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FLUCTUATIONS IN A JOSEPHSON CONTACT

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The behavior of a superconducting tunnel contact in the presence of thermal fluctuations is investigated. In contrast with the case considered in[1], the transverse dimensions of the transition are not assumed to be small. One must therefore assume that the fluctuations introduced from the external circuit and arising within the contact behave differently and in general cannot be taken into account by the introduction of some effective temperature. The dependences of the stationary current component on the magnetic field and the Josephson radiation spectra are calculated. The dependences may differ strongly from those known previously.[1-3] It is possible that some of the results obtained have been observed experimentally.[4,12,14]

1. INTRODUCTION

IN an earlier paper,[1] we considered tunneling in contacts with a small area of transverse cross section, with the Josephson coupling energy[2] commensurate with the fluctuation energy. The cross section was assumed to be so small that the fluctuation current change was the same over the whole junction. In most cases, evidently, this condition will not be satisfied. At the same time, analysis of some experimental results given in[1] (see also[3]) testifies to the fact that the effect of the fluctuations is reflected significantly in the experimental results. Here the fluctuations are expressed both in the stationary Josephson current[1,4] and in the radiation spectrum.[3] In this connection, it is of interest to study the fluctuation processes in superconducting tunnel contacts without such a strict limitation on the smallness of the transverse cross section of the contact.

In extended contacts, the fluctuations introduced from the external circuit and the fluctuations arising directly in the junction region act differently. The external fluctuations are fed to the edges of the junction in a layer whose dimensions are of the order of the penetration depth of the magnetic field in the superconductor. Consequently, the external fluctuations can be taken into account in the boundary conditions of the differential equation that determines the phase difference of the two superconductors.[5-8] The internal fluctuations arise through the area of contact and it is natural to consider them by the introduction of an inhomogeneous fluctuating term in the equation for the phase difference. On contacts of small dimensions, the fluctuations lead to the appearance of a finite slope of the volt-ampere characteristic,[1] i.e., to the appearance of a non-zero resistance. Consequently, the weak superconductivity is interrupted and a resistive state arises. This state is interesting in that it is determined to some extent by the parameters of the external circuit, which is at room temperature. A similar resistive state will also be observed in extended junctions. However, the very process of transition to the resistive state can have qualitative and quantitative differences in this case.

In the present work, we have considered the behavior of a superconducting tunnel contact whose characteristic dimensions are smaller than the Josephson penetration depth but are at the same time so large that it is necessary to take into account the inhomogeneity in the distribution of the fluctuations over the cross section. Here we have considered only the case of sufficiently strong fluctuations, in which the contact is definitely in the resistive region. The analysis is carried out without account of retardation effects[8,9] and the effect of the action of the quasi-particle current on the tunnel current of the Cooper pairs.

2. FUNDAMENTAL CONSIDERATIONS

For simplicity, we shall consider below a "linear" Josephson junction, i.e., a junction in which the phase difference φ is a function of time and a single space coordinate x. The experimental situation is such that φ always depends on two space variables. However, as a rule, the dependence on one of these variables is very slow. In this connection, theories developed for the description of "linear" junctions[6,7] are excellently confirmed by experiments (see, for example,[10]).

In the presence of fluctuations in the region of the tunnel contact, the phase difference will be determined from the equation

$$\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{\bar{c}^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{\gamma}{\bar{c}^2}\frac{\partial\varphi}{\partial t}=\frac{1}{\lambda_j^2}[\sin\varphi+j(x,t)/j_0], \qquad (1)$$

where $\bar{c}$ is the propagation velocity of waves in the tunnel structure, λ_j the Josephson penetration depth, j_0 the amplitude of the Josephson current, γ the effective damping inside the contact:

$$\gamma=(RC)^{-1}.$$

Here R is the resistance of the contact to the normal current,[3,8] $C = \epsilon S/4\pi d$ is the capacitance of the contact, S the area of the transverse cross section, d the thickness of the dielectric layer, ϵ its dielectric permittivity. Equation (1) differs from that generally used[6-8] only by the presence of the fluctuation current j(x, t) produced by the dissipation fluctuations in the junction region. It is assumed that this current has the properties of white noise,[11] i.e.,

$$\langle j(x,t)\rangle=0,\quad \langle j(x,t)j(x',t')\rangle=2\Theta\delta(x-x')\delta(t-t')/Rl, \qquad (2)$$

where Θ is the temperature of the contact, l its extent, $\langle ... \rangle$ the mean value over the random variable.

To determine φ, it is necessary to supplement Eq. (1) with boundary conditions. The problem of the choice of boundary conditions was discussed in[7], where it was shown that for the description of the electrodynamics of the tunnel contact it is convenient to carry out a division of the boundary conditions for the stationary component and the high-frequency oscillations. Such a division in the boundary conditions is essentially connected with the fact that the variable component of the Josephson current is strongly damped along the output of the junction. It is not difficult to see that in such a partition of the boundary conditions, it is not possible to take correctly into account the fluctuations introduced from the external circuit. In this connection, we again introduce the boundary conditions under the assumption that losses from radiation are lacking. It is not necessary to make this assumption, but it significantly simplifies the final results, changing them only slightly quantitatively and not at all qualitatively. Formally, the losses by radiation can be taken into account as in[7] by loading the boundaries of the junction by some effective complex impedance.

We shall assume that the junction is included in the closed circuit formed by the source of emf E and the internal resistance R_1. In some cases, the principal role is played by the inductance of the leads L. At the points of connection of the external circuit to the tunnel contact, the total current of the source will split in our model into the two currents I_1 and I_2 ($I_1 + I_2 = I$) flowing to the right and left boundaries of the contact. From the branch point of the current to the dielectric, each current I_1 and I_2 travels the distance l_1. The specific inductance of such a part of the superconductor $L_1 = 4\pi\lambda l_1$, where λ is the penetration depth. We use the induction law

$$\oint \mathbf{E} d\mathbf{l} + \frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{H} d\mathbf{S} = 0$$

for the two loops. The first loop travels along the path of the current I and I_1 and the second along the path I and I_2. Assuming quasistationarity for the external circuit and the edges of the junction region, we get

$$E + V(t) = IR_1 + \frac{L}{c^2}\frac{dI}{dt} + 2\frac{L_1}{c^2}\frac{dI_1}{dt} + U\left(-\frac{l}{2}\right) + \frac{1}{c}\frac{d\Phi_1}{dt}, \quad (3)$$

$$E + V(t) = IR_1 + \frac{L}{c^2}\frac{dI}{dt} + 2\frac{L_1}{c^2}\frac{dI_2}{dt} + U\left(\frac{l}{2}\right) + \frac{1}{c}\frac{d\Phi_2}{dt}, \quad (4)$$

where $U(l/2)$ and $U(-l/2)$ are the potentials at the ends of the junction, Φ_1 and Φ_2 the fluxes of the external magnetic field through the first and second loops. The presence of fluctuations in the external circuit is taken into account by the introduction of the fluctuating emf V(t), which has the properties of white noise:

$$\langle V(t) \rangle = 0, \quad \langle V(t) V(t_1) \rangle = 2\Theta_1 R_1 \delta(t - t_1), \quad (5)$$

where Θ_1 is the temperature of the external circuit.

For the formulation of the boundary conditions in terms of the phase, it is necessary to express the currents I_1, I_2 and the potentials $U(l/2)$ and $U(-l/2)$ in the relations (3) and (4) in terms of φ. The currents I_1, I_2 fall off in a distance of the order of λ from the edge of the superconductor, while the Josephson current $j_0 \sin\varphi$ and the fluctuating current j(x, t) are distributed along the junction at the distances λ_j or l. Integrating the continuity equation in the region of the junction from $-l/2$ to $-l/2 + \lambda$ and from $l/2 - \lambda$ to $l/2$ and taking into account the Josephson relations[2]

$$\frac{\partial\varphi}{\partial t} = \frac{2e}{\hbar}U, \quad \frac{\partial\varphi}{\partial x} = \frac{4e\lambda}{\hbar c}H, \quad (6)$$

we obtain

$$I_1 = \int_{-l/2}^{-l/2+\lambda} dx\,[j_0 \sin\varphi + j(x,t)] - \lambda_j^2 j_0 \varphi_x\left(-\frac{l}{2}\right),$$

$$I_2 = \int_{l/2-\lambda}^{l/2} dx\,[j_0 \sin\varphi + j(x,t)] + \lambda_j^2 j_0 \varphi_x\left(\frac{l}{2}\right).$$

With accuracy to terms of order λ/λ_j

$$I_1 = -\lambda_j^2 j_0 \varphi_x(-l/2), \quad I_2 = \lambda_j^2 j_0 \varphi_x(l/2). \quad (7)$$

Combining (3) and (4) with account of (6) and (7), after elementary transformations, we get

$$\frac{2e}{\hbar}\left[\frac{1}{c}\frac{\partial(\Phi_1 + \Phi_2)}{\partial t} + E + V(t)\right] = \frac{1}{2}\left[\varphi_t\left(\frac{l}{2}\right) + \varphi_t\left(-\frac{l}{2}\right)\right] + \frac{\bar{c}^2}{l\gamma_1}\left(1 + \frac{1}{\gamma_2}\frac{\partial}{\partial t}\right)\left[\varphi_x\left(\frac{l}{2}\right) - \varphi_x\left(-\frac{l}{2}\right)\right]. \quad (8)$$

We shall make some explanation in connection with (8). If there are no fluctuations ($V(t) = 0$, $f(x, t) = 0$) and we are in the region of the existence of a stationary regime, then it follows from (8) that

$$\frac{2e}{\hbar}E = \frac{\bar{c}^2}{l\gamma_1}\left[\varphi_x\left(\frac{l}{2}\right) - \varphi_x\left(-\frac{l}{2}\right)\right]. \quad (9)$$

Integrating Eq. (1) over x and substituting in (9), we get

$$\frac{E}{R} = j_0 \int_{-l/2}^{l/2} dx \sin\varphi,$$

a stationary Josephson current is possible, i.e., if $E/R < j_0 l$. Here the voltage at the junction is equal to zero.

Subtracting (3) and (4), we get

$$\frac{\partial}{\partial t}\left\{l_1\left[\varphi_x\left(\frac{l}{2}\right) + \varphi_x\left(-\frac{l}{2}\right)\right] + \varphi\left(\frac{l}{2}\right) - \varphi\left(-\frac{l}{2}\right)\right\} = 2\pi\frac{\partial}{\partial t}\frac{\Phi_1 - \Phi_2}{\Phi_0}, \quad (10)$$

where $\gamma_1 = (R_1 C)^{-1}$, $\gamma_2 = c^2 R_1/(L + L_1)$ and $\Phi_0 = \pi\hbar c/e$ is the flux quantum.

For a constant external magnetic field H, the term with the derivative with respect to the time of the sum of the fluxes Φ_1 and Φ_2 in Eq. (8) is equal to zero. Integrating (10) with respect to time and assuming the integration constant to be zero, inasmuch as in the absence of the external fields the open-circuit phase difference is equal to zero, we have, finally,

$$l_1\left[\varphi_x\left(\frac{l}{2}\right) + \varphi_x\left(-\frac{l}{2}\right)\right] + \varphi\left(\frac{l}{2}\right) - \varphi\left(-\frac{l}{2}\right) = 2\pi\frac{\Phi}{\Phi_0}. \quad (11)$$

Here Φ is the flux through the cross section of both superconductors:

$$\Phi = 2\lambda H(l + 2l_1).$$

Strictly speaking, the integration constant in (11) can be different from zero, since the possibility of trapping of the flux by the Josephson contact is not excluded in principle. In this case, the integration constant will simply be proportional to the trapped flux.

3. STATIONARY JOSEPHSON CURRENT IN THE INERTIA-FREE CAST

We shall first consider the case in which one can neglect the inductance of the leads, and the self inductance and capacitance of the tunnel contact. Experimentally, this case can be realized only in an ideal junction. For real contacts, however, there is always a region of small voltages, in which the characteristic frequencies are small and therefore one can neglect the reactive terms in the relations (1) and (8). This region is the narrower the more the measurement circuit differs from the ideal. Owing to the absence of inertia in the phase change, the solution for φ will consist of two terms, one of which will depend only on the space coordinate while the second will depend essentially on the time and very weakly on x. Inasmuch as we shall also limit ourselves to the case $l < \lambda_j$, it follows from (11) that φ has the form

$$\varphi = kx + \varphi_1(x, t),$$

where $k = 4eH\lambda/\hbar c$, and $\varphi_1(x, t)$ is a slowly changing function of x.

Integrating Eq. (I) over the coordinate x from $-l/2$ to $l/2$ and discarding the small inertial term with the second time derivative, inasmuch as it is proportional to C, we get

$$\varphi_{1x}\left(\frac{l}{2}\right) - \varphi_{1x}\left(-\frac{l}{2}\right) = \frac{l\gamma}{\bar{c}^2}\dot{\bar{\varphi}}_1 + \frac{l}{\lambda_j^2}\left(\frac{\sin \pi y}{\pi y}\right)\sin\bar{\varphi}_1 + \frac{1}{\lambda_j^2 j_0}\int_{-l/2}^{l/2} dx\, j(x, t), \tag{12}$$

where $\bar{\varphi}_1$ is the value of φ_1 at some point inside the contact, y is the current arising in the junction in the vicinity of the barrier, expressed in units of the flux quantum ($y = kl/2\pi$).

Substituting the relation (12) in Eq. (8), after neglect of small reactive corrections, we arrive at the equation

$$\frac{2e}{\hbar}[E+V(t)] = \frac{\partial\varphi_1}{\partial t}\left(1+\frac{R_1}{R}\right) + \frac{2e}{\hbar}R_1 l j_0\left(\frac{\sin \pi y}{\pi y}\right)\sin\varphi_1 + \frac{2e}{\hbar}R_1\int_{-l/2}^{l/2} dx\, j(x, t). \tag{13}$$

As is seen from (13), the spatial dependence of the internal fluctuations in the inertia-free case is negligible. In this case, one can introduce the "effective temperature"

$$\Theta_2 = (\Theta R_1 + \Theta_1 R)\,/\,(R + R_1)$$

and the "effective resistance" of the contact $R_2 = RR_1/(R + R_1)$.

The difference from the point contact considered in[1] lies in the fact that the magnetic field enters through the boundary condition (11), and the internal fluctuations enter through Eq. (1). Equation (13) differs from the equation considered in[1] only by certain coefficients and by the form of the fluctuation term. Therefore the solution of the problem in the given case is easily obtained by use of the method developed in[1]. The stationary component of the Josephson current through the junction has the form

$$I = I_0\left(\frac{\sin \pi y}{\pi y}\right)\mathrm{Im}\,[I_{1-i\mu}(z)/I_{-i\mu}(z)], \tag{14}$$

where

$$I_0 = j_0 l, \qquad z = \frac{2e}{\hbar}I_0 R_2 \frac{\sin \pi y}{D\pi y},$$

$$\mu = \frac{2e}{\hbar}\frac{ER_2}{DR_1}, \qquad D = \left(\frac{2e}{\hbar}\right)^2 \Theta_2 R_2,$$

I_α is the Bessel function of imaginary argument of order α.

The relation (14) connects the stationary current with the emf of the source. However, the dependence of the stationary current on the voltage at the contact is measured experimentally. In this connection, for comparison with the experimental curves, it is necessary to know the dependence of the voltage at the content of the emf of the source, which is obtained by averaging of Eq. (13)

$$U(y, E) = \frac{\hbar}{2e}\frac{d}{dt}\langle\varphi\rangle = \frac{R_2}{R_1}E - R_2 I(y). \tag{15}$$

The dependence of the current on the magnetic field has a rather complicated form. In a large magnetic field, or for sufficiently strong fluctuations, Eq. (14) is materially simplified, since $z \ll 1$ here:

$$I = \frac{I_0\mu z}{2(1+\mu^2)}\frac{\sin \pi y}{D\pi y}. \tag{16}$$

In this approximation, the potential at the junction coincides with the emf of the source.

As is seen from relations (14) and (15), the dependence of the current on the voltage for a given magnetic field is similar to the analogous dependence observed in[1]. The dependence of the stationary component of the current on the magnetic field for a fixed voltage will be oscillating with the same period as in the absence of fluctuations; however, with increase in the magnetic field, the current falls off more rapidly than usual, and for large fields we have $I \sim (\sin \pi y/\pi y)^2$ in place of $I \sim \sin \pi y/\pi y$.

4. GENERAL CASE

In the general case, it is necessary to solve Eq. (1) with the boundary conditions (8) and (11). The exact solution has not been obtained here, and therefore we consider different approximate solutions. We shall use the method of successive approximations, which in the given case is applicable if one of the following conditions is satisfied.

1. Strong fluctuations. This condition corresponds to the approximation in which the relation (16) is obtained.
2. Relatively large emf of the source, for which $I_0 \ll E/R_2$. This case is very important for the study of the spectral characteristics of the radiation.
3. The current induced by the external magnetic field is large in comparison with the amplitude of the Josephson current I_0.

In all these three cases, one can develop the perturbation theory for the nonlinear term in (1). We shall seek a solution of the zeroth approximation in the form

$$\varphi = \varphi_0 + kx + \Omega t + \frac{\gamma\Omega}{2\bar{c}^2}x^2 + \Psi(x, t), \tag{17}$$

where $\Omega = \mu D$.

The equation for Ψ and the boundary conditions are obtained directly from the relations (1), (8), and (11) if we substitute Eq. (17) in them and discard the nonlinear term in φ in Eq. (1). The linear equation thus obtained is conveniently solved with the help of the Laplace transform

$$\Psi(x, p) = \int_0^\infty dt\, e^{-pt}\Psi(x, t). \tag{18}$$

In the presence of fluctuations, the result of the calculation, after averaging over the random variables for sufficiently long times, will not depend on the initial conditions at the instant of connection. Therefore, for simplicity, we shall take zero initial conditions. Here, it is easy to obtain the following expression for $\Psi(x, p)$:

$$\Psi(x,p)=\frac{2e}{\hbar}V(p)f(x,p)+\frac{1}{\lambda_j^2 j_0}\int_{-l/2} dx_1 j(x_1,p)f_1(x,x_1,p).$$

In this case,

$$f(x,p)=\left[p\,\mathrm{ch}\,q\frac{l}{2}+\frac{2\bar{c}^2}{l\gamma_1}\left(1+\frac{p}{\gamma_2}\right)q\,\mathrm{sh}\,q\frac{l}{2}\right]^{-1}\mathrm{ch}\,qx, \quad (19)$$

$$f_1(x,x_1,p)=-\frac{f(x,p)}{2q}\left[p\,\mathrm{sh}\,q\left(\frac{l}{2}-x_1\right)+\frac{2\bar{c}^2}{l\gamma_1}\left(1+\frac{p}{\gamma_2}\right)q\,\mathrm{ch}\,q\left(\frac{l}{2}-x_1\right)\right]$$
$$-\frac{\mathrm{sh}\,qx}{2q}\left[\frac{p}{\bar{c}}\,\mathrm{ch}\,q\left(\frac{l}{2}-x_1\right)+\frac{1}{l_1}\mathrm{sh}\,q\left(\frac{l}{2}-x_1\right)\right]\left[q\,\mathrm{ch}\,q\frac{l}{2}+\frac{1}{l_1}\mathrm{sh}\,q\frac{l}{2}\right]^{-1}$$
$$+\theta(x-x_1)\frac{\mathrm{sh}\,q(x-x_1)}{q}, \quad (20)$$

where $q^2 = p(p+\gamma_2)/\bar{c}^2$.

To determine the stationary component of the current, it is necessary to know the correction of the first approximation $\Psi_1(x, t)$, which we shall obtain in similar fashion in the form

$$\Psi_1(x,p)=\frac{1}{\lambda_j^2}\int_{-l/2}^{l/2}dx_1 f_1(x,x_1,p)\int_0^\infty dt e^{-pt}\sin\left[\varphi_0+kx_1+\frac{\gamma\Omega}{2\bar{c}^2}x_1^2+\Omega t+\Psi(x,t)\right]. \quad (21)$$

If fluctuations are absent, then $\Psi(x, p) = 0$ and the solution of the problem of the determination of the form of the volt-ampere characteristic of the contact is given by Eq. (21), in which it is necessary to set $\Psi = 0$. From Eq. (21), with account of the fact that the stationary component of the current is equal to the mean value of $j_0 \sin\varphi$ over time and the coordinate x, it is not difficult to see that the current will reach maxima for three values of the voltage at the junction, for which Ω is close to the imaginary part of the poles of $f_1(x, x_1, p)$. For $|p| \gg \gamma$ this leads to a series of maxima in the vicinities of the voltages determined by the conditions $\Omega \sim \pi n\bar{c}/l$, where n is an integer. The experimental procedure for the measurement usually leads not to maxima but to steps which have been studied both theoretically and experimentally in a number of researches.[6,7,10] In addition to bursts of stationary current at high voltages, another burst is possible when $\Omega \approx (\Omega_0^2-\gamma_3^2)^{1/2}$, where $\Omega_0^2=\gamma_2(\gamma+\gamma_1)$, $\gamma_3=(\gamma+\gamma_2)/2$.

The dc component of the Josephson current in the presence of flucutations is equal to

$$I=j_0\,\mathrm{Im}\int_{-l/2}^{l/2}dx\lim_{t\to\infty}\langle\exp[i\varphi(x,t)]\rangle=\frac{j_0}{2\lambda_j^2}\mathrm{Im}\int_{-l/2}^{l/2}dx\int_{-l/2}^{l/2}dx_1\int_0^\infty dt\, f_1(x,x_1,t)\cdot$$
$$\times F(x,x_1,t)\exp\{i[k(x-x_1)+\gamma\Omega(x^2-x_1^2)/2\bar{c}^2+\Omega t]\}, \quad (22)$$

where

$$F(x,x_1,t)=\lim_{t_1\to\infty}\langle\exp\{i[\Psi(x,t+t_1)-\Psi(x_1,t_1)]\}\rangle. \quad (23)$$

The spectrum of Josephson radiation R_ω is very easily connected with the function F determined by the relation (23):

$$R_\omega=\frac{j_0^2}{\pi}\mathrm{Re}\int_0^\infty dt\cos\omega t\int_{-l/2}^{l/2}dx\int_{-l/2}^{l/2}dx_1 F(x,x_1,t)\cdot$$
$$\times\exp\{i[k(x-x_1)+\gamma\Omega(x^2-x_1^2)/2\bar{c}^2+\Omega t]\}. \quad (24)$$

Using relations (2) and (5), we carry out averaging in (23). As a result, we get for F

$$F(x,x_1,t)=\exp\left\{-\left[\left(\frac{2e}{\hbar}\right)^2\Theta_1 R_1 J(x,x_1,t)+\frac{\Theta}{Rl(\lambda_j^2 j_0)^2}J_1(x,x_1,t)\right]\right\}. \quad (25)$$

Equation (24) contains the quantities

$$J(x,x_1,t)=\lim_{t_1\to\infty}\left[\int_0^{t+t_1}d\tau f^2(x,\tau)+\int_0^{t_1}d\tau f^2(x_1,\tau)-2\int_0^{t_1}d\tau f(x,\tau+t)f(x_1,\tau)\right], \quad (26)$$

$$J_1(x,x_1,t)=\lim_{t_1\to\infty}\int_{-l/2}^{l/2}dx_2\left[\int_0^{t+t_1}d\tau f_1^2(x,x_2,\tau)+\int_0^{t_1}d\tau f_1^2(x_1,x_2,\tau)-2\int_0^{t_1}d\tau f_1(x,x_2,\tau+t)f_1(x_1,x_2,\tau)\right]. \quad (27)$$

To calculate (26) and (27), it is necessary to take the inverse Laplace transforms of the functions f(x, p) and $f_1(x, x_1, p)$ determined by the expressions (19) and (20). It is not difficult to see that all the singularities of these functions are simple poles. These all lie in the left half-plane with the exception of p = 0. In this connection, it is convenient, in carrying out the inverse Laplace transform, to expand f(x, p) and $f_1(x, x_1, p)$ in partial fractions. We denote the poles of (19) by p_k and the residues at these poles by $G(p_k, x)$. In the same way we introduce the notation p_i and $N(p_i, x, x_1)$ for the relation (20). Then, after elementary calculations for f(x, t) and $f_1(x, x_1, t)$, we find

$$f(x,t)=\frac{\gamma_1}{\gamma+\gamma_1}+\sum_k G(p_k,x)e^{p_k t}, \quad (28)$$

$$f_1(x,t)=-\frac{\bar{c}^2}{l(\gamma+\gamma_1)}+\sum_i N(p_i,x,x_1)e^{p_i t}. \quad (29)$$

The pole at the point p = 0 is extracted in Eqs. (28) and (29) for the sake of convenience. Substituting the resultant f(x, t) and $f_1(x, x_1, t)$ in Eqs. (26) and (27), we get, after carrying out the necessary transformations:

$$J(x,x_1,t)=t\left(\frac{\gamma_1}{\gamma+\gamma_1}\right)^2+\frac{\gamma\gamma_1^2}{(\gamma+\gamma_1)^2}\frac{x^2-x_1^2}{2\bar{c}^2}+\sum_{p_k}[G(p_k,x)f(x,-p_k)$$
$$+G(p_k,x_1)f(x_1,-p_k)-2G(p_k,x)f(x_1,-p_k)e^{p_k t}], \quad (30)$$

$$J_1(x,x_1,t)=tl\left(\frac{\bar{c}^2}{l(\gamma+\gamma_1)}\right)^2-\frac{\bar{c}^2\gamma_1}{l(\gamma+\gamma_1)^2}\frac{x^2-x_1^2}{2}$$
$$+\int_{-l/2}^{l/2}dx_1\sum_{p_i}[N(p_i,x,x_2)f_1(x,x_2,-p_i)+N(p_i,x_1,x_2)f_1(x_1,x_2,-p_i)-$$
$$-2N(p_i,x,x_2)f_1(x_1,x_2,-p_i)e^{p_i t}]. \quad (31)$$

We shall carry out further calculations of the stationary component of the current and the radiation spectrum for several more characteristic special cases. As a rule, the inequalities $\gamma l/\bar{c} \ll 1$, $\gamma_1 l/\bar{c} \ll 1$, $\gamma_2 l/\bar{c} \ll 1$ are very well satisfied experimentally. This means simply that the delay in the external circuit is much greater than the time of passage of the electromagnetic wave from one side of the junction to the other in the waveguide formed by the tunnel structure. In satisfying these inequalities, all the poles of the functions (19) and (20) are divided into two characteristic groups $p^{\mathrm{I}}_{i,k}$ and $p^{\mathrm{II}}_{i,k}$. To the first group belong those for which p_k and p_i are small in absolute value. They are identical for both functions f and f_1:

$$p^{\mathrm{I}}\approx-\gamma_3\pm\sqrt{\gamma_3^2-\Omega_0^2}. \quad (32)$$

To the second group belong the p_i and p_k of larger absolute value. In order of magnitude, they are equal to $2n\bar{c}/l$, where n is an integer. The principal contribution to the function F is made by the poles of the first group. The contribution due to the poles of the second group has the order $(\gamma_1 l/\bar{c})^2$.

We note that the small contribution due to the poles of the second group is obtained only in the calculation of the function F. The contribution made by these poles to the stationary component of the current is of the same order as from the poles (32) under the condition that the voltage at the junction is such that the imaginary part of p_i^{II} is close to Ω. However, we shall be interested in the stationary component of the current for small voltages and we shall not approach the region in which the step structure of the volt-ampere characteristic begins to appear, i.e., the region where $\Omega \sim \mathrm{Im}\, p_i^{II}$. The volt-ampere characteristic and the spectrum of Josephson radiation can experience appreciable quantitative and qualitative changes with increase in the temperatures Θ and Θ_1, the parameters of the external circuit and the junction. We shall first consider the case of very strong fluctuations, when the inequalities $D \gg \gamma_1, \gamma_2, \gamma_3$ are satisfied. In this case, integration over time in (22) can be carried out by the Laplace method. The coordinate dependence of $J(x, x_1, t)$ and $J_1(x, x_1, t)$ here can have an important effect on the change in the stationary current with the field. Using the relations (22), (23), (30) and (31), and the approximations mentioned above, we get for the current the expression

$$I(y,\Omega)=\frac{\sqrt{\pi}}{32}I_0\frac{\bar{c}^2\Omega}{\lambda_j^2B^3}e^{-\Omega^2/4B^2}\int_{-1}^{1}dx\int_{-1}^{1}dx_1\exp[i\pi y(x-x_1)+i(x^2-x_1^2)\tau+\beta(x^2-x_1^2)^2], \qquad (33)$$

where

$$\tau=\frac{\Omega l^2}{4\bar{c}^2}\alpha\gamma_3(\gamma_2+\alpha)^{-1},\qquad \beta=D\left(\frac{l}{2\bar{c}}\right)^4\frac{\Omega_0^2\gamma_2(\gamma-\alpha)^2}{16\gamma_3(\gamma_2+\alpha)}.$$

$$\bar{\gamma}=\gamma+\gamma_1,\qquad \alpha=\frac{\Theta}{\Theta_2}\gamma,\qquad B^2=D\frac{\Omega_0^2(\gamma_2+\alpha)}{4\gamma_3\gamma_2}.$$

The shape of the radiation line here will be Gaussian, with a width $\Delta\omega \sim 4B$.

Figure 1 shows the results of the numerical calcula-

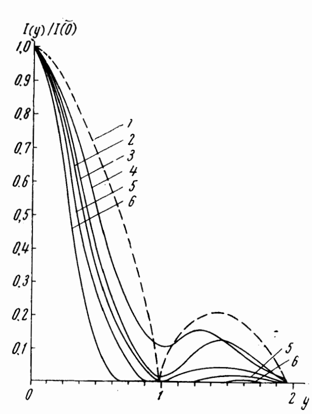

FIG. 1. Dependence of the stationary Josephson current component on the magnetic field: curve 1 – fluctuations are absent, 2 – inertia-free case in the limit of strong fluctuations; curves of the same type are obtained in cases considered in Sec. 4 if D is not very large. Curves 3–6 correspond to the case of very strong fluctuations: 3 – $\beta = 0, \tau = \pi/8$; 4 – $\beta = 0, \tau = \pi/4$; 5 – $\beta = 1.25, \tau = 0$; 6 – $\beta = 2.5, \tau = 0$.

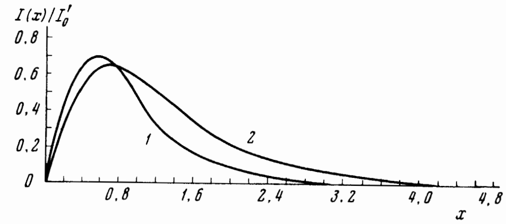

FIG. 2. Dependence of the stationary current component on the voltage for $\gamma_3^2 \gg \Omega_0^2$. Curves 1 – $\nu = 0.5$, 2 – $\nu = 1$.

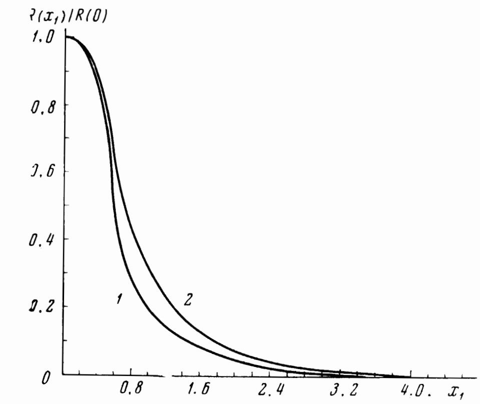

FIG. 3. Spectral intensity of radiation for $\gamma_3^2 \gg \Omega_0^2$. Curve 1 – $\nu = 0.5$; 2 – $\nu = 1$.

tion of the relation (33) for some values of the parameters β and τ. As is seen from the figure, with increase in τ at fixed β, the ratio I(y)/I(0) goes higher, smoothing the oscillatory part considerably. The change in the parameter β for fixed τ acts somewhat differently. With increase in β the current is rapidly damped with the magnetic field and for sufficiently large β only one maximum in the vicinity of y = 0 will be significant. If $\tau \ll 1$ and $\beta \ll 1$, then $I \sim (\sin \pi y/\pi y)^2$, just as in (16).

If there are no strong fluctuations, i.e., when D is not very large, it is of interest to consider the following four possibilities:

1) $\gamma_2^2 \gg \Omega_0^2 \gg \gamma_1^2$. In this case, a result is obtained which was found earlier by Larkin and Ovchinnikov[3] (see also[1]).

2) $\gamma_3^2 \gg \Omega_0^2$. Here, we get from the relations (22) and (24)

$$I(y,x)=I_0'\int_0^\infty dt\sin xt(1-e^{-\nu t})\exp\left\{-\left[t-\frac{1}{\nu}(1-e^{-\nu t})\right]\right\}, \qquad (34)$$

$$R(x_1)\sim\int_0^\infty dt\cos xt\cos x_1t\exp\left\{-\left[t-\frac{1}{\nu}(1-e^{-\nu t})\right]\right\}. \qquad (35)$$

The dependence of the current on the voltage and the spectral intensity are shown in Figs. 2 and 3 in reduced units

$$\nu=\frac{8\gamma\gamma_3^2}{D\alpha\bar{\gamma}},\qquad x=\frac{4\Omega\gamma_3^2}{D\alpha\bar{\gamma}},\qquad x_1=\frac{4\omega\gamma_3^2}{D\alpha\bar{\gamma}},$$

$$I_0'=\frac{I_0\gamma_3 z}{\alpha}\,\frac{\sin\pi y}{\pi y}.$$

The dependences obtained here are qualitatively of the same type as in the first case.

3) $\gamma_3^2 \sim \Omega_0^2$. For definiteness, we set $\gamma_3^2 = \Omega_0^2$. Here

$$I(y,\mu)=I_0''\int_0^\infty dt\sin\mu t\,[1-e^{-\zeta t}(1+\zeta a t)]\cdot$$

$$\times\exp\left\{-\left[t-\frac{3-\alpha/\gamma_2}{2\zeta}(1-e^{-\zeta t})+\frac{t}{2}\left(1-\frac{\alpha}{\gamma_2}\right)e^{-\zeta t}\right]\right\}, \quad (36)$$

$$R(\mu_1)\sim\int_0^\infty dt\cos\mu t\cos\mu_1 t\cdot$$

$$\times\exp\left\{-\left[t-\frac{3-\alpha/\gamma_2}{2\zeta}(1-e^{-\zeta t})+\frac{t}{2}\left(1-\frac{\alpha}{\gamma_2}\right)e^{-\zeta t}\right]\right\}. \quad (37)$$

These dependences are illustrated in Figs. 4 and 5. Here we have introduced the notation

$$\mu_1=\omega/D,\quad \zeta=\gamma_3/D,\quad a=1-\bar{\gamma}/\gamma_3,$$

$$I_0''=\frac{I_0 z}{2}\frac{\sin\pi y}{\pi y}.$$

As is seen from Figs. 2, 3, 4, and 5, the general character of the dependences in the second and third cases is very similar. However, a significant difference in the path of the curves is possible.

4) $\gamma_3^2 \ll \Omega_0^2$. Here,

$$I(y,\eta)=I_0'''\int_0 dt\left[1+\frac{\bar{\gamma}}{\Omega_0}\sin\frac{t}{\xi}-\cos\frac{t}{\xi}\right]\sin\eta\frac{t}{\xi}$$

$$\times\exp\left\{-\left[t+\frac{1}{2\zeta}\left(1-e^{-\zeta t}\cos\frac{t}{\xi}\right)-\frac{3}{2}\xi e^{-\zeta t}\sin\frac{t}{\xi}\right]\right\}, \quad (38)$$

$$R(\eta_1)\sim\int_0^\infty dt\cos\eta\frac{t}{\xi}\cos\eta_1\frac{t}{\xi}$$

$$\times\exp\left\{-\left[t+\frac{1}{2\zeta}\left(1-e^{-\zeta t}\cos\frac{t}{\xi}\right)-\frac{3}{2}\xi e^{-\zeta t}\sin\frac{t}{\xi}\right]\right\}. \quad (39)$$

The curves I/I_0'' and the spectral intensity are plotted in

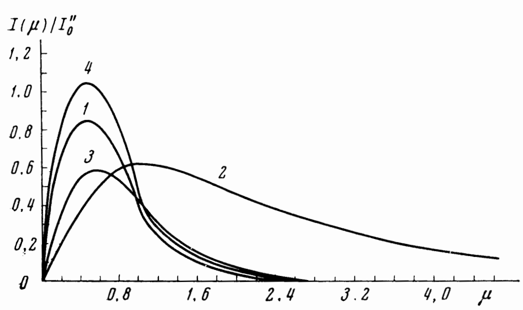

FIG. 4. Dependence of the stationary current component on the voltage for $\gamma_3^2=\Omega_0^2$. Curve 1 – a = 0.2, ζ = 0.5, $\alpha/2\gamma_2$ = 0; 2 – a = 0.2, ζ = 5, $\alpha/2\gamma_2$ = 0; 3 – a = 0.2, ζ = 0.5, $\alpha/2\gamma_2$ = 1; 4 – a = –0.2, ζ = 0.5, $\alpha/2\gamma_2$ = 0.

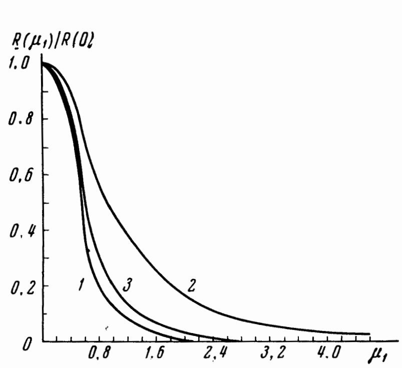

FIG. 5. Spectral intensity of radiation for $\gamma_3^2=\Omega_0^2$. Curve 1 – a = 0.2, ζ = 0.5, $\alpha/2\gamma_2$ = 0; 2 – a = 0.2, ζ = 5, $\alpha/2\gamma_2$ = 0; 3 – a = 0.2, ζ = 0.5, $\alpha/2\gamma_2$ = 1.

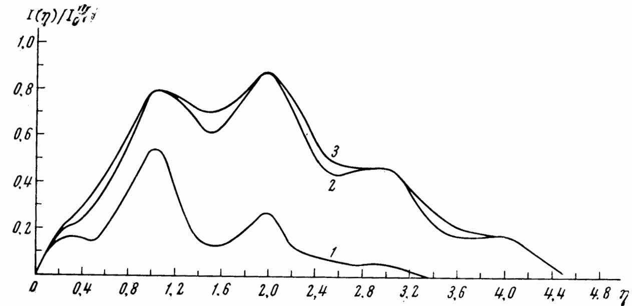

FIG. 6. Dependence of the stationary current component on the voltage for $\gamma_3^2 \ll \Omega_0^2$. Curve 1 – cosh $2\pi\xi$ = 2, ζ = 0.5; 2 – cosh $2\pi\xi$ = 4, ζ = 0.2; 3 – cosh $2\pi\xi$ = 9, ζ = 0.2.

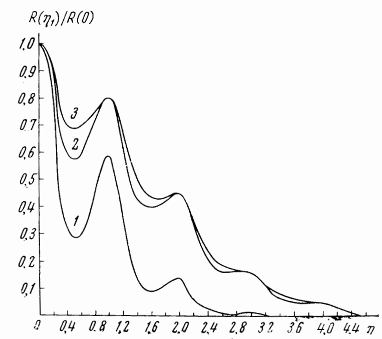

FIG. 7. Spectral intensity of radiation for $\gamma_3^2 \ll \Omega_0^2$. Curve 1 – cosh $2\pi\xi$ = 2, ζ = 0.5; 2 – cosh $2\pi\xi$ = 4, ζ = 0.2; 3 – cosh $2\pi\xi$ = 9, ζ = 0.2.

Figs. 6 and 7 as functions of $\eta_1=\omega/D$ for several values of the parameter $\xi = D/\Omega_0$.

$$I_0'''=I_0 z\left(\frac{\sin\pi y}{\pi y}\right)\frac{\gamma_3}{\gamma_2}\sqrt{\pi\zeta}\,\mathrm{cth}\,\pi\xi,$$

The frequency is reckoned from Ω in Figs. 3, 5, and 7. Here the curves are constructed only on one side, since, for $\Omega \gg D$, the spectral intensity will be symmetric relative to $\omega = \Omega$.

The case $D \gg \Omega$ does not present any special interest, since the width of the radiation line here will be such that its contribution will be scarcely observed experimentally ($\Delta\omega/\Omega < 1$).

5. CONCLUSION

At the present time, two experimental researches are known to us in which the effect of the fluctuations on the stationary Josephson current component was established in clear fashion. These are the works of Vant-Hull and Mercereau[4] and of Galkin et al.[12] Before these researches there was the communication of Shigi et al.[13] on the anomalous behavior of the Josephson current, which is possibly connected with the effect of fluctuations. However, the authors of[13] did not connect the observed anomalies with fluctuations. The appearance of a finite resistance in the superconducting tunnel contact in the presence of fluctuations was first noted in the work of Vant-Hull and Mercereau. Here a contribution similar in form to that shown in Fig. 6 was obtained to the quasiparticle current, associated with the Josephson current. The oscillating change in the current with the magnetic field is also convincing proof of this. The maxima on the curve of Fig. 6 appear at points where $\Omega \approx n\Omega_0$. In the experiment of[4], Ω_0 is close to the resonant frequency of the long line which the tunnel contact terminates. It would also be interesting to investigate the radiation spectrum in this system which, in accord with (39), ought to contain a series of maxima, located in symmetric fashion relative to the fundamental frequency (see Fig. 7). The distance between the nearest maxima will be $\approx \Omega_0$. A report on the observation of a similar spectrum was made by Dmitrenko et al. at the 14th All-Union Conference on Low Temperature Physics.[14] However, the results of a detailed study of such a spectrum are not present in the literature available to us.

In the work of Galkin et al.,[12] high-resistance junctions with $l < \lambda_j$ were studied, on which it was discovered that for certain magnetic fields, the stationary current flowed for finite voltages at the barrier, preserving a periodic dependence on the field. The magnetic field in the given case decreases the effective binding energy of the contact and in this connection, for some value of the magnetic field, the fluctuations destroy the coherent state of two superconductors with a fixed phase difference. Moreover, it was also discovered in[12] that the Josephson current at very high-ohmic contacts does not show an oscillating dependence on the magnetic field. The latter correlates well with the proposed theory, since the case of very strong fluctuations can be realized at high-ohmic contacts, when (see, for example, curve 4 in Fig. 1) the dependence on the field can be practically monotonic.

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Note added in proof (December 10, 1969). In a recently published paper of Ambegaokar and Halperin, [15] a dependence was observed of the potential at a small junction, connected to a source of constant current, on the current in the circuit. In spite of another method of solution, the results of Ambegaokar and Halperin agree analytically with the results obtained by us earlier in [1] and with Eq. (15). For comparison, it is convenient to write the voltage as a function of the mean current in the circuit I_1, using (14) and (15):

$$V = \hbar D \operatorname{sh} \pi\tilde{z}_0 / 2\pi e |I_{i\tilde{z}_0}(z)|^2,$$

where $\tilde{z}_0 = 2eR_2(I_1R_1 + V)/\hbar R_1 D$.

In the limit $y = 0$, $E \to \infty$, $R_1 \to \infty$ for $E/R_1 = 1$, this relation is identical with the result of Ambegaokar and Halperin, which is easily proved by expansion of both solutions in series in z.

The experimental proof of this relation, undertaken by Anderson and Goldman, [16] showed a qualitative agreement between theory and experiment. Evidently the agreement can be improved if we take into account the finite capacity of the junction and the difference in the external circuit from an ideal current source (the rather high effective temperature $\Theta_2 \sim 10°K$, which significantly exceeds the temperature of the junction, indicates this).

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