NONEQUILIBRIUM PROCESSES IN THE CRITICAL REGION

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The singular parts of the kinetic coefficients for He^4 near the λ-point are calculated. A new critical index defining the time correlation range enters the formulae. The analysis is performed on the basis of microscopic theory.

1. INTRODUCTION

A second order phase transition in any many-particle system is foreshadowed by the appearance of long-range space and time correlations. Long-range space correlation, as is well-known[1], leads to temperature singularities in the thermodynamic quantities. Long-range time correlation must influence substantially the nonequilibrium characteristics of a substance, in particular the kinetic coefficients.

Ferrell et al.[2], and also Halperin and Hohenberg[3], used the scaling hypotheses applied earlier to static effects to describe dynamical phenomena. It was assumed that the spectrum of the critical fluctuations is determined by one characteristic length, which is the same as in the static case. This hypothesis had no serious justification apart from the fact that it is the simplest possibility. To investigate the dynamical phenomena, Kadanoff and Swift[4,5] used a semi-microscopic method, which is also not quite satisfactory since it is assumed that the intensity of the dynamical fluctuations is the same as that of the static ones and this is not always physically justified.

In the present paper an attempt is made at a microscopic analysis of nonequilibrium phenomena on the basis of a diagram technique. The structure is found of time-dependent correlation functions of various quantities and a new unknown critical index determining the power of the frequency is introduced. Such an index was introduced earlier[2-5] but there it was connected with the static critical indices. Below it will be shown that there are no theoretical reasons for such a connection.

It is shown that the collective excitation spectrum is defined not by one but by several scaling lengths, expressed in terms of the new critical index. All kinetic coefficients of the system are expressed in terms of this same index.

The method of this paper is directly applicable to all phase transitions in which the order parameter is not a conserved quantity, i.e., to the λ-transition in He^4 where the order parameter is a Bose-field operator $\psi(x)$, to the antiferromagnetic transition (where the order parameter is the difference between the magnetic moments of the sublattices N), etc. Transitions in an isotropic ferromagnet, where the order parameter (the total magnetic moment M) satisfies a conservation law, or the liquid-gas critical point, where the order parameter is the number of particles, require some modification of the methods and are not treated in this article.

From considerations of methodological convenience, we analyze below the λ-transition. The generalization to the other enumerated systems involves no difficulty.

2. GENERAL RELATIONS AND SYMBOLS

A Bose system above the condensation point is described by the temperature Green function $G_n(p)$[6]. The Green function is expressed in terms of the self-energy part by the formula

$$G_n^{-1}(p) = i\varepsilon_n - p^2/2m + \mu - \Sigma_n\ (p,\ \mu,\ T) \qquad (2.1)$$

(μ is the chemical potential). The self-energy part Σ_n is in turn functionally expressed in terms of G_n by means of Matsubara diagrams. The phase transition line is defined by the equation[7]

$$\mu_\lambda = \Sigma_0(0,\ \mu_\lambda,\ T_\lambda). \qquad (2.2)$$

Near to the transition line, formula (2.1) may be rewritten thus:

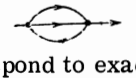
$$G_n^{-1} = i\varepsilon_n - p^2/2m - \tau - [\Sigma_n(p,\ \mu,\ T) - \Sigma_0(0,\ \mu_\lambda,\ T_\lambda)], \qquad (2.3)$$
$$\tau = a(\mu - \mu_\lambda) + b(T - T_\lambda)$$

(a and b are constants).

The dynamical properties of the system are defined by the retarded Green function $G_R(p, \epsilon)$ which is analytic in the lower half-plane and possesses the property that $G_R(p, i\epsilon_n) = G_n(p)$ for $\epsilon_n < 0$. To calculate $G_R(p, \epsilon)$ or $\Sigma_R(p, \epsilon)$, defined by the formula

$$G_R^{-1} = \varepsilon - p^2/2m - \tau - [\Sigma_R(p,\ \varepsilon,\ \mu,\ T) - \Sigma_R(0,\ 0,\ \mu_\lambda,\ T_\lambda)], \qquad (2.4)$$

it is necessary to continue the diagrams for $\Sigma_n(p)$ into the lower half-plane. Here it is not permissible to retain from the start only the zeroth terms in the sums over the frequencies. The separation of the most singular terms must be carried out only after analytic continuation of the whole sum. For a preliminary understanding of the situation, we shall consider, as in the static case[8], the simplified Dyson equation with $\Sigma_n(p)$ equal to the diagram



(2.3′)

where the lines correspond to exact Green functions.

We have

$$\Sigma_n(p) \propto \sum_{n_1 n_2} \int dp_1\, dp_2 G_{n_1}(p_1) G_{n_2}(p_2) G_{n-n_1-n_2}(p - p_1 - p_2). \qquad (2.5)$$

We put into (2.5) the spectral representation for the Green function[6]:

$$G_n(\mathbf{p}) = \frac{1}{\pi}\int \frac{\Delta G(\mathbf{p},\varepsilon)\,d\varepsilon}{i\varepsilon_n - \varepsilon}, \quad \Delta G = \operatorname{Im} G_R(\mathbf{p},\varepsilon). \tag{2.6}$$

After this the sums over the frequencies in (2.5) are easily calculated by standard methods and we obtain

$$\Sigma_n(\mathbf{p}) \infty \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\varepsilon_1\, d\varepsilon_2\, d\varepsilon_3 \frac{n_1 n_2 (1+n_3) - (1+n_1)(1+n_2) n_3}{\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - i\varepsilon_n} \times \Delta G(\mathbf{p}_1, \varepsilon_1)\Delta G(\mathbf{p}_2, \varepsilon_2)\Delta G(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2, \varepsilon_3), \tag{2.7}$$

where $n_i = (e^{\beta\epsilon_i} - 1)^{-1}$ and $i = 1, 2, 3$.

For the analytic continuation of (2.7) into the lower half-plane it is sufficient to replace $i\epsilon_n$ by $\epsilon - i0$. Further, the most singular part of $\Sigma(\mathbf{p}, \epsilon)$ for small ϵ and $\mathbf{p}$ will be obtained if we replace n_i by their classical limit $(\beta\epsilon_i)^{-1}$. As a result we have

$$\Sigma(\mathbf{p},\varepsilon) \infty \int \frac{d\mathbf{p}_1\, d\mathbf{p}_2\, d\varepsilon_1\, d\varepsilon_2\, d\varepsilon_3}{\varepsilon_1\varepsilon_2\varepsilon_3} \frac{\varepsilon_1 + \varepsilon_2 - \varepsilon_3}{\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon + i0} \cdot \times \Delta G(\mathbf{p}_1, \varepsilon_1)\Delta G(\mathbf{p}_2, \varepsilon_2)\Delta G(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2, \varepsilon_3). \tag{2.8}$$

We shall seek a solution of Eq. (2.4) for $\tau = 0$ and for small $\mathbf{p}$ and ϵ, with Σ given by (2.8), in the form

$$G(\mathbf{p}, \varepsilon) = |\mathbf{p}|^{-3/2} f(\varepsilon |\mathbf{p}|^{-\rho}). \tag{2.9}$$

Assuming that the bare terms $\epsilon - p^2/2m$ may be neglected and that in the integral (2.8) the important internal momenta and frequencies will be those of the order of the external ones, we put (2.9) into (2.8) and (2.4). Introducing the dimensionless variables $\kappa_i = \mathbf{p}_i/|\mathbf{p}|$ and $x_i = \epsilon_i |\mathbf{p}_i|^{-\rho}$ it is not difficult to reduce the equation to the form

$$f^{-1}(x) = \int \mathcal{K}(x, x_1, x_2, \rho)\Delta f(x_1)\Delta f(x_2)\Delta f(x_3)\frac{dx_1\, dx_2\, dx_3}{x_1 x_2 x_3}, \tag{2.10}$$

$$\mathcal{K}(x, x_1, x_2, \rho) = \int d\varkappa_1\, d\varkappa_2 \left\{ \frac{x_1|\varkappa_1|^\rho + x_2|\varkappa_2|^\rho - x_3|\mathbf{n} - \varkappa_1 - \varkappa_2|^\rho}{|\varkappa_1|^{3/2}|\varkappa_2|^{3/2}|\mathbf{n} - \varkappa_1 - \varkappa_2|^{3/2}} \cdot \frac{1}{x_1|\varkappa_1|^\rho + x_2|\varkappa_2|^\rho - x_3|\mathbf{n} - \varkappa_1 - \varkappa_2|^\rho - x + i0} - \frac{1}{|\varkappa_1|^{3/2}|\varkappa_2|^{3/2}|\varkappa_1 + \varkappa_2|^{3/2}} \right\}.$$

It is easy to check that the integrals (2.10) converge for $|\kappa_i| \sim 1$ and $x_i \sim 1$, provided that $\rho > 3/2$. Further, the discarded terms in the inverse Green function were of order ϵ, $p^2/2m$. The inverse Green function itself is of order $\epsilon^{3\rho/2}$, $|\mathbf{p}|^{3/2}$ and consequently, with the same condition $\rho > 3/2$, the proposed solution reproduces itself. The quantity $\rho > 3/2$ is determined from the solvability condition for Eq. (2.10). We note that, in another problem, equations of a similar type were considered by Gribov and Migdal[9].

It would seem that the terms connected with other diagrams, not taken into account in (2.8), are not small. As in the static case[8,9], substitution of the solution (2.9) into the other diagrams leads to logarithmic integrals. We may suppose that, by analogy with the static case, the logarithms are collected into the exponent and alter the powers in (2.9) while conserving the general self-similar form of the solution. To justify this supposition it is necessary to use exact solutions for $\Sigma_R(\mathbf{p}, \epsilon)$; we now turn to the formulation of these.

3. UNITARITY CONDITIONS FOR THE RETARDED GREEN FUNCTION AND PROPERTIES OF THE EXACT SOLUTION

The basis of the following analysis will be the unitarity condition for the retarded Green function. Here we shall write it out taking into account those simplifications which are brought in by the closeness to the phase transition and the importance of the small frequencies and momenta. A general formulation and proof of the unitarity condition is given in Appendix A.

The unitarity equation has the form

$\Delta\Sigma(\mathbf{p},\varepsilon) =$  $+ \cdots$ (3.1)

The diagram (3.1) and those similar to it are interpreted thus:

$$\Delta\Sigma(\mathbf{p},\varepsilon) \infty \varepsilon \int |\Gamma(\mathbf{p}_i, \varepsilon_i)|^2 \delta\Big(\mathbf{p} - \sum \mathbf{p}_i\Big)\delta\Big(\varepsilon - \sum \varepsilon_i\Big) \times \prod_i \Delta G(\mathbf{p}_i, \varepsilon_i)\frac{d\varepsilon_i\, d\mathbf{p}_i}{\varepsilon_i}, \tag{3.1'}$$

$$\Delta\Sigma \equiv \operatorname{Im}\Sigma_R.$$

Here the factors ϵ_i^{-1} have arisen from the expansion of the Bose functions $n(\epsilon_i)$. The vertex $\Gamma(\mathbf{p}_i, \epsilon_i)$ is the analytic continuation of the Matsubara four-point $\Gamma(\mathbf{p}_i, i\epsilon_n)$. The continuation path is determined from the condition that all frequencies entering from the right have a negative imaginary extra part and all outgoing frequencies a positive one. For example, the vertex in (3.1) coincides with the Matsubara vertex on the portions of the imaginary axes defined by the inequalities

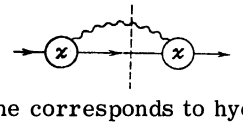
$$\operatorname{Im}\varepsilon_{1,2} \geqslant 0, \quad \operatorname{Im}\varepsilon_3 \leqslant 0. \tag{3.2}$$

Such a definition of Γ, as shown in Appendix A, corresponds to the fact that it is a retarded nonlinear susceptibility characterizing the change of $\langle\psi(\mathbf{x}, t)\rangle$ in the variable field $\int[\eta^*(\mathbf{x}, t)\psi(\mathbf{x}, t) + \eta(\mathbf{x}, t)\psi^*(\mathbf{x}, t)]d^3x$. This fact will be used below. It is not difficult to see that if we assume that the Green function and the vertex parts have the form

$$G(\mathbf{p}, \varepsilon) = \tau^{-\alpha\beta} g(\varepsilon\tau^{-\gamma}, \mathbf{p}\tau^{-\beta}),$$
$$\Gamma_n(\mathbf{p}_i, \varepsilon_i) = \tau^{[3-n(3-\alpha)/2]\beta}\gamma_n(\varepsilon_i\tau^{-\gamma}, \mathbf{p}_i\tau^{-\beta}), \tag{3.3}$$

then Eq. (3.1) and the equation for Γ_n, which we do not write out, will become dimensionless. For $\epsilon_i = 0$ we come back to the static results[8,10,11]. The parameters α and β are the known static critical indices and γ is the dynamical index. All these parameters are determined by the solvability of the equations (cf. Sec. 2) and are completely independent.

Thus, so long as we consider the self-interaction of the critical mode the self-similar solutions (3.3) are self-consistent and any amplitudes occurring in the theory will also be self-similar. Nevertheless, treating only the self-interaction of the critical modes is physically inadequate since in the system there exist other low-frequency modes, the hydrodynamic ones (sound, thermal conductivity, etc.). In the unitarity condition terms appear of the form

 (3.4)

(here the wavy line corresponds to hydrodynamic excitation); these terms will not be self-similar. To estimate the contribution of diagrams with hydrodynamic excitations one must know the amplitudes of the interaction between the hydrodynamic and critical modes. In the low-frequency limit such amplitudes may

be defined from general considerations. This problem is solved in the following section.

4. INTERACTION OF THE HYDRODYNAMIC MODES WITH ONE-PARTICLE EXCITATIONS

One can calculate the amplitude of this interaction using the fact that the hydrodynamic modes correspond to slow changes of temperature, chemical potential and velocity, obeying linearized Navier-Stokes equations. The interaction constant for the interaction of the hydrodynamic mode with one-particle states (ψ-modes) is proportional to the deviation of the Green function from its equilibrium value. In He^4 above the λ-point there are three types of hydrodynamic mode: sound with the spectrum $\omega_S = \pm u_S k + \frac{1}{2} i D_S k^2$, the thermal T-mode with the spectrum $\omega_T = i\lambda k^2/\rho C_p$ (where λ is the thermal conductivity coefficient and ρ is the density; below we assume $\rho = 1$) and the viscosity V-mode with the spectrum $\omega_V = i\eta k^2$, where η is the transverse (shear) viscosity.

We shall begin with the calculation of the interaction amplitude $V\bar{\psi}\psi$ (emission or absorption of the viscosity mode by one-particle excitations).

We place the system in a field of transverse vector potential $\delta\mathbf{A}_\perp$[1]. In the first approximation, this potential only induces velocity fluctuations with transverse gradient. In the system of coordinates linked to the medium the reference points of the momentum and particle energy fluctuate. That is,

$$\mathbf{p} \to \mathbf{p} - \delta\mathbf{v}_\perp - \delta\mathbf{A}_\perp, \quad \varepsilon \to \varepsilon - \mathbf{p}\delta\mathbf{v}_\perp \tag{4.1}$$

(the particle mass is chosen to be unity). Account is taken of the fact that in a field of vector potential $\mathbf{A}$ the momentum is $\mathbf{p} - \mathbf{A}$. The Green function is changed thus:

$$\delta G^{-1} = -\frac{\partial G^{-1}}{\partial \varepsilon}\mathbf{p}\delta\mathbf{v} - \frac{\partial G^{-1}}{\partial \mathbf{p}}(\delta\mathbf{v}_\perp + \delta\mathbf{A}_\perp). \tag{4.2}$$

From the Navier-Stokes equation

$$\delta\dot{\mathbf{v}}_\perp = \mathbf{f}_\perp + \eta\nabla^2\delta\mathbf{v}_\perp = -\delta\dot{\mathbf{A}}_\perp + \eta\nabla^2\delta\mathbf{v}_\perp \tag{4.3}$$

we find

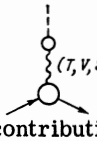
$$\delta\mathbf{v}_\perp(\mathbf{k},\omega) = -\frac{i\omega}{i\omega + \eta k^2}\delta\mathbf{A}_\perp. \tag{4.4}$$

Putting (4.4) into (4.2) we find

$$\vec{\mathcal{T}}_\perp(\mathbf{p},\varepsilon,\mathbf{k},\omega) \approx \mathbf{p}_\perp\frac{\partial G^{-1}}{\partial\varepsilon} - \frac{\eta k^2}{i\omega + \eta k^2}\left[\frac{\partial G^{-1}}{\partial \mathbf{p}_\perp} + \mathbf{p}_\perp\frac{\partial G^{-1}}{\partial\varepsilon}\right], \tag{4.5}$$

where the vector vertex $\vec{\mathcal{T}}_\perp$ is defined as the coefficient of proportionality between $-\delta G^{-1}$ and $\delta\mathbf{A}_\perp$[6]; $\mathbf{p}_\perp = \mathbf{p} - (\mathbf{p}\cdot\mathbf{k})\mathbf{k}/\mathbf{k}^2$.

To determine the emission vertex of the V-mode $\chi_V(p, \epsilon, k)$ we remark that the pole term in (4.5) arises because of the following diagram

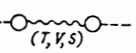


(4.6)

This diagram gives a contribution

$$\chi_V(p,\ \varepsilon,\ k)\, g_V(k) \,/\, (i\omega + \eta k^2) \tag{4.6'}$$

(where $g_V(\mathbf{k})$ is the transition amplitude of the flux into the V-mode). The quantity g_V determines the residue at the pole of the correlator of two transverse fluxes. The contribution of the corresponding diagram, having the form



(4.7)

is

$$g_V^2(\mathbf{k}) \,/\, (i\omega + \eta k^2). \tag{4.7'}$$

On the other hand, the vector correlator is given by formula (4.4), whence

$$g_V(\mathbf{k}) = \sqrt{\eta k^2}. \tag{4.8}$$

Putting (4.8) into (4.6') and comparing with (4.5) we find

$$\chi_V(\mathbf{p},\varepsilon,\mathbf{k}) = \sqrt{\eta k^2}\left[\frac{\partial G^{-1}}{\partial \mathbf{p}_\perp} + \mathbf{p}_\perp\frac{\partial G^{-1}}{\partial\varepsilon}\right] \tag{4.9}$$

(for $k \ll p$, $\eta k^2 \ll \epsilon$). Formula (4.9) solves the problem posed in the case of the V-mode.

We turn to consider the T-mode interaction. We place the system in a field of sclaar potential $\delta\varphi(\mathbf{k}, \omega)$, such that $\omega \ll u_S k$, where u_S is the velocity of sound. Such a field gives rise to fluctuations of temperature and chemical potential in the system, which are determined from the Navier-Stokes equation:

$$\begin{gathered}\delta\mathbf{v} = -\nabla(\delta P + \delta\varphi) + \eta\nabla^2\delta\mathbf{v} + \tfrac{\eta}{3}\nabla(\nabla\delta\mathbf{v}),\\ \delta\rho + \nabla\delta\mathbf{v} = 0, \quad T\delta S = \lambda\nabla^2\delta T.\end{gathered} \tag{4.10}$$

Here S is the entropy per unit mass and P is the pressure. We recall that we took an initial density $\rho = 1$. The solution of (4.10) for $\omega \ll u_S k$ is simple and leads to the result

$$\begin{gathered}\delta P \approx -\delta\varphi, \quad \delta\mathbf{v} \approx 0,\\ \delta\rho \approx -\frac{1}{u_T^2}\frac{i\omega C_V + \lambda k^2}{i\omega C_P + \lambda k^2}\delta\varphi,\\ \delta T \approx +\frac{1}{u_T}\sqrt{T(C_P - C_V)}\,\frac{i\omega}{i\omega C_P + \lambda k^2}\delta\varphi,\end{gathered} \tag{4.11}$$

$u_T^2 = (\partial P/\partial\rho)_T$.

The Green function is changed owing to the fluctuations of μ and T and the shift of the frequency reference point:

$$\varepsilon \to \varepsilon - \delta\varphi - \delta\mu. \tag{4.12}$$

The second term in (4.12) is connected with the fact that in the Green functions used the frequency is reckoned relative to the chemical potential.

We have

$$\begin{aligned}-\delta G^{-1} &= \frac{\partial G^{-1}}{\partial\varepsilon}(\delta\varphi + \delta\mu) - \frac{\partial G^{-1}}{\partial\mu}\delta\mu - \frac{\partial G^{-1}}{\partial T}\delta T\\ &= \left[s\left(\frac{\partial G^{-1}}{\partial\mu} - \frac{\partial G^{-1}}{\partial\varepsilon}\right) - \frac{\partial G^{-1}}{\partial T}\right]\delta T \neq \frac{\partial G^{-1}}{\partial\mu}\delta\varphi.\end{aligned} \tag{4.13}$$

(We have used the facts that $\delta P \approx -\delta\varphi$ and $\delta\mu = \delta P - s\delta T$, where s is the entropy per unit volume). Introducing the particle density vertex by the formula $\delta G^{-1} = -J\delta\varphi$ (cf.[6]) and putting (4.11) into (4.13), we find

$$\mathcal{T} = \frac{\partial G^{-1}}{\partial\mu} + \frac{\sqrt{T(C_P - C_V)}}{u_T}\,\frac{i\omega}{i\omega C_P + \lambda k^2}\left[s\left(\frac{\partial G^{-1}}{\partial\mu} - \frac{\partial G^{-1}}{\partial\varepsilon}\right) - \frac{\partial G^{-1}}{\partial T}\right] \tag{4.14}$$

for $k \ll p$, $\omega \ll \epsilon$ and $\omega \ll u_S k$.

To determine the emission amplitude $\chi_T(p, \epsilon, k)$ we shall proceed exactly as in the derivation of (4.9). That is, we shall consider diagram (4.6) and this time by the wavy line we shall understand a T-mode and by

[1] I.e., we add a term $\mathbf{J}\delta\mathbf{A}_\perp$ (where $\mathbf{J}$ is the mass flux density) to the Hamiltonian. By analogy with electrodynamics such an extra term is equivalent to the action on the particles of a "Lorentz force" $\mathbf{f} = \mathbf{E} + \mathbf{v} \times \mathbf{H}$; $\mathbf{E} = -\delta\dot{\mathbf{A}}_\perp$, $\mathbf{H} = \operatorname{curl}\delta\mathbf{A}_\perp$.

the dotted line the density operator. The contribution of this diagram is:

$$\frac{\chi_T(p, \varepsilon, k) f_T(k)}{i\omega + \lambda k^2 / C_P}, \tag{4.15}$$

where f_T is the amplitude of the transition to the T-mode. To calculate f_T we shall compare the pole contribution to the density correlator (cf. (4.7))

$$f_T^2(k) \,/\, (i\omega + \lambda k^2 / C_P) \tag{4.16}$$

with formula (4.11) for $\delta\rho/\delta\varphi$. Equating the residue at the pole of (4.11) to f_T^2, we find

$$f_T = \frac{1}{u_T} \sqrt{\frac{C_P - C_V}{C_P}} \sqrt{\frac{\lambda k^2}{C_P}}. \tag{4.17}$$

Putting (4.17) into (4.15) and comparing with (4.14), we get

$$\chi_T(p, \varepsilon, k) = \sqrt{\frac{T}{C_P}} \sqrt{\frac{\lambda k^2}{C_P}} \left[s\left(\frac{\partial G^{-1}}{\partial \mu} - \frac{\partial G^{-1}}{\partial \varepsilon} \right) - \frac{\partial G^{-1}}{\partial T} \right] \tag{4.18}$$

$$= -\sqrt{\frac{T}{C_P}} \sqrt{\frac{\lambda k^2}{C_P}} \left[\left(\frac{\partial G^{-1}}{\partial T} \right)_P - s\left(\frac{\partial G^{-1}}{\partial \varepsilon} \right) \right]$$

for

$$k \ll p, \quad \lambda k^2 / C_P \ll \varepsilon.$$

It is convenient to treat the sound mode emission by a slightly different method. To calculate $\chi_{S\pm}(p, \epsilon, k)$, where $\pm$ correspond to the poles $\omega_{S\pm} = {}^1/_2 i D_S k^2 \pm u_S k$, we note that in the frequency region $\lambda k^2/C_p \ll \omega \ll u_S k$ by virtue of (4.14) we shall have

$$\mathcal{T} = \frac{\partial G^{-1}}{\partial \mu} + \frac{1}{u_S} \sqrt{T\left(\frac{1}{C_V} - \frac{1}{C_P} \right)} \left[s\left(\frac{\partial G^{-1}}{\partial \mu} - \frac{\partial G^{-1}}{\partial \varepsilon} \right) - \frac{\partial G^{-1}}{\partial T} \right]. \tag{4.19}$$

(We have used the fact that $u_S/u_T = \sqrt{C_P/C_V}$). On the other hand at very high frequencies $\omega \gg u_S k$ the Ward identity[6]

$$\mathcal{T}^\omega = \partial G^{-1} / \partial \varepsilon. \tag{4.20}$$

must be satisfied. On account of the ω sound poles. Eqs. (4.19) and (4.20) can be matched. For $\omega \gg \lambda k^2/C_p$ we have

$$\mathcal{T} = \tilde{\mathcal{T}} + \frac{f_{S+}\chi_{S+}}{u_S k - \omega + {}^1/_2 i D_S k^2} + \frac{f_{S-}\chi_{S-}}{u_S k + \omega - {}^1/_2 i D_S k^2} \tag{4.21}$$

(where $\tilde{\mathcal{T}}$ has no sound poles). We can find the quantities $f_{S\pm}$ by noting that in the sound frequency region the density correlator has the well-known form

$$\Pi(k, \omega) \equiv \frac{\delta\rho}{\delta\varphi} = \frac{k^2}{\omega^2 - u_S^2 k^2 - i D_S k^2 \omega}. \tag{4.22}$$

Equating f_{S+}^2 and f_{S-}^2 to the residues at the corresponding poles in (4.22) we find

$$f_{S+} = f_{S-} = \sqrt{\frac{k}{2u_S}} + O(k^{3/2}). \tag{4.23}$$

Further, equating (4.21) to (4.20) for $\omega \gg u_S k$ and to (4.19) for $\omega \ll u_S k$ and taking account of the fact that the quantity $\tilde{\mathcal{T}}$ is the same in both frequency regions, we find

$$\chi_{S+} + \chi_{S-} = \sqrt{2u_S k} \left\{ u_S \left(\frac{\partial G^{-1}}{\partial \mu} - \frac{\partial G^{-1}}{\partial \varepsilon} \right) \right. \tag{4.24}$$

$$\left. + \sqrt{T\left(\frac{1}{C_V} - \frac{1}{C_P} \right)} \left[s\left(\frac{\partial G^{-1}}{\partial \mu} - \frac{\partial G^{-1}}{\partial \varepsilon} \right) - \frac{\partial G^{-1}}{\partial T} \right] \right\}.$$

To calculate the difference $\chi_{S+} - \chi_{S-}$ we shall consider the longitudinal vector vertex $\mathcal{T}_x(p, \epsilon, k, \omega)$ (the x axis is chosen in the direction of the vector k). It satisfies the Ward identities[6]:

$$\mathcal{T}_x^\omega = p_x \frac{\partial G^{-1}}{\partial \varepsilon}, \quad \mathcal{T}_x^k = -\frac{\partial G^{-1}}{\partial p_x}. \tag{4.25}$$

From (4.25) it is clear that $\mathcal{T}_x$ has a singular point at $\omega, k = 0$. This singularity arises by virtue of the diagram (4.6), in which the wavy line can correspond either to a T-mode or an S-mode. However, as was pointed out above (cf. (4.11)), in the propagation of the T-mode longitudinal velocity fluctuations do not arise. Because of this the residue at the thermal pole of the longitudinal vector vertex is small ($\sim k^3$) and this pole may be left out of consideration; it does not produce any different k- and ω-limits. The difference in the k- and ω-limits of is connected entirely with the sound pole. Writing

$$\mathcal{T}_x = \tilde{\mathcal{T}}_x + \frac{g_{S+}\chi_{S+}}{u_S k - \omega + {}^1/_2 i D_S k^2} + \frac{g_{S-}\chi_{S-}}{u_S k + \omega - {}^1/_2 i D_S k^2}, \tag{4.26}$$

where $g_{S\pm}$ are the transition amplitudes from the longitudinal flux to the modes $S_\pm$, taking into account that, by virtue of the conservation of flux,

$$\omega f_{S+} \equiv u_S k f_{S+} = k g_{S+}, \quad \omega f_{S-} = -u_S k f_{S-} = k g_{S-}, \tag{4.27}$$

and comparing (4.26) and (4.25), we get

$$\chi_{S+} - \chi_{S-} = \sqrt{2u_S k} \left(\frac{\partial G^{-1}}{\partial p_x} + p_x \frac{\partial G^{-1}}{\partial \varepsilon} \right). \tag{4.28}$$

From (4.24) and (4.28) we find the desired formulae:

$$\chi_{S\pm} = \sqrt{\frac{u_S k}{2}} \left\{ u_S \left(\frac{\partial G^{-1}}{\partial \mu} - \frac{\partial G^{-1}}{\partial \varepsilon} \right) + \sqrt{T\left(\frac{1}{C_V} - \frac{1}{C_P} \right)} \right. \tag{4.29}$$

$$\left. \times \left[s\left(\frac{\partial G^{-1}}{\partial \mu} - \frac{\partial G^{-1}}{\partial \varepsilon} \right) - \frac{\partial G^{-1}}{\partial T} \right] \pm \frac{\mathbf{k}}{|\mathbf{k}|} \left(\frac{\partial G^{-1}}{\partial \mathbf{p}} + \mathbf{p} \frac{\partial G^{-1}}{\partial \varepsilon} \right) \right\}.$$

The formulae (4.9), (4.18) and (4.29) solve the problem posed at the beginning of this section. It would be possible to obtain similar formulae (though not coinciding with ours) by the method of Kadanoff and Swift; however, in their purely static method all terms containing the derivatives $\partial G^{-1}/\partial \epsilon$ are lost.

5. ESTIMATION OF THE KINETIC COEFFICIENTS

The emission amplitudes of the hydrodynamic modes, found in the preceding Section, depend essentially on the kinetic coefficients λ, η, and ζ. Therefore, before we can estimate diagrams of the type (3.4), it is necessary to calculate these coefficients. In this calculation we shall have to assume beforehand that the diagrams (3.4) for $\epsilon \sim \epsilon_c$ and $p \sim p_c$ are small and that in this frequency region the formulae (3.3) are valid. The consistency of this assumption will be demonstrated later (Sec. 6).

The kinetic coefficients or, what is the same, the frequencies of the collective modes in principle should be determined from the Bethe-Salpeter equation. However, it is difficult using this method to obtain concrete results. It is more convenient to use as a basis the Kubo formulae[12], by which

$$\lambda = \omega^{-1} \operatorname{Im} \langle J_x^s J_x^s \rangle_{k\omega}, \quad \zeta = \omega^{-1} \operatorname{Im} \langle XX \rangle_{k\omega}, \tag{5.1}$$

$$\eta = \omega^{-1} \operatorname{Im} \langle \tau_{xy} \tau_{xy} \rangle_{k\omega}$$

$$\text{for } k = 0, \ \omega \to 0.$$

for $\mathbf{k} = 0$, $\omega \to 0$. Here the vector k is directed along the x-axis, τ_{xy} are the components of the stress tensor and $\mathbf{J}^S$ is the entropy flux, defined by the formula

$$\mathbf{J}^s = \mathbf{J}^E - w\mathbf{J}, \tag{5.2}$$

where J^E is the energy flux, J the mass flux and w the enthalpy per unit volume. (As before the density of the system $\rho = 1$). Further

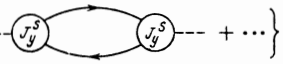
$$X = \tau_{xx} - \left(\frac{\partial P}{\partial \rho}\right)_E \hat{\rho} - \left(\frac{\partial P}{\partial E}\right)_\rho \hat{E}, \tag{5.3}$$

where $\hat{\rho}$ is the mass density operator and $\hat{E}$ is the energy density. The brackets in (5.1) are interpreted as follows:

$$\langle MN\rangle_{k\omega} = \int d^3x\, dt\, e^{i(\omega t - \mathbf{kx})}\theta(-t)\langle [M(0,0), N(\mathbf{x},t)]\rangle, \tag{5.4}$$

where M and N are any two operators. It is clear that $\langle MN\rangle_{k\omega}$ is the retarded Green function for these quantities, obeying the unitarity condition. This fact allows us to estimate the orders of the quantities appearing in the formulae (5.1).

We shall start with the determination of the thermal conductivity λ. Taking into account that for $\mathbf{k} \to 0$ the x and y directions are equivalent the unitarity condition for λ can be written in the form

$\lambda \backsim \frac{1}{\omega}$  (5.5)

The terms not written out will be discussed later. To evaluate diagram (5.5) we must know the order of magnitude of the vertex of the entropy flux:

$\mathcal{T}[J_y^S] \equiv$  $= \mathcal{T}[J_y^E] - w\mathcal{T}[J_y]$ (5.6)

We note that in $\mathcal{T}^S_y$ there are no hydrodynamic poles in ω. Indeed, the T-mode and S-mode are purely longitudinal and cannot enter into the transverse vertex J^S_y. The V-mode does not give a contribution, since the amplitude of its transition to the entropy flux is zero. The latter is obvious from the fact that change of the fluxes with change of velocity in the hydrodynamic approximation is

$$\delta J^E = w\delta v = w\delta J, \quad \delta J^S = 0. \tag{5.7}$$

The absence of hydrodynamic poles makes the ordering of the passage to the limit $\omega, \mathbf{k} \to 0$ in (5.5) and (5.6) immaterial; therefore, to evaluate the order of magnitude of $\mathcal{T}[J^S_y]$ we can use Ward identity for its k-limit. This identity is derived in Appendix B. It has the form

$$-\mathcal{T}^k[J_y{}^S p\varepsilon] = \varepsilon\frac{\partial G^{-1}}{\partial p_y} + G^{-1}p_y - Ts\frac{\partial G^{-1}}{\partial p_y} \tag{5.8}$$

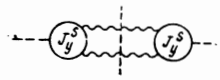
(s is the entropy density).

For $\epsilon \sim \epsilon_c$, $p \sim p_c$, the basic contribution is obtained from the last term in (5.8). Putting it into (5.5) we obtain

$$\lambda = \lim_{\mathbf{k},\omega\to 0}\int |\mathcal{T}[J_y{}^S]|^2\,\Delta G(\mathbf{k}-\mathbf{k}_1, \omega-\omega_1)\Delta G(\mathbf{k}_1,\omega_1)\frac{d\omega_1\, d\mathbf{k}_1}{\omega_1(\omega-\omega_1)}$$
$$\backsim T^2 s^2\left(\frac{G^{-1}}{p_c}\right)^2 G^2\frac{p_c^3}{\varepsilon_c} \backsim \frac{p_c}{\varepsilon_c} \backsim \tau^{\beta-\nu}. \tag{5.9}$$

In making this evaluation we have taken into account that in the integral the principal region is $k_1 \sim p_c$ and $\omega_1 \sim \epsilon_c$.

We shall consider the unevaluated terms in (5.5). It is not difficult to show that diagrams containing only continuous lines will give a contribution (5.9). In addition to this the contribution to λ of the hydrodynamic modes, i.e., of terms of the type

 (5.10)

will be of an essentially different order because of the fact that the characteristic frequencies of the hydrodynamic modes are not equal to ϵ_c.

We shall consider first the case when in (5.10) two T-modes are emitting. For small enough k the frequency of the T-mode is

$$\omega_T(k) = \frac{\lambda k^2}{C_P} \sim \frac{p_c}{\varepsilon_c}\frac{\tau^2}{p_c^3}k^2 \sim \frac{\tau^2}{\varepsilon_c}\frac{k^2}{p_c^2}. \tag{5.11}$$

We have taken into account that $C_p \sim p_c^3/\tau^{2}$ [8] and assumed, with a calculation for subsequent confirmation, that the main contribution to λ is of the order of (5.9). Formula (5.11) is applicable in order of magnitude provided that

$$\omega_T(k) < \varepsilon_c = \tau^\nu, \quad k < p_c = \tau^\beta \tag{5.12}$$

or

$$k < \min(k_T, p_c),$$

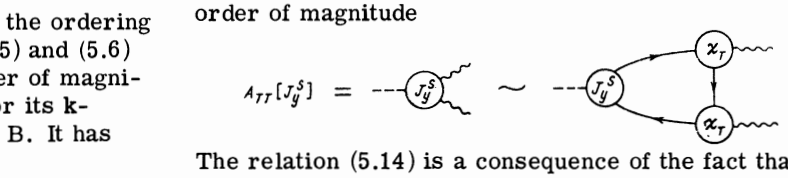
where

$$k_T = \frac{\varepsilon_c}{\tau}p_c = \tau^{\nu-1+\beta}. \tag{5.13}$$

For the subsequent analysis it is necessary to distinguish the cases $\gamma < 1$ and $\gamma > 1$. In the first of these the virtual momenta $\mathbf{k} \sim p_c$ make the main contribution to the integral (5.10). In this the characteristic frequency of the wavy line is

$$\omega_T(p_c) \sim \tau^2/\varepsilon_c \sim \tau^{2-\nu} \ll \varepsilon_c \sim \tau^\nu.$$

Since the emission amplitudes of the T-modes, occurring in (5.10), change at frequencies $\sim\epsilon_c$ we can assume that these amplitudes in (5.10) are taken at zero frequencies. It is not difficult to see that in this limit they are zero. In fact on time-reversal the flux J^S changes sign and consequently the amplitudes in (5.10) must be odd functions of the frequencies. The arguments adduced show that the contribution of diagram (5.10) is small. There is a similar assertion in[4]. To obtain a quantitative estimate we note that, by order of magnitude

$A_{TT}[J_y^S] =$  (5.14)

The relation (5.14) is a consequence of the fact that in the critical region all diagrams containing exact vertices and Green functions are of a similar order. The important internal momenta in the right hand side of (5.14) are of order p_c if all the external frequencies are of order ϵ_c and the momenta of order p_c. Consequently we may write

$$A_{TT}[J_y{}^S] \sim \mathcal{T}[J_y{}^S]G^3\chi_T{}^2 p_c{}^3 \sim$$
$$\sim \frac{G^{-1}}{p_c}G^3\frac{G^{-2}}{\tau^2}\frac{\omega_T(p_c)}{C_P}p_c{}^3 \sim \frac{\omega_T(p_c)}{p_c}. \tag{5.15}$$

In the derivation of (5.15) we took account of the Ward identity (5.8), the expression for χ_T and the fact that, since $\epsilon_i \gg \tau$, in (4.18) we may leave out of consideration terms containing $\partial G^{-1}/\partial\epsilon$.

The estimate (5.15) is incorrect for external frequencies low compared with ϵ_c. As was pointed out above, the amplitude A_{TT} is zero in the limit of zero frequencies; therefore, for low frequencies, (5.15) is multiplied by a small factor $\sim\omega/\epsilon_c$. Since the frequen-

cies in (5.10) are of order $\omega_T(p_c)$, the amplitude occurring in this integral are as follows:

$$A_{TT} \sim \omega_T^2(p_c) / p_c\varepsilon_c. \tag{5.16}$$

By analogy with (5.9) the integral itself can be written down in the form

$$\lambda_{TT} = \int \frac{d\omega_1\, dk_1}{\omega_1^2} |A_{TT}|^2 \Delta F(k_1, \omega_1) \Delta F(k - k_1, \omega - \omega_1), \tag{5.17}$$

where λ_{TT} is the contribution of diagram (5.10) to λ and

$$\Delta F(k, \omega) \sim \mathrm{Im}\,[i\omega + \lambda k^2 / C_P]^{-1}.$$

For $\omega \lesssim \epsilon_c$, $k \lesssim p_c$ the frequencies $\omega_1 \sim \omega_T(p_c)$ and the momenta $k_1 \sim p_c$ are the important ones in integral (5.17). Therefore, taking (5.16) into account,

$$\lambda_{TT} \sim \frac{p_c}{\varepsilon_c} \frac{\omega_T(p_c)}{\varepsilon_c} \ll \lambda \sim \frac{p_c}{\rho}. \tag{5.18}$$

Consequently, as we asserted, the TT-contribution to the thermal conductivity in the case $\gamma < 1$ (or $\epsilon_c \gg \tau$) is small.

We shall consider now the case $\gamma > 1$ ($\epsilon_c \ll \tau$). In this case the integral over the momenta in (5.17) will begin to converge when the characteristic frequency of the wavy line $\omega_T(k)$ approaches ϵ_c. On further increase of $\omega_T(k)$, the decrease of the amplitude A_T, occurring at frequencies greater than ϵ_c, cuts off the integral. The contribution of the region $\omega_T(k) \sim \epsilon_c$ or $k \sim k_T \sim \epsilon_c p_c/\tau$ to the integral (5.17) has the form

$$\lambda_{TT} \sim A_{TT}^2 k_T^3 / \varepsilon_c^3. \tag{5.19}$$

By an estimate analogous to (5.10), at external frequencies $\sim\epsilon_c$ and momenta $\sim k_T$, A_{TT} is given by

$$A_{TT} \sim \varepsilon_c / p_c. \tag{5.20}$$

Putting (5.20) into (5.19), we find

$$\lambda_{TT} \sim \left(\frac{k_T}{p_c}\right)^3 \frac{p_c}{\varepsilon_c} \sim \left(\frac{\varepsilon_c}{\tau}\right)^3 \frac{p_c}{\varepsilon_c} \ll \lambda. \tag{5.21}$$

The region $k \sim p_c \gg k_T$, as can be verified, also gives a small contribution, if we assume that $\omega_T(p_c) \gg \epsilon_c$ (or $\sim\epsilon_c$). This assumption means that $\omega_T(k)$, having attained the value ϵ_c for $k \sim k_T \ll p_c$ does not begin to decrease on further increase of k.

Investigation of the contribution of the sound and viscosity modes to λ is carried out analogously and this contribution is found to be small. Therefore, the final estimate for λ is

$$\lambda \sim p_c / \varepsilon_c \sim \tau^{\beta-\nu}. \tag{5.22}$$

The qualitative meaning of formula (5.22) is that, owing to the fluctuations, the system can be found, for a time $t_c \sim \epsilon_c^{-1}$, in a superfluid state. Then heat transfer will occur on account of the motion, without mass transfer, of the superfluid and normal components to different sides. Consequently,

$$\mathbf{J}^S \sim (s/\rho)\rho_s \mathbf{v}_s, \tag{5.23}$$

where ρ_S and s_S are the superfluid density and velocity. From the equations of two-fluid hydrodynamics it follows that

$$\dot{\mathbf{v}}_s \sim \nabla\mu = (\partial\mu/\partial T)_P \nabla T, \quad \mathbf{v}_s \sim t_c(\partial\mu/\partial T)_P \nabla T \sim \varepsilon_c^{-1}(\partial\mu/\partial T)_P \nabla T. \tag{5.24}$$

(We have taken account of the fact that the heat conduction occurs at constant pressure).

Putting (5.24) into (5.23), we find

$$\lambda = \frac{|\mathbf{J}^S|}{|\nabla T|} \sim \frac{s}{\rho}\rho_s \left(\frac{\partial\mu}{\partial T}\right)_P \varepsilon_c^{-1}. \tag{5.25}$$

Since $(\partial\mu/\partial T)_p = s/\rho$ and $\rho_S \infty p_c$[1], this formula coincides with (5.22).

We turn to the evaluation of η. By the third of the formulae (5.1), for this it is necessary to know the vertex of the stress tensor τ_{xy}. As shown in Appendix B, for zero transferred momentum, this vertex satisfies the Ward identity

$$\mathcal{T}[\tau_{xy}] = p_x \partial G^{-1} / \partial p_y. \tag{5.26}$$

Putting (5.26) into formula (5.9) together with $T[J_y^S]$, we find

$$\eta \sim \mathcal{T}^2 G^2 \frac{n^3}{\varepsilon} \sim \frac{n^3}{\varepsilon} \sim \tau^{3\beta-\nu}. \tag{5.27}$$

It would seem that (5.27) gives only the singular part of η; therefore, for $\gamma < 3\beta$ (which, evidently, is fulfilled for real systems)

$$\eta \sim \text{const}. \tag{5.28}$$

The estimate does not change when the hydrodynamic modes are taken into account.

It remains to consider the quantity ζ. It is convenient to take immediate account of the characteristics of He^4, in particular the fact that $\beta \approx 2/3$[1] and, in addition, that the compressibility of He^4 is anomalously small. As was shown by Batyev, Patashinskiĭ and Pokrovskiĭ[13], the latter fact means that in the formula

$$\tau = a(\mu - \mu_\lambda) + b(T - T_\lambda) \tag{5.29}$$

the coefficient

$$b \gg a.$$

Therefore in the critical region the variables depend weakly on the chemical potential and the derivatives with respect to μ are far smaller than the derivatives with respect to T.

We shall begin with an evaluation of the contribution to ζ of the one-particle excitations. In Appendix B the Ward identity for the vertex $\mathcal{T}[X]$ is derived

$$\mathcal{T}[X] = p_x \frac{\partial G^{-1}}{\partial p_x} + G^{-1} - \frac{\partial G^{-1}}{\partial\mu}\left[\left(\frac{\partial P}{\partial\rho}\right)_E + \mu\left(\frac{\partial P}{\partial E}\right)_\rho\right] - \left(\frac{\partial P}{\partial E}\right)_\rho \left(\frac{\partial G^{-1}}{\partial T} - \varepsilon\frac{\partial G^{-1}}{\partial\varepsilon} - G^{-1}\right). \tag{5.30}$$

Transforming to the variables μ and T, we may write down the coefficients in (5.30) in the form

$$\left(\frac{\partial P}{\partial E}\right)_\rho = \frac{1}{T}(sP_{\mu\mu} - P_{T\mu})(P_{TT}P_{\mu\mu} - P_{T\mu}^2)^{-1}, \tag{5.31}$$

$$\left(\frac{\partial P}{\partial\rho}\right)_E + \mu\left(\frac{\partial P}{\partial E}\right)_\rho = (P_{TT} - sP_{T\mu})(P_{TT}P_{\mu\mu} - P_{T\mu}^2)^{-1}.$$

The function $P(T, \mu)$ has the form[10]

$$P(T, \mu) = D\tau^2 \ln\tau^{-1} + P_{Reg}(T, \mu)$$

(where P_{reg} is a continuous function). In[13] it is shown that in the practicably attainable region of temperatures the derivatives $(P_{\mu\mu})_{reg} \gg Da^2 \ln\tau^{-1}$. Taking this into account, we find

$$\left(\frac{\partial P}{\partial E}\right)_\rho \approx -\frac{Dab\ln\tau^{-1}}{Db^2 T P_{\mu\mu}\ln\tau^{-1}} \approx -\frac{a}{b}\frac{1}{(P_{\mu\mu})_{Reg}},$$

$$\left(\frac{\partial P}{\partial \rho}\right)_E + \mu\left(\frac{\partial P}{\partial E}\right)_\rho \approx \frac{1}{(P_{\mu\mu})_{Reg}}. \quad (5.32)$$

Thus the coefficient of $\partial G^{-1}/\partial T$ contains the small quantity a/b and the derivative $\partial G^{-1}/\partial \mu$ is itself small. As a result, the estimate of $\mathcal{T}[X]$ is

$$\mathcal{T}[X] \sim (P_{\mu\mu})^{-1}_{Reg}\, \partial G^{-1}/\partial \mu. \quad (5.33)$$

Putting this estimate into a formula of the type (5.9), we find

$$\zeta \sim \frac{1}{\varepsilon_c}\frac{(P_{\mu\mu})_{sing}}{(P_{\mu\mu})^2_{Reg}} \sim \frac{u_T^2}{\varepsilon_c} D a^2 u_T^2 \quad (5.34)$$

(since $u_T^2 \approx p^{-1}_{\mu\mu}$). In the derivation of formula (5.34) it is essential that the integral (5.9) converge and give no new logarithms.

We shall now consider the contribution to ζ of the hydrodynamic modes and in particular of the T-mode. Using diagram (5.14) we can evaluate $A_{TT}[X]$, the amplitude for the conversion of X into two T-modes. Replacing $\mathcal{T}[J^S_y]$ in (5.15) by $T[X]$ and taking (5.33) into account, we find

$$A_{TT} \sim (P_{\mu\mu})^{-1}_{Reg} \frac{\partial G^{-1}}{\partial \mu} G^3 \chi_T^2 p_c^3. \quad (5.35)$$

As before, we shall consider the cases $\gamma < 1$ and $\gamma > 1$ separately. In the first case, in the integral of the type (5.27), $k \sim p_c$ and $\omega_1 \sim \omega_T(p_c)$ are important. Taking (5.35) into account and the fact that $F \approx 1/\omega_T$, we find the estimate

$$\zeta_{TT} \sim D a^2 u_T^2 u_T^2 / \varepsilon_c. \quad (5.36)$$

Further,

$$\omega_T(p_c) \sim \lambda p_c^2 / C_P \sim \tau^{2-\gamma} \ln^{-1} \tau. \quad (5.37)$$

Consequently

$$\zeta_{TT} \sim D a^2 u_T^2 \frac{u_T^2 \ln \tau}{\tau^{2-\gamma}} \equiv \alpha \frac{u_T^2 \ln \tau}{\tau^{2-\gamma}}. \quad (5.38)$$

It is possible to show that the V- and S-modes always give a small contribution. Therefore, for $\gamma < 1$ the thermal modes give the main contribution to ζ, and ζ is given by formula (5.38). For $\gamma > 1$ one-particle excitations have effect and ζ is determined by (5.34).

To evaluate the dimensionless quantity $\alpha = Da^2u_T^2$, we note that, firstly, $Cp = T_\lambda Db^2 \ln \tau^{-1}$ and, secondly, $d\mu/dT|_\lambda = a/b$ (where $d\mu/dT|_\lambda$ is the slope of the λ-curve). Consequently, we can write

$$\alpha = \frac{m u_T^2}{\varkappa_B T_\lambda}\left(\frac{d\mu}{dT}\Big|_\lambda\right)^{-1} \frac{C_P}{\varkappa_B \ln \tau^{-1}} \quad (5.39)$$

(m is the mass of a helium atom and κ_B is the Boltzmann constant). Inserting numerical data from[13,2] and using the fact that $u_T^2 \approx u_S^2$, we find

$$\alpha \sim 10^{-3}. \quad (5.40)$$

6. CONTRIBUTION OF THE HYDRODYNAMIC MODES TO THE GREEN FUNCTION

We shall show that, as was stated at the beginning of Sec. 5, the hydrodynamic modes have little effect on the Green function. For this we shall evaluate the correction to Σ given by diagram (3.4), with the wavy line implying, at first, a T-mode. We have

$$\mathrm{Im}\, G_{TT}^{-1} = \varepsilon \int \frac{dk\, d\omega\, |\chi_T|^2}{\omega(\varepsilon - \omega)} \Delta F(k, \omega)\, \Delta G(p - k, \varepsilon - \omega). \quad (6.1)$$

Let $\epsilon \sim \epsilon_c$ and $p \sim p_c$. As above, we distinguish two cases $\gamma < 1$ and $\gamma > 1$. In the first case $k \sim p_c$ and $\omega \sim \omega_T(p_c) \ll \epsilon_c$ are important in (6.1). Therefore, we can rewrite (6.1) in the form

$$\mathrm{Im}\, G_T^{-1} \approx \int \frac{dk\, d\omega}{\omega} |\chi_T(p, \varepsilon, k)|^2\, \Delta F(k, \omega)\, \Delta G(p - k, \varepsilon - \omega) \approx$$
$$\approx \int dk\, |\chi_T(p, \varepsilon, k)|^2 F_{st}(k) \Delta G(p - k, \varepsilon); \quad (6.2)$$
$$F_{st}(k) = \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \Delta F(k, \omega).$$

The quantity (6.2) as a function of ϵ is determined by the scale of ϵ_c and may, with precision up to the terms of order $\omega_T(p_c)/\epsilon_c$ which were discarded in the derivation, be represented in the form (3.3).

We shall consider now the case $\gamma > 1$. The region $k \sim k_T$ and $\omega \sim \epsilon_c$ gvies to (6.1) the contribution

$$\mathrm{Im}\, G_T^{-1} \sim k_T^3 \chi_T^2 FG \sim (k_T / p_c)^3 G^{-1} \ll G^{-1}. \quad (6.3)$$

The region $k \sim p_c$ gives a small contribution, if $\omega_T(p_c) \gg \omega_c$, and a contribution of the type (3.3) if $\omega_T(p_c) \sim \epsilon_c$. (We recall that we can investigate $\omega_T(k)$ only for $k < k_T \ll p_c$). The hydrodynamic formulae give no information about the quantity $\omega_T(k)$ for $k \sim p_c$.

The estimates made show that in the principal region $\epsilon \sim \epsilon_c$ and $p \sim p_c$, the Green function has the form (3.3). (We do not discuss the completely analogous arguments for the S- and V-modes).

7. THE REGION BELOW THE TRANSITION (He II)

The rigorous consideration of the He II problem is extremely complicated owing to the fact that the Green function and vertex parts at sufficiently low ϵ and p are subject to the equations of hydrodynamics and therefore do not have a definite dimensionality. Using unitarity, it is difficult to trace how self-consistency comes about in the region of small ϵ and p. Nevertheless, we can perform an estimate of the kinetic coefficients using considerations of the correspondence between the regions $T < T_\lambda$ and $T > T_\lambda$.

The point is that the spectra of He II and He I at $p \sim p_c$ must be matched, since the temperature fluctuations in a volume of dimensions p_c^{-3} are of order $T - T_\lambda$ and in these ranges the difference between He II and He I loses its meaning. Consequently, in He II at $k \sim p_c$ we have one-particle relaxation with frequency ϵ_c and thermal relaxation with frequency $\omega_T(p_c)$ (and, of course, S- and V-modes). Further, in the region of extremely low p (it will be stated below which p must be regarded as small), two-fluid hydrodynamics is applicable and the spectrum consists of second sound:

$$\omega_2 = \pm u_2 k + {}^1/_2 i D_2 k^2 \quad (7.1)$$

and, as before, the S- and V-modes. We shall assume that the two frequencies (7.1) are transformed into ϵ_c and $\omega_T(p_c)$ as k increases. Formula (7.1) ceases to be true as soon as $\omega_2(k)$ becomes of the order of the smallest of ϵ_c and $\omega_T(p_c)$. Consequently, two-fluid hydrodynamics is applicable when

$$k < k_2 = u_2^{-1} \min\,(\omega_T(p_c), \varepsilon_c) = \tau^{|\gamma-1|} \ll p_c \quad (7.2)$$

(account has been taken of the fact that the velocity of

the second sound $u_2 \sim \sqrt{\rho_S/C\rho}$, $u_2 p_C \sim \tau$). Knowing the character of the spectrum, one can use the Kubo formulae to evaluate the kinetic coefficients for a superfluid liquid[14]:

$$\begin{gathered}\lambda = \omega^{-1}\operatorname{Im}\langle J_x^S J_x^S\rangle, \quad \eta = \omega^{-1}\operatorname{Im}\langle \tau_{xy}\tau_{xy}\rangle,\\ \zeta_1 = \omega^{-1}\operatorname{Im}\langle \hat{\mu}\hat{X}\rangle, \quad {}^4/_3\eta + \zeta_2 = \omega^{-1}\operatorname{Im}\langle \hat{X}\hat{X}\rangle,\\ \zeta_3 = \omega^{-1}\operatorname{Im}\langle \hat{\mu}\hat{\mu}\rangle;\\ \text{when } k = 0 \quad \omega \to 0.\end{gathered} \tag{7.3}$$

Here J_x^S, τ_{xy} and X are the same as in Sec. 6; ζ_1, ζ_2 and ζ_3 are bulk viscosities and $\hat{\mu}$ is the chemical potential operator, related to the superfluid velocity v_S by the formula

$$\mathbf{v}_s = \nabla\hat{\mu}. \tag{7.4}$$

The main contribution to the kinetic coefficients arises from the region $k \sim p_C$, where the characteristic frequencies are the same as in He I. (The region $k \lesssim k_2$ has small statistical weight and is therefore insignificant). Consequently, the most singular parts of λ, η and ζ_2 are the same in He II as in He I.

To estimate ζ_1 and ζ_3 we make use of the fact that at low frequencies we may consider the operator $\hat{\mu}$ to be an equilibrium operator, i.e.,

$$\hat{\mu} \approx \left(\frac{\partial\mu}{\partial\rho}\right)_E \hat{\rho} + \mu\left(\frac{\partial\mu}{\partial E}\right)_\rho \hat{\rho} + \left(\frac{\partial\mu}{\partial E}\right)_\rho (\hat{E} - \mu\hat{\rho}). \tag{7.5}$$

Simple transformations lead to the formulae

$$\begin{gathered}\left(\frac{\partial\mu}{\partial E}\right)_\rho = -\frac{P_{T\mu}}{T(P_{TT}P_{\mu\mu} - P_{T\mu}^2)},\\ \left(\frac{\partial\mu}{\partial\rho}\right)_E + \mu\left(\frac{\partial\mu}{\partial E}\right)_\rho = P_{TT}(P_{TT}P_{\mu\mu} - P_{T\mu}^2)^{-1}.\end{gathered} \tag{7.6}$$

Comparing (7.6) with (5.32) and (5.30) we come to the conclusion that the singular parts of the operators $\hat{\mu}$ and X are identical. Consequently,

$$\zeta_1 \sim \zeta_2 \sim \zeta_3. \tag{7.7}$$

8. CONCLUSION

In conclusion we shall enumerate the results obtained and compare them with experiment. In the above-critical region (He I) there is a long-lived one-particle relaxation mode, the spectrum $\epsilon(p)$ of which obeys the similarity relation:

$$\varepsilon(p) = \tau^{-\nu}\varphi(p\tau^{-\beta}), \tag{8.1}$$

where γ is the new dynamical index and $\tau^{-\beta}$ is the static correlation range. The mode $\epsilon(p)$ has an essential effect on the spectrum of the hydrodynamic modes. Thus, the temperature relaxation frequency $\omega_T(k)$ is

$$\omega_T(k) = \frac{\lambda k^2}{C_P} \sim \frac{1}{\tau^{\gamma\beta-2}}\frac{1}{\tau^{\gamma-\beta}}k^2. \tag{8.2}$$

Formula (8.2) is applicable if

$$k < \min(\tau^\beta, \tau^{1-\gamma+\beta}). \tag{8.3}$$

The anomalous thermal conductivity (8.2) is physically linked with the fact that, with changes of temperature in the system, displacement of the distribution of the Bose field ψ from the equilibrium distribution arises. The succeeding, and, according to (8.1), very slow, relaxation of this distribution gives rise to a large energy dissipation.

Experimentally[15], the thermal conductivity

$$\lambda \infty (T - T_\lambda)^{-1/2}. \tag{8.4}$$

Putting the value $\beta \approx {}^2/_3$ into (8.2), we find

$$\gamma \approx 1.$$

The relation $\gamma = 1$ was assumed in[2,3,16] Theoretical reasons for the exact fulfillment of this equality (as for the static equality $\beta = {}^2/_3$ too) apparently do not exist.

Anomalous damping of sound, connected with the second viscosity ρ, can be induced by two different mechanisms. For $\gamma < 1$ the main one is the decay of the sound into one-particle excitations, and ζ is determined by (5.34). For $\gamma < 1$ decays to the T-modes are dominant and ζ is (5.38).

The experimental data on sound absorption[17] agree poorly with the hypothesis $\gamma = 1$. The reason for the discrepancy is not clear at the present time. Experiments at lower frequencies than were used in[17] and at $T - T_\lambda \sim 10^{-6}$ would be able to clarify the situation.

In He II the experiments on first and second sound absorption agree well with the formulae of Sec. 7, if $\gamma \approx 1$.

We emphasize, finally, that the theory developed enables us to calculate the spectra of the collective modes only in the hydrodynamic regime, i.e., for $k < p_C$, $\omega < \epsilon_S$ and says nothing about the behavior of the spectra at large k.

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APPENDIX A

To derive the unitarity condition it is convenient to use the ordered diagram technique demonstrated in[19,20]. By the rules of this technique each diagram for Σ is proportional to a product of energy denominators

$$\Sigma_R \infty \prod_i \frac{1}{(\omega - E_i - i0)}, \tag{A.1}$$

where each denominator corresponds to a possible split of the diagram and E_i is the sum of the energies of the particles going across the point at which the diagram is split. To calculate the imaginary part of the expression (A.1) we use the equality

$$\operatorname{Im}\prod_{i=1}^{n}\frac{1}{\omega - E_i - i0} = \sum_{a=1}^{n}\pi\delta(\omega - E_a)\prod_{i<a}\frac{1}{\omega - E_i + i0}\cdot\prod_{i>a}\frac{1}{\omega - E_i - i0}. \tag{A.2}$$

Thus, to calculate the imaginary part of a diagram it is necessary to replace the denominator at each successive split by a δ-function, convert the denominators on the left into their complex conjugates and leave those on the right unchanged.

We shall show that the left amplitude Γ, expressed by the product with $i < a$ and depending on $z_i \ldots z_k$ (where k is the number of particles at the split a and z_i are the energies of these particles), is analytic for $\operatorname{Im} z_i > 0$ (all particles are assumed outgoing from Γ). For the proof we note that among the splits are some where the outgoing lines from Γ are not cut and some

where these lines are cut. The contribution of the denominators of the first type is

$$[\omega - E + i0]^{-1} = \Big(\sum z_i - E + i0\Big)^{-1} \tag{A.3}$$

(E are the energies of the internal particles). The denominators (A.3), clearly, are analytic for Im $z_i > 0$. The denominators of the second type have the form

$$\Big(\omega - E - \sum{}' z_i + i0\Big)^{-1} = \Big(\sum{}'' z_i - E + i0\Big)^{-1} \tag{A.4}$$

(where Σ' is a sum over the cut outgoing lines and Σ'' a sum over the uncut ones). The denominators (A.4) are also analytic for Im $z_i > 0$; thus our assertion is proved.

The ordered diagrams are the analytic continuation of the Matsubara diagrams and the condition Im $z_i > 0$ uniquely defines the path of the continuation. In the time representation the analyticity condition when Im $z_i > 0$ leads to the fact that the left amplitude is advanced in these times and the right one is, correspondingly, retarded.

We arrive at the following rule for calculating the imaginary part of the diagram for Σ_R. We successively carry out all possible divisions of the diagram. With a division we associate $\pi\delta(\omega - \Sigma\epsilon_i)$ and the factor

$$\prod_i n(\varepsilon_i) - \prod_i (1 + n(\varepsilon_i)), \tag{A.5}$$

where $n(\epsilon_i) = (e^{\beta\epsilon_i} - 1)^{-1}$. The creation amplitudes are the analytic continuation of the Matsubara amplitudes by the path indicated above.

These rules differ from the rules of the relativistic theory[21] by the unimportant factors $n(\epsilon)$ in the statistical weight and therefore enable us to obtain the unitarity condition. Collecting diagrams and replacing $n(\epsilon_i)$ by $(\beta\epsilon_i)^{-1}$ we arrive at the diagrammatic equations (3.1). In the general case it is necessary to insert (A.5) in place of $\epsilon/\pi\epsilon_i$ in Eq. (3.1).

APPENDIX B

To derive the identities used in the text we note that, since for our purposes the structure of the interaction between the He^4 atoms is not important, this interaction may be assumed to be local. The technique for obtaining identities of the Ward type for the vertices of conserved fluxes in local field theory is well-known[22]. In our case the vertex parts are analytic continuations of Matsubara T-products[6]. For example, $\mathcal{T}[J^\epsilon]$ is related to the quantity

$$\langle T_\tau J^\epsilon(x\tau)\psi(x_1\tau_1)\psi^+(x_2\tau_2)\rangle.$$

Using the conservation law: $\epsilon + \nabla J^\epsilon = 0$ and the canonical commutation relations, we find

$$\begin{aligned} -G(\mathbf{p},\varepsilon)G(\mathbf{p}+\mathbf{q}\,\varepsilon+\omega)\{\omega\mathcal{T}^\epsilon(\mathbf{p},\varepsilon,\mathbf{q},\omega) - \mathbf{q}\vec{\mathcal{T}}^\epsilon(\mathbf{p},\varepsilon,\mathbf{q},\omega)\} \\ = \frac{\mathbf{p}(\mathbf{p}+\mathbf{q})}{2}[G(\mathbf{p}+\mathbf{q},\varepsilon+\omega) - G(\mathbf{p},\varepsilon)] \\ + \Big[\Big(\varepsilon+\omega-\frac{(\mathbf{p}+\mathbf{q})^2}{2}\Big)G(\varepsilon+\omega,\mathbf{p}+\mathbf{q}) - \Big(\varepsilon-\frac{\mathbf{p}^2}{2}\Big)G(\mathbf{p},\varepsilon)\Big]. \end{aligned} \tag{B.1}$$

Here $\mathcal{T}^\epsilon$ is the energy density vertex and $\vec{\mathcal{T}}^\epsilon$ is the energy flux density vertex.

We must bear in mind that by the energy here is meant the quantity

$$\varepsilon = E - \mu\rho, \tag{B.2}$$

where E is the energy density in the usual sense. Therefore, the entropy flux J^S is

$$\mathbf{J}^S = \mathbf{J}^E - \frac{w}{\rho}\mathbf{J} = \mathbf{J}^\varepsilon - \frac{w-\mu\rho}{\rho}\mathbf{J} = \mathbf{J}^\varepsilon - T\frac{s}{\rho}\mathbf{J} \tag{B.3}$$

(w is the enthalpy and s the entropy per unit volume).

Thus

$$\lim_{\mathbf{k},\,\omega\to 0}\mathcal{T}[J_y^S] = \lim_{\mathbf{k},\,\omega\to 0}\mathcal{T}[J_y^\varepsilon] + T\frac{s}{0}\frac{\partial G^{-1}}{\partial p_y}. \tag{B.4}$$

From (B.1) it follows that

$$\mathcal{T}[\mathbf{J}^\varepsilon] = -[\varepsilon\partial G^{-1}/\partial\mathbf{p} + G^{-1}\mathbf{p}]; \quad \text{for } \omega = 0,\ \ \mathbf{q}\to 0. \tag{B.5}$$

Formulas (B.4) and (B.5) yield the identity (5.8).

To derive formula (5.26) we need a Ward identity for the energy-momentum tensor; this is obtained by the standard methods:

$$\begin{aligned} \omega\mathcal{T}[J_\alpha] - q_\beta\mathcal{T}[\tau_{\alpha\beta}] \\ = p_\alpha G^{-1}(p,\varepsilon) - (p+q)_\alpha G^{-1}(p+q,\varepsilon+\omega). \end{aligned}$$

Assuming in (B.5) that $\omega = 0$ and $\mathbf{q} \to 0$, we obtain (5.26).

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