GENERAL SOLUTION OF THE GRAVITATIONAL EQUATIONS WITH A PHYSICAL SINGULARITY

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An attempt is made to set up a general solution of the Einstein equations with a physical singularity with respect to time. The general solution possesses the same time behavior as in the case of a metric with a three-parameter motion group of the ninth Bianchi type.

OUR investigations (together with E. M. Lifshitz)[1,2] have made it possible to conclude that there exists a general solution of the gravitational equations with a physical singularity, and to clarify the qualitative character of the evolution of the metric on approaching the singular point.

In this paper we investigate a general solution, containing a simultaneous physical singularity in time, of the Einstein equations. By general solution we mean a solution in which the physical arbitrariness is determined by four arbitrary functions of three spatial coordinates in vacuum and by eight such functions in a space with matter[3]. The obtained solution contains as a particular case the solution investigated by us in[1], and has qualitatively the same time behavior.

1. INTRODUCTION

We recall here brieflv the main results of[1]. We consider a metric in the form

$$-ds^2 = -dt^2 + (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\, dx^\alpha dx^\beta, \tag{1.1}$$

where a, b, and c depend only on the time t, while the vectors **l**, **m**, and **n** depend only on the spatial coordinates x, y, and z. These vectors satisfy the following conditions[1)]:

$$\begin{aligned} &\mathbf{l}\,\mathrm{rot}\,\mathbf{l} = \lambda, && \mathbf{m}\,\mathrm{rot}\,\mathbf{l} = 0, && \mathbf{n}\,\mathrm{rot}\,\mathbf{l} = 0,\\ &\mathbf{l}\,\mathrm{rot}\,\mathbf{m} = 0, && \mathbf{m}\,\mathrm{rot}\,\mathbf{m} = \mu, && \mathbf{n}\,\mathrm{rot}\,\mathbf{m} = 0,\\ &\mathbf{l}\,\mathrm{rot}\,\mathbf{n} = 0, && \mathbf{m}\,\mathrm{rot}\,\mathbf{n} = 0, && \mathbf{n}\,\mathrm{rot}\,\mathbf{n} = \nu,\\ &\mathbf{l}[\mathbf{mn}] = 1, && \lambda, \mu, \nu = \mathrm{const.} \end{aligned} \tag{1.2*}$$

(The solution of Eqs. (1.2) is given in the appendix of[1]). The Einstein equations $R_{ik} = 0$ then reduce to a system of ordinary differential equations in time for the functions a, b, and c. If we define a new variable τ by means of the relation

$$d\tau = dt / abc, \tag{1.3}$$

then the indicated system of equations takes the form

$$(\ln a^2)_{\tau\tau} = (\mu b^2 - \nu c^2)^2 - \lambda^2 a^4, \tag{1.4}$$

$$(\ln b^2)_{\tau\tau} = (\lambda a^2 - \nu c^2)^2 - \mu^2 b^4,$$

$$(\ln c^2)_{\tau\tau} = (\lambda a^2 - \mu b^2)^2 - \nu^2 c^4;$$

$$(\ln a)_\tau (\ln b)_\tau + (\ln a)_\tau (\ln c)_\tau + (\ln b)_\tau (\ln c)_\tau = \tfrac{1}{2}(\ln abc)_{\tau\tau}. \tag{1.5}$$

The solution of these equations, which were investigated also by Misner[4], has the following properties.

1. Since three diagonal scalar products **l** · curl **l**, **m** · curl **m**, and **n** · curl **n** do not vanish simultaneously, the singularity[2)] cannot be of the Kasner type[3] $a^2, b^2, c^2 \sim t^{2p_1}, t^{2p_2}, 2^{2p_3}$, since the latter arises only when at least one of the indicated scalar products vanishes, for example **l** · curl **l** = 0. This gives rise to a preferred direction **l**, along which the scales expand without limit as $t \to 0$ (the corresponding projection of the metric tensor is $g_{\alpha\beta} l^\alpha l^\beta = a^2 \sim t^{2p_1}$, $p_1 < 0$).

2. The solution has likewise no singularity of fictitious character $(p_1, p_2, p_3) = (0, 0, 1)$. The only exception is the particular case when $\lambda a^2 = \mu b^2$. However, as shown in[1,5], such a solution is unstable against a small perturbation of the type $\lambda a^2 - \mu b^2 \neq 0$.

3. When τ (1.3) changes from 0 to $+\infty$ ($\tau \to +\infty$ corresponds to $t \to 0$), the solution runs through an infinite sequence of qualitatively identical periods, which are joined by relatively short intermediate regions. The start of each such period is characterized by the fact that one of the functions, a, b, or c, begins to decrease approximately like t^2 and subsequently (during the extent of the entire period) it becomes much smaller than the two others, which oscillate against a background of a slow decrease (compared with t^2). During the final stage of the period, the small function begins to increase and the solution enters into a region (which is short compared with the length of the period itself) in which it is described by the Kasner asymptotic form. After such a short region, an analogous period again sets in, but another function becomes small. The lengths of these periods increase as $\tau \to \infty$. Thus, on approaching the singularity, an infinite number of changes takes place in the direction of the axis, along which the relative rapid contraction of the three-space whose volume tends to zero like abc, takes place.

The period described above is in accord with the "ground state" in which the system spends most of the time, and an analytic solution in its region has been considered by us in Sec. 4 of[1].

1)The algebraic and differential vector operations are carried out as if the three dimensional space were flat.

*[mn] ≡ m × n.

2)The singularity (which is inevitable in the synchronous reference frame) takes place at the point where the determinant $a^2 b^2 c^2$ vanishes. Taking into account the possibility of the transformation $t \to t + c - t$, we assume that it corresponds to the point t = 0. In terms of the variable τ this corresponds to $\tau \to \pm\infty$.

2. CASE OF ARBITRARY GRAVITATIONAL FIELD

In connection with the foregoing, the following fundamental question arises.

To what degree is the time picture of the metric evolution, which takes place in the particular solution of (1.1)—(1.5) investigated by us, a characteristic of the general solution of the gravitational equations?

The only way of answering this question is to attempt to construct for Einstein's equations a general solution having the same character of the behavior as in the particular case considered here. We have seen that the investigated solution consists of an infinite sequence of qualitatively indistinguishable periods, in each of which one of the functions a, b, or c is much smaller than the other two, and which are connected by short intermediate regions (compared with the length of the period itself). We shall now show that the general solution also has similar properties.

We define the coordinate system by means of the conditions

$$-g_{00}=g_{33}, \quad g_{0\alpha}=0. \tag{2.1}$$

Henceforth, the Greek indices assume the values 1, 2, and 3, the Latin indices i and k the values 0, 1, 2, and 3, and the Latin indices a, b, and c the values 1 and 2. The time variable will be denoted by the letter ξ (to distinguish it from the synchronous time t), and the coordinate x^3 by the letter z. Differentiation with respect to these coordinates is designated by a dot and by a prime, respectively.

Thus, the metric takes the form

$$-ds^2=g_{33}(dz^2-d\xi^2)+g_{ab}dx^a dx^b+2g_{a3}dx^a dz. \tag{2.2}$$

We seek the solution in that region of variation of ξ (assuming its existence), where the components of the metric tensor are satisfied by the conditions

$$g_{33}\ll g_{ab}, \quad g_{a3}\ll\sqrt{g_{33}g_{aa}}. \tag{2.3}$$

This means that the components g_{a3} are small and the first approximation to the solution will be the metric (2.2) under the condition $g_{a3}=0$. In this approximation it is meaningless to write out the components of the Ricci tensor R^0_a and R^3_a, since they determine g_{a3} in their first order. Writing out the remaining components R^0_0, R^0_3, R^3_3, and R^b_a (calculated from the metric (2.2) under the condition $g_{a3}=0$), it is easy to verify that all their terms containing differentiation with respect to the coordinates x^a are small compared with the terms containing the differentiation with respect to ξ and z (the ratio of the former to the latter is just of the order g_{33}/g_{ab}).

Thus, to obtain the principal-approximation equations, it is necessary to put $g_{a3}=0$, and the components g_{33} and g_{ab} must be differentiated as if they were independent of the variables x^a. We note that if we extend this rule also to R^0_a and R^3_a, then we obtain simply $R^0_a\equiv 0$ and $R^3_a\equiv 0$.

Putting

$$g_{33}=c^2, \quad |g_{ab}|=G, \quad \dot g_{ab}=\varkappa_{ab}, \quad g_{ab}'=\lambda_{ab}, \tag{2.4}$$

we obtain the following principal-approximation equations

$$2c^2R_a{}^b=\frac{1}{\sqrt{G}}(\sqrt{G}\,\varkappa_a{}^b)^{\cdot}-\frac{1}{\sqrt{G}}(\sqrt{G}\,\lambda_a{}^b)'=0, \tag{2.5}$$

$$2c^2R_3{}^0=-\varkappa(\ln c)'-\lambda(\ln c)^{\cdot}+\varkappa'+{}^1\!/_2\varkappa_b{}^a\lambda_a{}^b=0, \tag{2.6}$$

$$2c^2(R_0{}^0-R_3{}^3)=-2\lambda(\ln c)'-2\varkappa(\ln c)^{\cdot}+\dot\varkappa+\lambda'+{}^1\!/_2\varkappa_a{}^b\varkappa_b{}^a+{}^1\!/_2\lambda_a{}^b\lambda_b{}^a=0. \tag{2.7}$$

The raising and lowering of the indices is carried out here with the aid of g_{ab}. The quantities $\varkappa$ and λ are the contractions of $\varkappa^a_a$ and λ^a_a, with

$$\varkappa=(\ln G)^{\cdot}, \quad \lambda=(\ln G)'. \tag{2.8}$$

We have not written out here the equation

$$2c^2(R_0{}^0+R_3{}^3)=4(\ln c)^{\cdot\cdot}-4(\ln c)''-\lambda'+\dot\varkappa+{}^1\!/_2\varkappa_a{}^b\varkappa_b{}^a-{}^1\!/_2\lambda_a{}^b\lambda_b{}^a=0.$$

However, it is easy to verify that it is a direct consequence of the system (2.5)—(2.7) in the case when $\dot G\neq 0$ or $G'\neq 0$. But when $\dot G=G'=0$ we arrive at a flat space, and this case will therefore not be considered.

Taking the contraction of (2.5), we obtain

$$(\sqrt{G})^{\cdot\cdot}-(\sqrt{G})''=0; \tag{2.9}$$

Consequently

$$\sqrt{G}=f_1(x,y,\xi+z)+f_2(x,y,\xi-z). \tag{2.10}$$

Different cases can occur here, depending on the character of the variable $\sqrt{G}$, i.e., on the value of the norm $N=g^{ik}(\sqrt{G})_{,i}(\sqrt{G})_{,k}$.

In this approximation, $g^{00}=-g^{33}\gg g^{ab}$, and consequently

$$N=g^{00}(\sqrt{G})^{\cdot 2}+g^{33}(\sqrt{G})'^2=-4g^{33}f_1f_2.$$

If $N>0$, then $\sqrt{G}$ is a space-like variable and it is easy to show that by means of the remaining coordinate transformations, which do not violate the conditions (2.1) and the inequalities (2.3), it is possible to choose $\sqrt{G}$ in this approximation in the form $\sqrt{G}=f(x,y)z$.[3)] Such a case leads to the generalization of the well known Einstein-Rosen metric[6]. In the case when $N=0$, we arrive at the Robinson-Bondi wave metric[7], which depends only on $\xi+z$ or only on $\xi-z$. On the other hand, if $N<0$, then $\sqrt{G}$ is timelike and, using the remaining transformations, we can, without loss of generality, choose

$$\sqrt{G}=f(x,y)\xi, \tag{2.11}$$

where f(x, y) is an arbitrary function. This is the case that we shall investigate further, since it is connected with the time singularities of interest to us. We emphasize that (2.11) is possible only in the principal approximation. When further corrections are taken into account the determinant, of course, will not have such a simple form.

Under the condition (2.11), Eqs. (2.5)—(2.7) take the form

$$\dot\varkappa_a{}^b+\xi^{-1}\varkappa_a{}^b-\lambda'_a{}^b=0, \tag{2.12}$$

$$\dot\psi=-\xi^{-1}+{}^1\!/_4\xi(\varkappa_a{}^b\varkappa_b{}^a+\lambda_a{}^b\lambda_b{}^a), \tag{2.13}$$

$$\psi'={}^1\!/_2\xi\varkappa_a{}^b\lambda^a{}_b, \tag{2.14}$$

3) In the case when the metric does not depend on x and y, this possibility follows directly from the presence of other allowed transformations of the coordinates of the type $z'=f_1(\xi+z)+f_2(\xi-z)$ and $\xi'=f_1(\xi+z)-f_2(\xi-z)$, which do not violate the form of $g_{33}(dz^2-d\xi^2)$. Such a choice remains possible also when the metric depends on x and y, but only in the principal approximation, as can be readily shown by operating with infinitesimally small transformations.

where

$$c^2 = e^{\psi}. \tag{2.15}$$

The fundamental equations here are (2.12), which determine the components g_{ab}, with which ψ is then obtained from (2.13) and (2.14) by simple integration. If Eqs. (2.12) are solved, then Eqs. (2.13) and (2.14) do not contradict each other, and the role of one of them reduces to a definition of the arbitrary integration function of the other.

To investigate the solution (2.12), we consider the variation of ξ from $+\infty$ to 0 and the corresponding asymptotic regions $\xi \gg 1$ and $\xi \ll 1$.

In the region $\xi \gg 1$, Eqs. (2.12) are best replaced by a single matrix equation. We take the matrix g (with components g_{ab}) in the form[4)]

$$g = f(x, y)\xi e^{H}, \tag{2.16}$$

where the matrix H is symmetrical and its trace is equal to zero. This ensures symmetry of g and the condition (2.11). Expressing H in the form

$$H = \begin{pmatrix} \chi & \varphi \\ \varphi & -\chi \end{pmatrix}, \quad \chi^2 + \varphi^2 = \sigma^2, \tag{2.17}$$

we can readily obtain an expression for g in terms of the components:

$$g = f\xi \begin{pmatrix} \operatorname{ch}\sigma + \chi\sigma^{-1}\operatorname{sh}\sigma & \varphi\sigma^{-1}\operatorname{sh}\sigma \\ \varphi\sigma^{-1}\operatorname{sh}\sigma & \operatorname{ch}\sigma - \chi\sigma^{-1}\operatorname{sh}\sigma \end{pmatrix}. \tag{2.18}$$

The system (2.12)—(2.15), expressed in terms of H, yields

$$\begin{aligned} \ddot{H} + \xi^{-1}\dot{H} - H'' = {} & (I - H\operatorname{cth} H)H^{-2}(H'S - \dot{H}T) \\ & + {}^1/_4(I - 2H\operatorname{cth} H + {}^1/_2H^{-1}\operatorname{sh} 2H)H^{-3}(S^2 - T^2), \end{aligned} \tag{2.19}$$

$$\dot{\psi} = -{}^1/_2\xi + {}^1/_4\xi\,\mathrm{Sp}[(\dot{H} + QT)^2 + (H' + QS)^2], \tag{2.20}$$

$$\psi' = {}^1/_2\xi\,\mathrm{Sp}[(\dot{H} + QT)(H' + QS)], \tag{2.21}$$

where

$$T = \dot{H}H - H\dot{H}, \; S = H'H - HH', \quad Q = {}^1/_2(I - H^{-1}e^{-H}\operatorname{sh} H)H^{-1} \tag{2.22}$$

and I is a unit matrix.

Such a representation itself leads to relatively complicated equations (for which simpler forms exist), but for the approximate construction as needed by us this representation is the best. We consider for (2.19) a solution that is periodic in the variable z and can be represented by a Fourier series in terms of this variable. It can be understood as a solution that is periodic in z in the entire range of variation of z, or else as a Fourier expansion of a nonperiodic solution, but in a finite region of variation of z.

It is easy to see that in this case all the components of the matrix H tend to zero like $1/\sqrt{\xi}$ when $\xi \to \infty$. Indeed, by virtue of the obvious expansions

$$\begin{aligned} (I - H\operatorname{cth} H)H^{-2} &= -{}^1/_3I + {}^1/_{45}H^2 + O(H^4), \\ {}^1/_4(I - 2H\operatorname{cth} H + {}^1/_2H^{-1}\operatorname{sh} 2H)H^{-3} &= {}^2/_{45}H + O(H^3) \end{aligned} \tag{2.23}$$

the right side of (2.19) appears only in the order H^3. Then the linear approximation, under the condition of periodicity in z, yields $H \sim 1/\sqrt{\xi}$, since we obtain the Bessel equation for the Fourier coefficients of $H_k(\xi)$.

Thus, the asymptotic solution of (2.19) at large values of ξ must be sought in the form

$$H = \frac{1}{\sqrt{\xi}} \sum_{k=-\infty}^{+\infty} [A_k e^{ik\omega(\xi+z)} + B_k e^{ik\omega(\xi-z)}] + O(\xi^{-3/2}), \tag{2.24}$$

where A_k and B_k are symmetrical matrices of the amplitudes, having a zero trace and satisfying the condition of H be real: $A_{-k} = A_k^*$, $B_{-k} = B_k^*$.

A more detailed analysis of Eq. (2.19) shows that the linear approximation is not perfectly valid, and that allowance for the nonlinear terms in the right side of the equation causes the amplitudes A_k and B_k themselves to be functions of ξ, but slowly varying compared with $e^{ik\omega\xi}$. The final asymptotic form of the solution can be written as

$$H = \begin{bmatrix} h + h^* & i(h - h^*) \\ i(h - h^*) & -(h + h^*) \end{bmatrix}, \tag{2.25}$$

$$h = \frac{1}{\sqrt{\xi}} \sum_{k=-\infty}^{+\infty}{}' [\chi_k e^{ik\omega(\xi+z)+ia\ln\xi} + \varphi_k e^{ik\omega(\xi-z)+ib\ln\xi}] + O(\xi^{-3/2}). \tag{2.26}$$

Here χ_k and φ_k with $k \neq 0$ are arbitrary complex functions of the variables x and y. The fundamental frequency ω is an arbitrary real function of x and y. The quantities a and b are real and are expressed when $k \neq 0$ in terms of ω, χ_k, and φ_k in the following manner:

$$a = -4\omega \sum_{-\infty}^{+\infty}{}' k\varphi_k\varphi_k^*, \quad b = -4\omega \sum_{-\infty}^{+\infty}{}' k\chi_k\chi_k^*. \tag{2.27}$$

The components χ_0 and φ_0 are also uniquely expressed in terms of ω, χ_k, and φ_k (for $k \neq 0$)[5)]

$$\chi_0 = \frac{8\omega(2a + i)}{1 + 4ab + 2i(b - a)} \sum_{k,k'=-\infty}^{+\infty}{}' k(\chi_{-k}\chi_{k'}^* + \chi_{-k'}\chi_k^*)\chi_{k+k'},$$

$$\varphi_0 = \frac{8\omega(2b + i)}{1 + 4ab + 2i(a - b)} \sum_{k,k'=-\infty}^{+\infty}{}' k(\varphi_{-k}\varphi_{k'}^* + \varphi_{-k'}\varphi_k^*)\varphi_{k+k'}. \tag{2.28}$$

In formulas (2.27) and (2.28) the symbol Σ' denotes that the terms containing χ_0 and φ_0 have been excluded from the sum (i.e., as if we had $\chi_0 = \varphi_0 = 0$).

Thus, in the region $\xi \gg 1$ the components of the matrix of the metric tensor g_{ab} oscillate with decreasing ξ against the background of a slow decrease (this decrease is due to the factor ξ in (2.16)).

We now consider the region $\xi \ll 1$. It is easy to show that in the region of small ξ the principal approximation of the solution of (2.12) is obtained from the assumption (confirmed by the result) that the principal terms in the equation are those containing differentiation with respect to ξ. Consequently, we have here

$$\dot{\varkappa}_a^{\,b} + \xi^{-1}\varkappa_a^{\,b} = 0, \quad |g_{ab}| = f^2\xi^2. \tag{2.29}$$

The solution of (2.29) is

$$\begin{aligned} g_{ab} &= l_a l_b \xi^{2p_1} + m_a m_b \xi^{2p_2}, \\ l_1 m_2 - l_2 m_1 &= f, \quad p_1 + p_2 = 1. \end{aligned} \tag{2.30}$$

Here l_a, m_a, and p_a are arbitrary functions of x, y, and z. The subsequent terms of the expansion can be readily

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4)The functions of matrices are defined by their Taylor series.

5)The differential equations for the slowly varying amplitudes A_0 and B_0 (unlike the equations for A_k, B_k, $k \neq 0$) are linear and inhomogeneous equations, all the coefficients of which as well as the free part are expressed in terms of A_k and B_k with $k \neq 0$. The general solution of the homogeneous equations (containing arbitrary integration functions) does not satisfy the requirement that the variation of A_0 and B_0 be slow. This requirement is satisfied only by a particular solution of these equations, which indeed leads to the conditions (2.28).

obtained in the form of expansions in powers of ξ. The condition for the applicability of the approximation (2.30) is obtained from the requirement that the terms $\lambda_a^{\prime b}$ which have been discarded from (2.12) be small compared with those retained terms $\dot{\kappa}_a^b$, namely $\xi^{-1}\kappa_a^b \sim \xi^{-2}$. Calculating $\lambda_a^{\prime b}$ from (2.30), we get

$$\lambda_a^{\prime b} \sim (l^b m_a l^c m_c' \xi^{-2+4p_2} + m^b l_a m^c l_c' \xi^{-2+4p_1})'.$$

It follows therefore that the condition $\lambda_a^{\prime b} \ll \xi^{-2}$ will be satisfied if

$$p_1 > 0, \quad p_2 > 0. \tag{2.31}$$

Thus, it is necessary to take for p_a the parametric representation

$$p_1 = 1/2 + 1/2 \sin u, \quad p_2 = 1/2 - 1/2 \sin u, \tag{2.32}$$

where u is an arbitrary three-dimensional function.

We now can obtain the function ψ from (2.20) and (2.21) for the region $\xi \gg 1$ and from (2.13) and (2.14) for $\xi \ll 1$. A simple calculation yields in the region $\xi \gg 1$:

$$\psi = 2\omega^{2\xi} \sum_{-\infty}^{+\infty}{}' k^2(\chi_k\chi_k^* + \chi_{-k}\chi_{-k}^* + \varphi_k\varphi_k^* + \varphi_{-k}\varphi_{-k}^*) + s\ln\xi + 2\omega^2 z \sum_{-\infty}^{+\infty}{}' k^2(\chi_k\chi_k^* + \chi_{-k}\chi_{-k}^* - \varphi_k\varphi_k^* - \varphi_{-k}\varphi_{-k}^*) + \psi_0(x, y)$$

+ (terms of order of unity, oscillating in ξ and periodic in z) + $O(\xi^{-1})$. (2.33)

For the case of a metric that is periodic with respect to the variable z, it is necessary to require that the coefficient of z in formula (2.33) vanish, i.e., the two-dimensional functions χ_k and φ_k (which depend on x and y) should satisfy the condition

$$\sum_{-\infty}^{+\infty}{}' k^2(\chi_k\chi_k^* + \chi_{-k}\chi_{-k}^* - \varphi_k\varphi_k^* - \varphi_{-k}\varphi_{-k}^*) = 0.$$

It is obvious that such a requirement does not limit the generality of the solution described by the three-dimensional functions of the variables x, y, and z. The entire small term denoted by $O(\xi^{-1})$ also contains only a periodic dependence on z. The function $\psi_0(x, y)$ is arbitrary. The quantity s is a function of x and y only, and is expressed in terms of ω, χ_k, and φ_k. Its explicit form, however, can be obtained only when account is taken of terms of order $\xi^{-3/2}$ in the matrix H. It suffices for us to know only that s = s(x, y) (and this fact can be readily established also without the succeeding terms of the expansion in H).

In the region $\xi \ll 1$ we have

$$\psi = (p_1^2 + p_2^2 - 1)\ln\xi + \bar{\psi}_0(x, y) + 2\int (p_1 l^a l_a' + p_2 m^a m_a')\,dz, \tag{2.34}$$

where $\bar{\psi}_0(x, y)$ has an arbitrary integration function.

From the asymptotic formulas (2.33) and (2.34) we see quite clearly the character of the behavior of the function $c^2 = e^{\psi}$ ($g_{33} = -g_{00} = c^2$). When ξ decreases from a certain value, the function c^2 first decreases (in the region $\xi \gg 1$) practically exponentially:

$$c^2 \sim \xi^s e^{q\xi}, \tag{2.35}$$

where q is an essentially positive function of x and y:

$$q = 2\omega^2 \sum_{-\infty}^{+\infty}{}' k^2(\chi_k\chi_k^* + \chi_{-k}\chi_{-k}^* + \varphi_k\varphi_k^* + \varphi_{-k}\varphi_{-k}^*). \tag{2.36}$$

The presence of the arbitrary function $\psi_0(x, y)$ in ψ makes it possible to choose a sufficiently arbitrary initial value from which c^2 decreases.

The components g_{ab} hardly decrease in the region $\xi \gg 1$ compared with the exponential function, as follows from formulas (2.16) and (2.25) and (2.26). Thus, a region in which the inequality $c^2 \ll g_{ab}$, which is the first fundamental inequality of the entire approximation considered here, occurs still at large values of ξ.

At a certain value $\xi \lesssim 1$, we fall into the region described by formulas (2.30)–(2.32) and (2.34). Since $p_1, p_2 > 0$, the components g_{ab} decrease when $\xi \to 0$, whereas c^2 increases:

$$c^2 \sim \xi^{p_1^2+p_2^2-1}, \quad p_1^2 + p_2^2 - 1 = 1/2(\sin^2 u - 1) < 0. \tag{2.37}$$

Thus, in the region of small ξ there always sets in an instant at which c^2 becomes larger than the components g_{ab}, and the considered solution can only be used.

The foregoing analysis shows therefore that a solution satisfying the condition $c^2 \ll g_{ab}$, which was proposed from the very beginning, actually exists in a certain region of variation of ξ, which is bounded from both above and below[6].

It is now necessary to show that in this region there is satisfied also the main condition of applicability of the constructed approximation:

$$g_{a3} \ll \sqrt{g_{33}g_{aa}}.$$

To this end, it is necessary to consider the equation $R_a^0 = 0$, $R_a^2 = 0$, in their first approximation (i.e., it is necessary to assume that the inequality $g_{a3} \ll \sqrt{g_{33}g_{ab}}$ is satisfied), and after obtaining a solution for the components g_{a3} it is necessary to verify the validity of our assumption.

Calculating R_a^0 and R_a^3, we observe that they have the form

$$2c^2 R_a^3 = \frac{1}{\sqrt{G}}\left[\frac{\sqrt{G}}{c^2} g_{ac}(g^{cb}g_{b3})^{\cdot}\right]^{\cdot} + \lambda_{a;c}^{c} - \lambda_{;a} + \lambda(\ln c)_{;a} - 2(\ln c)_{;a}' + O(c^2), \tag{2.38}$$

$$2c^2 R_a^0 = -\frac{1}{\sqrt{G}}\left[\frac{\sqrt{G}}{c^2} g_{ac}(g^{cb}g_{b3})^{\cdot}\right]' - \varkappa_{a;c}^{c} + \varkappa_{;a} - \varkappa(\ln c)_{;a} + 2(\ln c)_{;a}^{\cdot} + O(c^2). \tag{2.39}$$

Taking into account the fact that $\sqrt{G} = f\xi$ ($\lambda = 0$, $\kappa = 2/\xi$) and the assumed notation $c^2 = e^{\psi}$, we obtain the equations of the principal approximation:

$$\xi^{-1}[\xi e^{-\psi} g_{ac}(g^{cb}g_{b3})^{\cdot}]^{\cdot} = \psi_{,a}' - \lambda_{a;c}^{c}, \qquad [e^{-\psi} g_{ac}(g^{cb}g_{b3})^{\cdot}]' = \psi_{,a} - \xi^{-1}\dot{\psi}_{,a} - \varkappa_{a;c}^{c}. \tag{2.40}$$

Here g^{cb} is the inverse of g_{cb}, and the covariant differentiation is carried out relative to g_{cb}, the solution from which was already obtained above.

Equations (2.40) are not contradictory and, by virtue of the Bianchi identities, one of these equations reduces to a determination of an arbitrary function of the first

[6] The upper bound of the region of applicability corresponds to a certain large value ξ_1 at which $c^2 \sim g_{ab}$ (and beyond which c^2 becomes much smaller than g_{ab}). The lower bound ξ_2 is found from the same condition in the region of small ξ.

integration of the other equation. Inasmuch as we are interested only in the behavior of g_{a3} relative to the variable ξ, we shall consider the first of the equations (2.40). Its solution can be readily expressed in terms of quadratures:

$$g_{a3} = g_{ca} \int j^c \, d\xi + g_{ca} \Phi^c(x, y, z),$$
$$j^c = \frac{1}{\xi} e^{\psi} g^{ca} \left[\int \xi \psi'_{,a} \, d\xi - \int \xi \lambda^{b}_{a;\,b} \, d\xi + F_a(x, y, z) \right]. \qquad (2.41)$$

Here Φ^c and F_a are four arbitrary functions of the spatial variables, $F_a(x, y, z)$ are the arbitrary functions of the first integration, and when (2.41) is substituted in the first equation of (2.40) we obtain the conditions $F'_a = 0$, i.e., $F_a = F_a(x, y)$. As to the functions Φ^c, we set them equal to zero, since in the region $\xi \gg 1$ the terms $g_{ca}\Phi^c$ in the component g_{a3} patently violate the condition $g_{a3} \ll \sqrt{g_{33}g_{aa}}$. It will become clear later, in the analysis of the degree of generality of the obtained solution, that the vanishing of Φ^c is not connected with a physical limitation, and is the result of the remaining permissible coordinate transformations.

It is now easy to see that in the region $\xi \gg 1$ (taking into account the asymptotic solutions obtained above for g_{ab} and ψ), the behavior of g_{a3} is determined by the common factor $c^2 (= e^{\psi})$, i.e., $g_{a3} \sim c^2$. We shall not consider in detail the complicated integrals that enter in (2.41), and it suffices to carry out a simple qualitative estimate, which shows that the terms of the highest order, which arise in the components g_{a3}, are of the form

$$g_{a3} = c^2 u_a, \qquad (2.42)$$

where u_a are bounded functions of ξ, and can be represented by Fourier series[7)].

In the region $\xi \ll 1$ it is possible to simply neglect the right side of the first equation of (2.40), since its order of magnitude is smaller than the order of magnitude of the left side $1/\xi^2$. In this case we have

$$g_{a3} = \left[\frac{l_a l^c}{2(p_2^2 - 1)} + \frac{m_a m^c}{2(p_1^2 - 1)} \right] F_a \xi^{p_1^2 + p_2^2 - 1} = c^2 f_a(x, y, z), \qquad (2.43)$$

We see therefore that in the region of large ξ the components g_{a3} decrease practically exponentially approximately like $g_{a3} \propto c^2$, and the second of the inequalities proposed by us, $g_{a3} \ll \sqrt{g_{33}g_{aa}}$, is satisfied in the region where $c^2 \ll g_{ab}$. In the region of small ξ, the components g_{a3} begin to increase, but again like c^2, i.e., the second inequality is violated as a result of violation of the first, and consequently also in the same region as the first.

We thus obtain the solution, satisfying our initial assumptions, of the equations $R_{ik} = 0$ in a certain limited region of ξ.

We now consider the question of the degree of its generality. The coordinate system is fixed in our case by the following conditions: 1) the four-dimensional conditions (2.1) hold; 2) the determinant $G = |g_{ab}|$ is equal to $f^2\xi^2$ in the principal approximation; 3) the terms $\sim g_{ab}$ are eliminated from the components g_{a3}[8)]; 4) the frequency ω is assumed constant; 5) the solution for g_{ab} is constructed in the first approximation in such a way

$$g_{ab} \to f\xi\delta_{ab} \text{ when } \xi \to \infty.$$

By carrying out a small four-dimensional coordinate transformation $x^i = \bar{x}^i + \eta^i(\bar{x}^k)$, we can readily verify that there are no transformations left containing any three-dimensional (and of course also four-dimensional) arbitrary functions without violating the five conditions indicated above. Among the transformations containing arbitrary two-dimensional functions, the only allowed remaining one is

$$z \to z + z_0(x, y).$$

Thus, one of the arbitrary functions of x and y in the constructed solution is not physical, and the entire three-dimensional arbitrariness will therefore be physical.

We see that the three-dimensional arbitrariness is contained precisely in the components g_{ab}. For the solution to be general it is necessary that the number of arbitrary functions of the spatial coordinates be equal to four. Turning to formula (2.30), we note that this is the case. In the region $\xi \ll 1$ we have six functions l_a, m_a, and p_a, which are connected by two algebraic conditions. In the region $\xi \gg 1$, it is necessary to consider the corresponding formulas (2.25)–(2.28). Taking for simplicity a small region of variation of ξ, in which it is possible to neglect the slow functions (the logarithmic phases), we obtain two independent real components of the matrix H:

$$H_{11} = h + h^* = \sum_j (\chi_k + \chi_{-k}^*) e^{ik\omega(\xi+z)} + \sum_j (\varphi_k + \varphi_{-k}^*) e^{ik\omega(\xi-z)},$$
$$H_{12} = i(h - h^*) = \sum_j i(\chi_k - \chi_{-k}^*) e^{ik\omega(\xi+z)} + \sum_j i(\varphi_k - \varphi_{-k}^*) e^{ik\omega(\xi-z)}. \qquad (2.44)$$

Inasmuch as χ_k and φ_k are fully arbitrary on the entire k axis (from $-\infty$ to $+\infty$), the equations (2.44) yield four arbitrary functions (two of x, y, $\xi + z$ and two of x, y, $\xi - z$). This, of course, is equivalent to four functions of x, y, and z. Thus, the physical arbitrariness in the solution is determined by four arbitrary functions of the variables x, y, and z, in full agreement with the requirement that the solution be general (see[3,8]).

3. CONCLUSION

Thus, the solution investigated by us is general. On the other hand, it contains as a particular case the metric (1.1)–(1.5), the explicit form of which is given in the appendix in[1]. We present it here again, but in somewhat different coordinates[9)]:

$$-ds^2 = \frac{c^2}{4}(dz^2 - d\xi^2) + \frac{1}{\operatorname{ch}^2 y}\left[\left(a^2 \sin^2\frac{z}{2} + b^2\cos^2\frac{z}{2}\right)dx^2 + \left(a^2\cos^2\frac{z}{2} + b^2\sin^2\frac{z}{2}\right)dy^2\right.$$

[7)]In the derivation of (2.42) we took into account the fact that ω = const (this can always be done by means of the remaining permissible coordinate transformations). In addition, we took into account the condition that the function ψ (2.33) is periodic in z.

[8)]By performing a small coordinate transformation $x^i = \eta^i(\bar{x}^k)$, we observe that the new components g_{a3} acquire terms of the form $g_{ac}\eta'^c$. The requirement that the coordinate conditions (2.1) must be conserved yields in this case $\eta^c = \eta^c(x, y, z)$. Thus, elimination of the terms $\sim g_{ab}$ from g_{a3} means that the foregoing transformations are forbidden.

[9)]In formula (A.11) of [1] we put for simplicity $\lambda = \mu = \nu = \beta = 1$. Carrying then the transformation $dt = (1/2)cd\xi$, $y = \tanh \bar{y}$, and $z = \bar{z}/2$, we obtain (3.1), where the bars over y and z have been omitted.

$$+(a^2-b^2)\sin z\,dx\,dy\Big]+c^2\,\mathrm{th}^2 y\,dx^2-c^2\,\mathrm{th}^2 y\,dx\,dz. \quad (3.1)$$

It is easy to see that the variable ξ in (3.1) coincides exactly with the variable ξ introduced in Sec. 4 of[1], in terms of which the corresponding asymptotic formulas have been written out. In the region $\xi \gg 1$ we had

$$a^2 = {}^1/_4 p\xi \exp[2A\xi^{-1/2}\sin(\xi-\xi_0+A^2\ln\xi)]+O(\xi^{-1/2}), \quad (3.2)$$
$$b^2 = {}^1/_4 p\xi \exp[-2A\xi^{-1/2}\sin(\xi-\xi_0+A^2\ln\xi)]+O(\xi^{-1/2}),$$
$$c^2 = c_0^2\xi^{A^{-1/2}}e^{2A\xi}[1+O(\xi^{-1})].$$

Here p, A, ξ_0, and c_0 are arbitrary constants. It is easy to verify that this result is obtained from the general case if we put in (2.16)

$$f(x,y)=p/4\,\mathrm{ch}^2 y, \quad (3.3)$$

and if we take only the two following non-vanishing Fourier components of χ_k and φ_k in (2.25):

$$\chi_1 = {}^1/_2 iAe^{-i\xi_0}, \quad \varphi_{-1} = -{}^1/_2 iAe^{i\xi_0}. \quad (3.4)$$

A similar comparison can be carried out also in the region $\xi \ll 1$.

This circumstance allows us to conclude that the subsequent behavior of the solution in the general case (after going outside the region of applicability of the approximation constructed by us) will be qualitatively the same as in the particular case of the metric (3.1).

That is to say, a prolonged period, analogous to that considered here, will again set in after a relatively short intermediate region, but another diagonal component of the metric tensor (g_{11} or g_{22}) will now become small, i.e., the axis along which the relative rapid compression of the three-space takes place will change. This change of axis will be repeated an infinite number of times up to the singular point.

Such a solution will be a general solution with a physical singularity.

We note in this connection that the assumption made by us in[1] that the physical singularity in the metric (3.1) is eliminated after inclusion of the arbitrary perturbations in the region where it is close to the form

$$-ds^2=-dt^2+dx^2+dy^2+t^2dz^2, \quad (3.5)$$

is not valid in the present case.

The general solution investigated here is a generalization of the metric (3.1) with a three-parameter group of motions of the ninth Bianchi type. The space described by the metric (3.1) is closed. In this connection, the question of the stability of the metric (3.5) should be solved with allowance for the topology of that space on which it is realized. A preliminary investigation, the details of which we hope to publish later, shows that in the case when the coordinate lines z in (3.5) are homomorphic to a circle (as in the metric (3.1)), the solution (3.5) is unstable. On the other hand, if the lines z are homomorphic to an infinite straight line, then there exists a general solution, with a fictitious singularity, close to (3.5) (see[9]).

In the foregoing investigation, we assumed the space to be empty. A simple analysis shows that inclusion of matter does not change the qualitative character of the solution. During the final stage of the period considered here, the dependence of the metric on the time has a Kasner (power-law) character. In this region, as shown in[9], inclusion of matter leads only to small corrections to the solution for vacuum.

In the initial region of the period (large ξ), this also takes place. We have seen that at large ξ the principal approximation is described by the metric (2.2), in which $g_{a3}=0$ and g_{33}, g_{ab} can be regarded as independent of the variables x and y. It is therefore sufficient to consider only such a simplified variant. In this case $R_a^0 \equiv R_a^3 \equiv 0$, and for the ultrarelativistic equation of state $\epsilon = 3p$ (which we assume for concreteness; actually, the performed analysis is valid for any equation of state) we obtain the following system of equations:

$$R_0^0-R_3^3=T_0^0-T_3^3=4p(u^0u_0-u^3u_3), \quad (3.6)$$
$$R_3^0=T_3^0=4pu^0u_3, \quad (3.7)$$
$$R_a^b=T_a^b=p\delta_a^b, \quad (3.8)$$
$$u^3u_3+u^0u_0=-1. \quad (3.9)$$

The velocity components $u_a = 0$ (in accordance with the requirement $T_a^0 \equiv T_a^3 \equiv 0$), while u_0 and u_3 as well as the pressure p depend on ξ and z. The equation $R_0^0 + R_3^3 = T_0^0 + T_3^3$, as before, is the consequence of the system (3.6)–(3.8). The equations of motion $T_{i;k}^k = 0$ and the identity (3.9) yield (we retain here the entire previous notation)

$$(\sqrt{G}u_3p^{3/4})'-(\sqrt{G}u_0p^{3/4})^{\cdot}=0, \quad (3.10)$$
$$(u_0p^{1/4})'-(u_3p^{1/4})^{\cdot}=0, \quad (3.11)$$
$$u_0^2-u_3^2=c^2. \quad (3.12)$$

Since c^2 is small in this approximation, we make the assumption (subsequently confirmed by the result) that in the principal approximation it follows from (3.12) that

$$u_0=u_3. \quad (3.13)$$

Then (3.10) and (3.11) yield

$$p=G^{-1}e^{\Phi_1(\xi+z)}, \quad u_0^2=u_3^2=\sqrt{G}e^{\Phi_2(\xi+z)}, \quad (3.14)$$

where Φ_1 and Φ_2 are arbitrary regular functions of their variable. The components of the energy-momentum tensor are now

$$T_0^0-T_3^3=-\frac{8}{\sqrt{G}\,c^2}e^{\Phi_1+\Phi_2}, \quad T_3^0=-\frac{4}{\sqrt{G}\,c^2}e^{\Phi_1+\Phi_2}, \quad (3.15)$$
$$T_a^b=\frac{1}{G}e^{\Phi_1}.$$

It is easy to see that in Eqs. (3.8) the right side of T_a^b is small compared with the left side R_a^b and can be discarded. Indeed, in this case the solution for g_{ab} does not change and formulas (2.16) and (2.26), in which all the arbitrary functions of x and y should be regarded as constants, are valid. Then $\sqrt{G} = \xi$ and, as follows from (2.5), $R_a^b \sim 1/\xi^2c^2$, whereas $T_a^b \sim 1/\xi^2$. Since c^2 is small, we get $T_a^b \ll R_a^b$. On the other hand, in Eqs. (3.6) and (3.7), the right side is of the same order as the left. From (2.6) and (2.7) it follows that R_3^0, $R_0^0 - R_3^3 \sim 1/\xi c^2$. Thus, Eqs. (3.6) and (3.7) take on the form

$$\dot\psi=-\xi^{-1}+{}^1/_4\xi(\varkappa_a^b\varkappa_b^a+\lambda_a^b\lambda_b^a)+8e^{\Phi_1+\Phi_2}, \quad (3.16)$$
$$\psi'={}^1/_2\xi\varkappa_a^b\lambda_b^a+8e^{\Phi_1+\Phi_2}. \quad (3.17)$$

The right sides of these equations are calculated from the solution (2.16), (2.26) for g_{ab}, and contain a periodic dependence on z. The arbitrary terms $8\exp[\Phi_1(\xi + z) + \Phi_2(\xi + z)]$ will also be expanded in a Fourier series in z. As a result we obtain

$$8e^{\Phi_1+\Phi_2} = \gamma^2 + \text{terms oscillating in } \xi, \quad (3.18)$$

where γ^2 is an arbitrary positive constant. It follows therefore that in the integration of (3.16), the principal term in ψ will be

$$\psi = q\xi + \gamma^2\xi, \quad (3.19)$$

where q is determined by the previous formula (3.36).

Thus, inclusion of matter leads to an increase of the argument of the exponential q in the asymptotic expression for c^2 (when $\xi \gg 1$). In other words, the decrease of c^2 at the beginning of the period becomes sharper, but the qualitative behavior of the solution does not change.

Finally, we note that in the region $\xi \gg 1$ the energy density of the matter, according to (3.14), increases like $\xi^{-2}(G \sim \xi^2)$ with increasing ξ, whereas the component g_{33} decreases exponentially. In synchronous time t, this component decreases like t^2, and the energy density increases like $[\ln(t/t_0)]^{-2}$, where t decreases from a certain instant $t \gg t_0$.

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