p}^2 - \mathbf{q}^2 + i\delta}$$

To the second approximation of perturbation theory we have

$$4\pi f(\mathbf{p}, \mathbf{p}') = U(\mathbf{p}' - \mathbf{p}) + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{U(\mathbf{p}' - \mathbf{q}) U(\mathbf{q} - \mathbf{p})}{\mathbf{p}^2 - \mathbf{q}^2 + i\delta}, \quad (1)$$

from which, since $R \gg a$, we obtain

$$4\pi f(\mathbf{p}, \mathbf{p}') = U(\mathbf{p}' - \mathbf{p}) - U_1(\mathbf{p}' - \mathbf{p}) + 4\pi f_1(\mathbf{p}, \mathbf{p}'),$$

where $f_1(\mathbf{p}, \mathbf{p}')$ is the amplitude for the scattering of two particles interacting by means of the potential V_1.

In the model[1] it is assumed that

$$|f(0.0) / f_1(0.0)| = |\beta| \ll 1. \quad (2)$$

The smallness of β indicates almost complete cancellation of repulsion and attraction in the scattering amplitude for small p and p′.

Since the characteristic momentum for U_1 is 1/a, then $f_1(\mathbf{p}, \mathbf{p}') \approx f_1(0, 0)$ for $p, p' \ll 1/a$. Therefore, for $p \ll 1/a$ we shall have

$$4\pi f(\mathbf{p}, \mathbf{p}) = [U(0) - U_1(0)] + 4\pi f_1(0.0) = 4\pi\beta f_0, \quad (3)$$

whereas for $1/R \ll |\mathbf{p} - \mathbf{p}'| \ll 1/a$ one will find

$$4\pi f(\mathbf{p}, \mathbf{p}') = [U(\mathbf{p}' - \mathbf{p}) - U_1(\mathbf{p}' - \mathbf{p})] + 4\pi f_1(0.0) \approx 4\pi f_0 \quad (4)$$

(one can neglect the quantity $U_2(\mathbf{p})$ for $p \gg 1/R$). The notation $f_0 = f_1(0, 0)$ is used here and in what follows.

Thus, in the present model the scattering amplitude varies rapidly during a change of $|\mathbf{p} - \mathbf{p}'|$ from zero to a value $\sim 1/R$, then remaining constant up to $p, p' \approx 1/a$.

We shall assume that the number density of particles n is small, i.e., the gas parameter $(nf_0^3)^{1/2} \ll 1$. The gas approximation is valid over the entire temperature range with the exception of only the immediate neighborhood of the Bose condensation point T_c.

The thermodynamics of a rarefied Bose gas were studied in articles[4-6] for a model with a repulsive potential. In the articles by Singh[4] and Popov[5] the thermodynamic potentials Ω and F were evaluated by means of direct integration of relations of the type $\partial\Omega/\partial\lambda = \lambda^{-1}\langle H_{\text{int}}\rangle$. Such a procedure enables the authors to calculate the thermodynamic functions only in the limiting cases of low and high temperatures. As we see below, in the model of a Bose gas under investigation a first-order phase transition takes place over a broad range of temperatures. An incorrect formula for the chemical potential is obtained in the article by Glassgold, Kaufman, and Watson.[6] In addition, their use of Bogolyubov's canonical transformation is not convenient for a determination of the higher-order approximations.

In the general case for the thermodynamic potential of a system of interacting Bose particles, Dominicis and Martin constructed a certain functional, expressed in terms of "occupation numbers," with the aid of which it was later shown that at low temperatures the heat capacity in such a system is determined by the phonon part of the excitation spectrum,[8] and the number density of supercondensed particles varies with temperature according to a quadratic law.[9] For specific calculations, in particular, in the model of a Bose gas under investigation, it is much more convenient to have an expression for Ω in terms of the exact Green's functions, analogous to the results which were obtained by Luttinger and Ward for the Fermi gas[10] and by Éliashberg for superconductivity.[11] Therefore, first (Sec. 1) we derive general formulas for the basic thermodynamic functions, assuming for simplicity that there is only a repulsive interaction in the Bose gas. The results obtained will then be used (Sec. 2) for an investigation of the properties of the model of interest to us of a Bose gas with attraction. The existence of a first-order phase transition in such a model will be shown under rather general assumptions about the relation between attraction and repulsion (Sec. 2).

1. THERMODYNAMIC POTENTIAL OF AN IMPERFECT BOSE GAS

According to Belyaev,[2,3] a system of Bose particles is described by several Green's functions whose Fourier transforms satisfy the following formulas:[1)]

$$G(p)=\frac{i\omega_k+\varepsilon_0(\mathrm{p})+S(p)-A(p)-\mu}{[i\omega_k-A(p)]^2-[\varepsilon_0(\mathrm{p})+S(p)-\mu]^2+\Sigma_{02}^2(p)},$$

$$\hat{G}(p)=\check{G}(p)=-\frac{\Sigma_{02}(p)}{[i\omega_k-A(p)]^2-[\varepsilon_0(\mathrm{p})+S(p)-\mu]^2+\Sigma_{02}^2(p)} \tag{1.1}$$

$$S(p)={}^1/_2(\Sigma_{11}(p)+\Sigma_{11}(-p)),\quad A(p)={}^1/_2(\Sigma_{11}(p)-\Sigma_{11}(-p)),$$
$$\varepsilon_0(\mathrm{p})=\mathrm{p}^2/2m.$$

Above we assumed that $(nf_0^3)^{1/2}\ll 1$ (for the model of a Bose gas considered in this section, f_0 denotes the value of the scattering amplitude f(p, p′) for p, p′ ≪ 1/a, where 1/a is the characteristic dimension of the repulsive interaction). It is not difficult to verify that here $Tf_0^2\ll 1$ over the entire temperature range of interest. We take the diagrams shown in Figs. 1a through 1c as the terms of the first approximation for Σ_{11}, Σ_{02}, and μ. In this approximation Σ_{11}, Σ_{02}, and μ have the form

$$\Sigma_{11}(p)=nU(0)+n_0U(\mathrm{p})-T\sum_k\int\frac{d\mathrm{q}}{(2\pi)^3}U(\mathrm{p}+\mathrm{q})G(q)e^{i\omega_k\delta},\quad \delta\to+0, \tag{1.2}$$

$$\Sigma_{02}(p)=n_0U(\mathrm{p}), \tag{1.3}$$

$$\mu=nU(0)-T\sum_k\int\frac{d\mathrm{q}}{(2\pi)^3}U(\mathrm{q})G(q)e^{i\omega_k\delta},\quad \delta\to+0, \tag{1.4}$$

where

$$n=n_0-T\sum_k\int\frac{d\mathrm{q}}{(2\pi)^3}G(q)e^{i\omega_k\delta},\quad \delta\to+0. \tag{1.5}$$

Substituting (1.2)–(1.4) into (1.1), we have

$$G(p)=-\frac{i\omega_k+\varepsilon_0(\mathrm{p})+n_0U(\mathrm{p})}{\omega_k^2+\varepsilon^2(\mathrm{p})},\quad \hat{G}(p)=\check{G}(p)=\frac{n_0U(\mathrm{p})}{\omega_k^2+\varepsilon^2(\mathrm{p})}, \tag{1.6}$$

where

$$\varepsilon(\mathrm{p})=[(\varepsilon_0(\mathrm{p})+\Sigma_{11}(p)-\mu)^2-\Sigma_{02}^2(p)]^{1/2}=[\varepsilon_0^2(\mathrm{p})+2n_0U(\mathrm{p})\varepsilon_0(\mathrm{p})]^{1/2}. \tag{1.7}$$

The second-order graphs for μ are shown in Fig. 1d. Taking these diagrams into account, we obtain the following expression for μ:

$$\mu=nU(0)-T\sum\int\frac{d\mathrm{q}}{(2\pi)^3}U(\mathrm{q})(\check{G}(q)+G(q))e^{i\omega_k\delta},\quad \delta\to+0. \tag{1.8}$$

The question of the contribution from diagrams of the next order for Σ_{11} and Σ_{02} arises in connection with the substitution of the expressions for G and $\hat{G}$ into Eq. (1.8). A typical diagram for Σ_{02} is shown in Fig. 1e. For it we write down the expression

$$\Sigma_{02}(p)=-n_0TU^2(\mathrm{p})\sum_k\int\frac{d\mathrm{q}}{(2\pi)^3}\hat{G}(q)G(p-q). \tag{1.9}$$

For T = 0 an estimate of similar diagrams was made in[2]. There it was shown that the expansion takes place in powers of the parameter $(n_0f_0^3)^{1/2}\ll 1$ (n_0 denotes the number density of the particles with p = 0). For temperatures $T\sim T_c$ the terms in the sums over k in which $\omega_k = 0$ are essential for estimates of the diagrams. It is easy to verify that in this case the expansion parameter is $Tf_0^2(n_0f_0^3)^{-1/2}\ll 1$. For $T\sim n^{2/3}$ one can write this condition either as $(n_0f_0^3)^{1/6}\ll 1$ or as $(Tf_0^2)^{1/4}\ll 1$. In what follows, both of these parameters will always be regarded as small. This property allows us to neglect the contribution of the higher-order approximations. At the same time it should be noted that expressions of the type (1.9) for $\Sigma_{11}(p)$ and $\Sigma_{02}(p)$ diverge at small momenta. For $p\ll(n_0U_0)^{1/2}$ Eq. (1.9) gives a contribution to $\Sigma_{02}(p)$ of order TU_0/p with respect to (1.3). However, as we see below the contribution to the thermodynamic functions coming from the region $p\ll(n_0U_0)^{1/2}$ is not important. Therefore expressions (1.6) and (1.7) for G and $\hat{G}$ are valid for $p\gg TU_0$. With their aid we can evaluate the second-order correction for the chemical potential (1.8).

It is convenient to use the following expression in order to calculate the thermodynamic potential Ω(T, 𝒱, μ) (𝒱 denotes the volume occupied by the Bose gas):

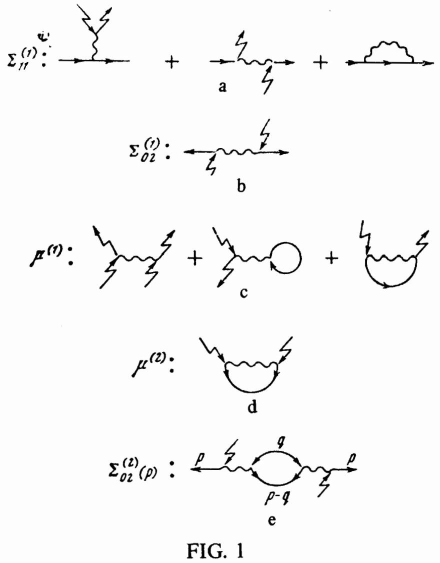

FIG. 1

[1)] The Green's functions and the mass operators depend on the 4-momentum p; in the remaining cases p = |p|. All notation agrees with the notation used in [3]

$$\Omega = \Omega_0 + T\mathcal{V}\sum_k \int \frac{d\mathbf{p}}{(2\pi)^3} e^{i\omega_k\delta}\Big|_{\delta\to+0}$$
$$\times\left\{\frac{1}{2}\ln\frac{\omega_k^2+(\varepsilon_0(\mathbf{p})+\Sigma_{11}(p)-\mu)^2-\Sigma_{02}^2(p)}{\omega_k^2+(\varepsilon_0(\mathbf{p})-\mu)^2}+\Sigma_{11}(p)\operatorname{Re}G(p)\right.$$
$$+\Sigma_{02}(p)G(p)-n_0[U(0)+U(\mathbf{p})]G(p)-n_0U(\mathbf{p})\hat{G}(p)$$
$$\left.-\frac{1}{2}T\sum_{k'}\int\frac{d\mathbf{p}'}{(2\pi)^3}[U(0)+U(\mathbf{p}-\mathbf{p}')]G(p)G(p')\right\}$$
$$+\frac{1}{2}\mathcal{V}U(0)n_0^2-\mathcal{V}n_0\mu, \tag{1.10}$$

where Ω_0 denotes the thermodynamic potential in the absence of any interaction:

$$\Omega_0 = T\mathcal{V}\int\frac{d\mathbf{p}}{(2\pi)^3}\ln\left(1-\exp\left[\frac{\mu-\varepsilon_0(\mathbf{p})}{T}\right)\right]. \tag{1.11}$$

Expression (1.10) may be obtained in analogy to the way this was done in[9,10]. The first variations of Ω with respect to Σ_{11}, Σ_{02}, and n_0 vanish provided that Σ_{11}, Σ_{02}, G, $\hat{G}$, and μ are determined by formulas (1.2), (1.3), and (1.6)–(1.8). Substituting Σ_{11}, Σ_{02} from (1.2), (1.3) and Ω_0 from (1.11) into expression (1.10) and carrying out the summation, we obtain the thermodynamic potential Ω to second order in the gas parameter:

$$\Omega = T\mathcal{V}\int\frac{d\mathbf{p}}{(2\pi)^3}\ln(1-e^{-\varepsilon(\mathbf{p})/T})-\frac{1}{2}\mathcal{V}\int\frac{d\mathbf{p}\,d\mathbf{p}'}{(2\pi)^6}U(\mathbf{p}-\mathbf{p}')n(\mathbf{p})n(\mathbf{p}')$$
$$+\frac{\mathcal{V}}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}[\varepsilon(\mathbf{p})-\varepsilon_0(\mathbf{p})-n_0U(\mathbf{p})]$$
$$+\frac{\mathcal{V}}{2}[U(0)n_0^2-U(0)(n-n_0)^2-2n_0\mu], \tag{1.12}$$
$$n(\mathbf{p}) = \frac{\varepsilon_0(\mathbf{p})+n_0U(\mathbf{p})}{\varepsilon(\mathbf{p})}\frac{1}{e^{\varepsilon(\mathbf{p})/T}-1}+\frac{\varepsilon_0(\mathbf{p})+n_0U(\mathbf{p})-\varepsilon(\mathbf{p})}{2\varepsilon(\mathbf{p})}$$

where n(**p**) is the number density of particles with momentum **p** ≠ 0.

In order to evaluate the entropy $S = -(\partial\Omega/\partial T)_{\mathcal{V},\mu}$, in Eq. (1.10) one should differentiate only with respect to the explicit dependence on T:

$$S = \frac{\mathcal{V}}{T}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{e^{\varepsilon(\mathbf{p})/T}-1}\left(\varepsilon(\mathbf{p})+\frac{\mathbf{p}}{3}\frac{\partial\varepsilon(\mathbf{p})}{\partial\mathbf{p}}\right). \tag{1.13}$$

Then in Eqs. (1.8), (1.5), (1.12), and (1.13) we express the Fourier components of the potential in terms of the amplitude for the scattering of two particles. In the first approximation $U(\mathbf{p}'-\mathbf{p}) = 4\pi f(\mathbf{p},\mathbf{p}') = 4\pi f_0$ (see Eq. (1)). But some of the integrals in Eqs. (1.8) and (1.12) diverge for large momenta. It is easy to eliminate the divergence if the second term in Eq. (1) is taken into consideration.[12] After this we obtain

$$\mu = 4\pi nf(0,0)+4\pi\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{f(\mathbf{p},0)\varepsilon_0(\mathbf{p})}{\varepsilon(\mathbf{p})}\frac{1}{e^{\varepsilon(\mathbf{p})/T}-1}$$
$$+2\pi\int\frac{d\mathbf{p}}{(2\pi)^3}f(\mathbf{p},0)\frac{\varepsilon_0^2(\mathbf{p})-\varepsilon_0(\mathbf{p})\varepsilon(\mathbf{p})+4\pi n_0f(\mathbf{p},0)\varepsilon(\mathbf{p})}{\varepsilon_0(\mathbf{p})\varepsilon(\mathbf{p})}, \tag{1.14}$$
$$n = n_0+\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\varepsilon_0(\mathbf{p})+4\pi n_0f(\mathbf{p},0)}{\varepsilon(\mathbf{p})}\frac{1}{e^{\varepsilon(\mathbf{p})/T}-1}$$
$$+\frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\varepsilon_0(\mathbf{p})+4\pi n_0f(\mathbf{p},0)-\varepsilon(\mathbf{p})}{\varepsilon(\mathbf{p})}, \tag{1.15}$$
$$\Omega = T\mathcal{V}\int\frac{d\mathbf{p}}{(2\pi)^3}\ln(1-e^{-\varepsilon(\mathbf{p})/T})-2\pi\mathcal{V}\int\frac{d\mathbf{p}\,d\mathbf{p}'}{(2\pi)^6}U(\mathbf{p}-\mathbf{p}')n(\mathbf{p})n(\mathbf{p}')$$
$$+\frac{1}{2}\mathcal{V}\int\frac{d\mathbf{p}}{(2\pi)^3}\left(\varepsilon(\mathbf{p})-\varepsilon_0(\mathbf{p})-4\pi n_0f(\mathbf{p},0)+\frac{(4\pi f(\mathbf{p},0))^2}{2\varepsilon_0(\mathbf{p})}\right)$$
$$+2\pi\mathcal{V}f(0,0)n_0^2-2\pi\mathcal{V}f(0,0)(n-n_0)^2-\mathcal{V}n_0\mu, \tag{1.16}$$

where

$$\varepsilon(\mathbf{p}) = [\varepsilon_0^2(\mathbf{p})+8\pi n_0f(\mathbf{p},0)\varepsilon_0(\mathbf{p})]^{1/2}. \tag{1.17}$$

The integrals in (1.13) and (1.16), containing an exponential in the denominator, are collected together for $\epsilon(\mathbf{p}) \sim T$. Values of $p \sim (n_0f_0)^{1/2}$ are essential for all remaining integrals in Eqs. (1.14)–(1.16) (since $(n_0U_0)^{1/2} \gg TU_0$ and $f_0 \sim U_0$, we have $p \gg TU_0$). Considering f(**p**, 0) to be constant everywhere, $f(\mathbf{p}, 0) \approx f_0$, and performing the possible integrations in Eqs. (1.13)–(1.16), we finally obtain

$$\mu = 4\pi f_0\left[n+\frac{32}{3\sqrt{\pi}}n_0(n_0f_0^3)^{1/2}+T^{3/2}I_2(\alpha)\right], \tag{1.18}$$
$$n = n_0+\frac{8}{3\sqrt{\pi}}n_0(n_0f_0^3)^{1/2}+T^{3/2}I_3(\alpha), \tag{1.19}$$
$$P = -\frac{\Omega}{\mathcal{V}} = 2\pi f_0\left[n^2+\frac{64}{5\sqrt{\pi}}n_0^2(n_0f_0^3)^{1/2}+(n-n_0)^2\right]+T^{5/2}I_1(\alpha), \tag{1.20}$$
$$S = \mathcal{V}T^{3/2}I_4(\alpha). \tag{1.21}$$

Here $\alpha = 4\pi n_0f_0/T$,

$$I_1(\alpha) = \frac{\sqrt{2}}{3\pi^2}\int_0^\infty\frac{[{}^3/_2\alpha+(z^2+\alpha^2)^{1/2}]z^3\,dz}{(z^2+\alpha^2)^{1/2}[\alpha+(z^2+\alpha^2)^{1/2}]^{3/2}(e^z-1)}$$
$$I_2(\alpha) = \frac{1}{\sqrt{2}\pi^2}\int_0^\infty\frac{z^3\,dz}{(z^2+\alpha^2)^{1/2}[\alpha+(z^2+\alpha^2)^{1/2}]^{3/2}(e^z-1)}, \tag{1.22}$$
$$I_3(\alpha) = \frac{1}{\sqrt{2}\pi^2}\int_0^\infty\frac{z\,dz}{[\alpha+(z^2+\alpha^2)^{1/2}]^{1/2}(e^z-1)},$$
$$I_4(\alpha) = \frac{1}{\sqrt{2}\pi^2}\int_0^\infty\frac{[\alpha+{}^5/_3(z^2+\alpha^2)^{1/2}]z^3\,dz}{(z^2+\alpha^2)^{1/2}[\alpha+(z^2+\alpha^2)^{1/2}]^{3/2}(e^z-1)}.$$

At low temperatures, $Tf_0^2 \ll n_0f_0^3$ ($\alpha \gg 1$), according to Eqs. (1.18) and (1.22) we have

$$\mu = 4\pi f_0\left(n+\frac{32}{3\sqrt{\pi}}n(nf_0^3)^{1/2}+\frac{\pi^2B_2T^4}{2c^5}\right),$$
$$n_0 = n-\frac{8}{3\sqrt{\pi}}n(nf_0^3)^{1/2}-\frac{B_1T^2}{2c},$$
$$P = 2\pi f_0\left(n^2+\frac{64}{5\sqrt{\pi}}n^2(nf_0^3)^{1/2}\right)+\frac{5}{6}\pi^2B_2\frac{T^4}{c^3}, \tag{1.23}$$
$$S = \frac{4}{3}\frac{\pi^2B_2\mathcal{V}T^3}{c^3}\qquad c = (4\pi nf_0)^{1/2}$$

(B_1 and B_2 are Bernoulli numbers). Thus, at low temperatures, $Tf_0^2 \ll n_0f_0^3$, only the phonon spectrum[4] gives a contribution to the entropy, and the quantity $n - n_0$ varies with temperature according to a quadratic law.[9]

At high temperatures, $Tf_0^2 \gg n_0f_0^3$, according to Eqs. (1.18) and (1.22) one will have

$$n_0 = n-n_{\text{cr}}+T\left[\frac{f_0}{\pi}(n-n_{\text{cr}})\right]^{1/2},\qquad n_{\text{cr}} = I_2(0)T^{3/2},$$
$$\mu = 4\pi f_0\left\{n+n_{\text{cr}}-2T\left[\frac{f_0}{\pi}(n-n_{\text{cr}})\right]^{1/2}\right\}, \tag{1.24}$$
$$P = I_1(0)T^{5/2}+2\pi f_0(n^2+n_{\text{cr}}^2)-{}^8/_3\sqrt{\pi}T[f_0(n-n_{\text{cr}})]^{3/2}$$
$$-4\sqrt{\pi}Tn_{\text{cr}}f_0[f_0(n-n_{\text{cr}})]^{1/2},$$
$$S = {}^5/_2I_1(0)\mathcal{V}T^{3/2}-6\pi I_2(0)\mathcal{V}T^{1/2}f_0(n-n_{\text{cr}}).$$

(here $I_1(0) = \zeta(5/2)/(2\pi)^{3/2}$ and $I_2(0) = I_3(0) = \zeta(3/2)/(2\pi)^{3/2}$; $\zeta(5/2)$ and $\zeta(3/2)$ denote zeta functions). It is not necessary to take into account terms of higher order in the expansion of the integrals $I_1(\alpha)$

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[2] The last term in the equation for the pressure (1.24) is not present in Popov's work. Everywhere in Singh's work, only the first-order terms in the gas parameter are evaluated.

through $I_4(\alpha)$ in powers of α, since for $T \sim T_c = n^{2/3}$ they become of the order of the contribution which comes from the discarded diagrams of perturbation theory. Finally, the quantity $n_0 \sim 1 - (T/T_c)^{3/2}$ becomes small quite close to the transition line $T_c = [n/I_2(0)]^{2/3}$ for an ideal Bose gas. As one can easily verify, the condition $T_c f_0^2/(n_0 f_0^3)^{1/2} \ll 1$ will be the criterion for the validity of all formulas with regard to nearness to the transition line.

Formulas (1.23) and (1.24) derived above basically agree with the results of Singh[4] and Popov.[5] 2)

2. THERMODYNAMICS OF A MODEL OF AN IMPERFECT BOSE GAS WITH ATTRACTION

In this Section we investigate the phase transition between "liquid" (characterized by the presence of a condensate) and "gaseous" phases. In equilibrium the usual conditions must be satisfied at the boundary between the two phases:

$$T = T', \ \mu = \mu', \ P = P'. \tag{2.1}$$

(We shall everywhere denote the thermodynamic functions pertaining to the gaseous phase by primes.)

Below we shall seek the region of phase stratification in the (T, n) plane. We shall use the results of the previous section for the quantities μ, P, and P′ which enter into Eq. (2.1).

In the liquid phase the diagrams for the mass operators Σ_{11}, Σ_{02} (Figs. 1a and 1b) and for the chemical potential μ (Figs. 1c and 1d), and also the analytic expressions (1.2), (1.3), and (1.8) corresponding to these diagrams remain valid. It was indicated above that $\Sigma_{11}(p)$ and $\Sigma_{02}(p)$ given by Eq. (1.9) vary significantly only for small momenta $p \lesssim TU_0$. Therefore, assuming $TU_0 \ll 1/R$ and taking the smallness of the quantity U(0) given by (2) into consideration, we obtain the result that to the lowest approximation, in the range of momenta $1/\alpha \gg p \gtrsim 1/R$, $\Sigma_{11}(p)$, $\Sigma_{02}(p)$, and μ are given by

$$\Sigma_{11}(p) = n_0 U(\mathbf{p}) - T \sum_k \int \frac{d\mathbf{q}}{(2\pi)^3} U(\mathbf{p}+\mathbf{q}) G(q) e^{i\omega_k \delta}, \quad \delta \to +0,$$

$$\Sigma_{02}(p) = n_0 U(\mathbf{p}),$$

$$\mu = -T \sum_k \int \frac{d\mathbf{q}}{(2\pi)^3} U(\mathbf{q}) G(q) e^{i\omega_k \delta}, \quad \delta \to +0.$$

Here, as usual, the Green's functions of the supercondensed particles for $1/a \gg p \gtrsim 1/R$ are described by formulas (1.6) and (1.7) obtained above.

It is impossible to perform the integration in formulas (1.13)–(1.16) for the thermodynamic functions without specifying the specific form of the scattering amplitude f(p, 0). However, for sufficiently large values of R one can significantly simplify expressions (1.13)–(1.16). In fact, if one assumes $(n_0 f_0)^{1/2} \gg 1/R$, i.e., $a/R \ll \beta$, then for $T \gg (n_0 f_0)^{1/2}/R$ everywhere in the integrals one can neglect the region of momenta $q \lesssim 1/R$ in which f(p, 0) varies rapidly, i.e, in all of the integrals (1.13)–(1.16) one can set $f(p, 0) = f_0$. Finally we obtain

$$P = 2\pi f_0 \left[\beta n^2 + \frac{64}{5\sqrt{\pi}} n_0^2 (n_0 f_0^3)^{1/2} + (n - n_0)^2\right] + T^{5/2} I_1(\alpha). \tag{2.2}$$

$$\mu = 4\pi f_0 \left[\beta n + \frac{32}{3\sqrt{\pi}} n_0 (n_0 f_0^3)^{1/2} + T^{3/2} I_2(\alpha)\right]. \tag{2.3}$$

$$n = n_0 + \frac{8}{3\sqrt{\pi}} n_0 (n_0 f_0^3)^{1/2} + T^{3/2} I_3(\alpha), \tag{2.4}$$

$$s = T^{3/2} I_4(\alpha) / n \tag{2.5}$$

where s is the entropy per particle, and $I_1(\alpha)$, $I_2(\alpha)$, $I_3(\alpha)$, and $I_4(\alpha)$ are determined by expressions (1.22).

In the "gaseous" phase $n_0 = 0$, $\Sigma_{02} = 0$, $\Sigma_{11} = \Sigma$, $\hat{G} = \check{G} = 0$, and according to Eq. (1.1) G has the form

$$G(p) = [i\omega_k - \varepsilon'(\mathbf{p})]^{-1}, \tag{2.6}$$

where

$$\varepsilon'(\mathbf{p}) = \varepsilon_0(\mathbf{p}) + \Sigma(p) - \mu'. \tag{2.7}$$

The perturbation theory formula for Σ is valid everywhere except in the immediate neighborhood of a phase transition, i.e., for $Tf_0 \ll |\mu'_{\text{eff}}|^{1/2}$ (μ'_{eff} denotes the "effective" chemical potential, $\mu'_{\text{eff}} = \mu' - \Sigma$). In this temperature range and for $1/a \gg p \gtrsim 1/R$ the mass operator $\Sigma(p)$ is given by

$$\Sigma(p) = -T \sum_k \int \frac{d\mathbf{q}}{(2\pi)^3} U(\mathbf{p}+\mathbf{q}) G(q) e^{i\omega_k \delta} \approx 4\pi n' f_0, \quad \delta \to +0. \tag{2.8}$$

The entropy, the number density of particles, and the pressure in the gaseous phase are calculated according to formulas (1.13), (1.15), and (1.16), where it is necessary to substitute $\epsilon(p) = \epsilon'(p)$ from (2.7), and n_0 is set equal to zero.

As already mentioned above, the condition $\mu = \mu'$ must be fulfilled on the equilibrium curve for the two phases. The quantity $\mu'/T \ll 1$ in the temperature range of interest to us, $T \gg (n_0 f_0)^{1/2}/R$. Therefore the quantity $\mu'/T \ll 1$ near the equilibrium curve, and therefore one also has $\mu'_{\text{eff}}/T \ll 1$. In this connection $Tf_0/|\mu'_{\text{eff}}|^{1/2} \ll 1$, and therefore the boundaries of phase stratification in the (T, n) plane are located far from the curve $T_c = [n/I_2(0)]^{2/3}$ which describes the transition in an ideal Bose gas. For $\mu'_{\text{eff}}/T \ll 1$ an evaluation of the thermodynamic functions in the gaseous phase is not difficult. Near phase equilibrium we have

$$P' = I_1(0) T^{5/2} + n_{\text{cr}} \mu'_{\text{eff}} + 2\pi f_0 n'^2, \tag{2.9}$$

$$s' = 5 I_1(0) / 2 I_2(0), \tag{2.10}$$

$$n' = n_{\text{cr}} - T |\mu'_{\text{eff}}|^{1/2} / \sqrt{2}\pi. \tag{2.11}$$

In order to understand the situation and show the order of magnitude of all quantities, let us consider the boundary of the stable states in the liquid phase, which is determined by the condition $(\partial P/\partial n)_T = 0$. In the region of temperatures and densities where $\alpha = 4\pi n_0 f_0/T \sim 1$ and $|n - n_0| \ll n$ one can neglect the term $(n - n_0)^2$ in Eq. (2.2). According to Eq. (2.2) we have

$$\frac{\partial I_1}{\partial \alpha} + \frac{2}{\pi^2} \alpha^{3/2} + \frac{\beta \alpha}{4\pi \tau^{1/2}} = 0, \tag{2.12}$$

where τ is the temperature in dimensionless variables, $\tau = Tf_0^2$, and according to Eq. (1.22) the derivative $\partial I_1/\partial \alpha$ is given by

$$\frac{\partial I_1}{\partial \alpha} = -\frac{\alpha}{\sqrt{2}\pi^2} \int_0^\infty \frac{[\alpha + {}^3/_2 (z^2 + \alpha^2)^{1/2}] z^3\, dz}{(z^2 + \alpha^2)^{1/4} [\alpha + (z^2 + \alpha^2)^{1/2}]^{3/2} (e^z - 1)}$$

The sum of the first two terms in Eq. (2.12), $(\partial I_1/\partial \alpha) + 2(\alpha^{3/2}/\pi^2)$, changes sign at $\alpha = 0.288$. Therefore, at finite temperatures a transition between the liquid and gaseous phases will exist both for $\beta < 0$ as well as for $\beta > 0$. For comparatively high temperatures, $\tau \gg \beta^2$, one finds

$$\nu = 0.0229\, \tau \tag{2.13}$$

as is evident from Eq. (2.12) for the cases $\beta < 0$ and $\beta > 0$ (ν is the number density of particles in dimensionless units, $\nu = nf_0^3$). For low temperatures, $\tau \ll \beta^2$, and $\beta < 0$ will give

$$\nu = \pi(\beta/16)^2.$$

For $\beta > 0$ the boundary of stability $\partial P/\partial n = 0$ for lower temperatures, $\tau \lesssim \beta^2$, falls into the region $n - n_0 \sim n_0$ and $\alpha \ll 1$. In this case for $\tau \ll \beta^2$ we obtain

$$\nu = \nu_{cr} + \pi^{-1}(\tau/\beta)^2.$$

After substituting P and μ from Eqs. (2.2) and (2.3) and P′ from (2.9) into them, the equilibrium conditions (2.1) for the two phases take the following form:

$$2\pi f_0[\beta n^2 + {}^{64}/{}_5\pi^{-1/2}n_0^2(n_0f_0^3)^{1/2} + (n-n_0)^2] + I_1(\alpha)T^{5/2} = I_1(0)T^{5/2} + n_{cr}\mu'_{\text{eff}} + 2\pi f_0 n'^2, \tag{2.14}$$

$$4\pi f_0\{\beta n + {}^{32}/{}_3\pi^{-1/2}n_0(n_0f_0^3)^{1/2} + I_2(\alpha)T^{3/2}] = \mu'_{\text{eff}} + 4\pi f_0 n'^2, \tag{2.15}$$

where n and μ'_{eff} are determined, respectively, from Eqs. (2.4) and (2.11). By solving the system of Eqs. (2.14) and (2.15) with the aid of Eqs. (2.4) and (2.11), we obtain the boundary of phase stratification in the T, n diagram. In this connection, as is clear from the preceding discussion, it is necessary to consider the cases $\beta < 0$ and $\beta > 0$ separately.

The case $\beta < 0$. In this case, as will be seen below, in the region $T \gg (n_0f_0)^{1/2}/R$ one always has $\alpha \gtrsim 1$ and $n_0 \approx n$. Therefore, neglecting the small terms of order β and the terms containing the superfluous power $\tau^{1/2}$, one can reduce Eqs. (2.14) and (2.15) to the form

$$I_1(\alpha) - I_1(0) + \frac{4}{5\pi^2}\alpha^{5/2} + \frac{1}{8\pi}\frac{\beta\alpha^2}{\tau^{1/2}} = 0, \tag{2.16}$$

$$\nu' = \nu_{cr}\left[1 - \frac{\tau^{1/4}}{\sqrt{\pi/2}I_2(0)}\left|I_2(\alpha) - I_2(0) + \frac{4}{3\pi^2}\alpha^{3/2} + \frac{\alpha\beta}{4\pi\tau^{1/2}}\right|^{1/2}\right]. \tag{2.17}$$

At comparatively high temperatures, $\tau \gg (5\pi\beta/32)^2$, the value of the parameter α is of the order of unity, as is clear from Eq. (2.16). In this connection the term containing $\beta/\tau^{1/2}$ becomes unimportant in both equations. Neglecting these terms in Eqs. (2.16) and (2.17), we obtain the result that the parameter α is determined by the equation

$$I_1(\alpha) - I_1(0) + \frac{4}{5\pi^2}\alpha^{5/2} = 0, \tag{2.18}$$

which gives numerically $\alpha = 0.501$. Thus, asymptotically

$$\nu = \frac{0.501}{4\pi}\tau = 0.0398\tau. \tag{2.19}$$

Near the asymptote we have

$$\nu = 0.0398\tau(1 - 0.472\beta/\tau^{1/2}). \tag{2.20}$$

In connection with the evaluation of ν' given by Eq. (2.17) in the temperature range $\tau \gg (5\pi\beta/32)^2$, the correction coming from the term $\beta/\tau^{1/2}$ is unimportant, and therefore

$$\nu = \nu_{cr}(1 - 1.34\tau^{1/4}). \tag{2.21}$$

Then substituting the value $\alpha = 0.501$ into Eq. (2.5), we find that the entropy in the liquid phase is small:

$$s = 3.7\tau^{1/2}. \tag{2.22}$$

In the limiting case of low temperatures, $\tau \ll (5\pi\beta/32)^2$, the parameter α becomes large, $1/\beta \gg \alpha \gg 1$. Neglecting for $\alpha \gg 1$ the terms $I_1(\alpha)$ and $I_2(\alpha)$ in Eqs. (2.16) and (2.17), we find

$$\frac{4}{5\pi^2}\alpha^{5/2} + \frac{1}{8\pi}\frac{\beta\alpha^2}{\tau^{1/2}} - I_1(0) = 0, \tag{2.23}$$

$$\nu' = \nu_{cr}\left[1 - \frac{\tau^{1/4}}{\sqrt{\pi/2}I_2(0)}\left|\frac{4}{3\pi^2}\alpha^{3/2} + \frac{\alpha\beta}{4\pi\tau^{1/2}}\right|^{1/2}\right]. \tag{2.24}$$

The last term in Eq. (2.23) is small; therefore

$$\alpha = \left(\frac{5\pi\beta}{32\tau^{1/2}}\right)^2\left[1 + \frac{5}{2}\pi^2 I_1(0)\left(\frac{32\tau^{1/2}}{5\pi|\beta|}\right)^5\right]. \tag{2.25}$$

Substituting the value (2.25) into expressions (2.4), (2.24), and (2.5), we find that

$$\nu = \frac{1}{4\pi}\left(\frac{5\pi\beta}{32}\right)^2\left[1 + \frac{5}{2}\pi^2 I_1(0)\left(\frac{32\tau^{1/2}}{5\pi|\beta|}\right)^5\right]$$

$$\nu' = \nu_{cr}\left[1 - \frac{5}{64\sqrt{3}I_2(0)}\left(\frac{|\beta|^3}{\tau}\right)^{1/2}\right], \tag{2.26}$$

$$s = \frac{4B_2}{3\sqrt{\pi}}\left(\frac{64}{5|\beta|}\right)^5\tau^3.$$

At T = 0 Eq. (2.26) gives a value for the density which in article[1] corresponds to the pressure being equal to zero.

From the formulas obtained above it is clear that over the entire range $\tau \gg (n_0f_0)^{1/2}/R$ under consideration, the entropy s in the liquid phase is small in comparison with the entropy s′ in the gaseous phase. Therefore, for $\tau \gg (n_0f_0)^{1/2}/R$ the latent heat of transition q per particle is given by

$$q = \frac{5}{2}\frac{I_1(0)}{I_2(0)}T. \tag{2.27}$$

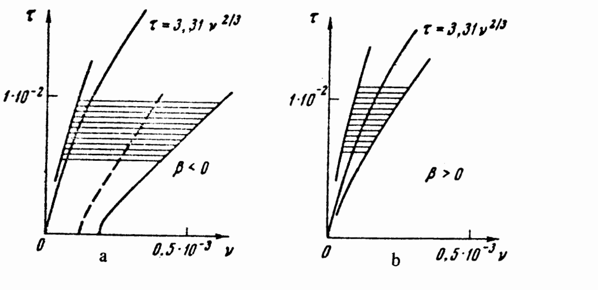
For the same reason the equilibrium curve of the phases on the P, T-diagram differs very little from the corresponding curve $P = I_1(0)T^{5/2}$ for ideal Bose condensation.

The case $\beta > 0$. For this case $\alpha \lesssim 1$ everywhere. At not too low temperatures ($\tau \gtrsim (5\pi\beta/32)^2$) Eqs. (2.14) and (2.15) have the form (2.16) and (2.17). Therefore expressions (2.19)–(2.22) and (2.27), which we derived in the region of high temperatures ($\tau \gg (5\pi\beta/32)^2$) for the case $\beta < 0$, are also valid for the case $\beta > 0$.

In the limiting case of low temperatures ($\tau \ll (5\pi\beta/32)^2$) and for a positive value of β, values of the parameter $\alpha \ll 1$ are essential in the equations. One can represent expressions (1.22) in the following form:

$$I_1(\alpha) = I_1(0) - \alpha^{3/2}/3\pi, \quad I_2(\alpha) = I_2(0) - \alpha^{1/2}/\pi,$$
$$I_3(\alpha) = I_2(0) - \alpha^{1/2}/2\pi, \quad I_4(\alpha) = I_1(0) - {}^3/{}_2 I_2(0)\alpha. \tag{2.28}$$

Taking (2.28) into account and substituting n_0 from (2.4) and μ'_{eff} from (2.11) into Eqs. (2.14) and (2.15), we find



ν and ν'. They are given by

$$\nu = {}^5/_2\nu_{\rm cr}, \quad \nu' = \nu_{\rm cr}\left[1 - 3\frac{\tau^{3/8}}{(3\pi I_2(0)/2)^{3/4}}\right]. \qquad (2.29)$$

According to Eqs. (2.5) and (2.28), for $\alpha \ll 1$ the entropy in the liquid phase is determined by the expression

$$s = I_1(0)\,/\,I_2(0). \qquad (2.30)$$

From Eqs. (2.29), (2.30), and (2.10) we obtain the following result for the latent heat of transition q:

$$q = \frac{3}{2}\frac{I_1(0)}{I_2(0)}T.$$

T, n-diagrams are shown in Figs. 2a and 2b, where the region of phase stratification is cross-hatched for the cases $\beta < 0$ and $\beta > 0$. The boundary of metastability on the side of the liquid phase is given by the dashed line for $\beta < 0$. The boundary of metastability of the gaseous phase is located in the immediate vicinity of the Bose condensation line, i.e., in a region where the formulas of perturbation theory are not applicable. All curves are obtained as the result of a numerical solution of Eqs. (2.12), (2.16), (2.17), (2.26), and (2.29) for $|\beta| = 0.1$.

Here it is appropriate to emphasize that all formulas are valid only for very small values of ν, τ, and β. This circumstance is associated with the fact that in order for our approach to be valid at high temperatures it is necessary that $\tau^{1/4} \ll 1$ and $\nu^{1/6} \ll 1$. The interplay of numerical coefficients also restricts the domain of validity of almost all asymptotic formulas. In particular, the transition line $\tau = 3.31\,\nu^{2/3}$ for Bose condensation and the asymptote $\tau = 25\nu$ for the stratification region now intersect at $\nu = 2.4 \times 10^{-3}$.

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