A METHOD FOR OBSERVING PERIODICITY OF A VORTEX LATTICE IN ROTATING He II

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A method is suggested for observing the periodicity of a vortex lattice by scattering light from HeII ions. Negative ions in rotating HeII tend to concentrate on the vortices. Thus coherent scattering of light in a plane perpendicular to the axis of rotation will have a diffraction maximum, providing the vortex lines form a regular lattice. In order to assess the feasibility of such an experiment the behavior of a quasi-neutral mixture of ions in rotating HeII is considered.

ROTATION produces in superfluid helium vortex filaments parallel to the rotation axis. The properties of the vortex lattices were considered by Tkachenko[1,2]. It has turned out that among the simple lattices, the energetically most convenient is a triangular lattice. In addition, it was shown that a triangular vortex lattice is stable. However, observation of order in the arrangement of the vortices encounters difficulties, owing to the smallness of the cross section for the scattering of light by the vortex filaments.

The periodicity of the vortex lattices likewise does not influence the magnitude of the mutual friction, since the amplitude of the thermal oscillations of the vortex filaments greatly exceeds the length of the proton wave. Therefore, in order for the scattering of the excitations by the vortex lattice to become coherent, it is necessary to lower the temperature to such an extent as to leave only phonons with a wavelength exceeding the amplitudes of the vortex oscillations. But this calls for such temperatures ($\sim 0.01°$K) at which second sound cannot be observed. For similar reasons, certain other methods of observation, for example neutron scattering, are likewise not feasible.

In connection with the foregoing difficulties, we proposed a method of observing the periodicity of the lattice by using ions.

1. Negative ions are described in helium with the aid of the bubble model[3]. According to this model, a negative ion is an electron situated in a localized state in a potential well of depth ~ 1 eV[4,5] and radius 15–20 Å.

It is known that the vortex filaments capture the bubbles, and the depth of the interaction potential of the ion with the vortex turns out to be in this case $\sim 45°$K[6]. Thus, the negative in rotating HeII tend to concentrate on the vortices.

If the vortex filaments form a periodic lattice, then the arrangement of the captured bubbles in a plane perpendicular to the rotation axis will be ordered. Therefore coherent scattering of light in this plane will have diffraction maxima as a result of scattering by the captured ions.

In order for the cross sections for scattering by negative ions to be sufficiently large, it is necessary that the frequency of the incident light be close to any one of the frequencies of absorption of light by the electron in the bubble. The first excited state of the electron in the bubble, 1p, is located a distance $\Delta E \approx 0.1$ eV from the ground state, 1s[7], corresponding to an absorption wavelength $\lambda_{1s\to 1p} \approx 10^{-3}$ cm.

In considering the scattering of light with such a wavelength, the thermal oscillations of the vortices are not significant, since, according to[8], at $T \sim 1°$K and $\Omega \sim 1$ revolution/sec, the rms deviation $[\langle u^2\rangle_{av}]^{1/2}$ of the vortex filament from the equilibrium position, due to the thermal oscillations of the vortices, will be $\sim 10^3$Å, which is much smaller than $\Lambda_{1s\to 1p}$. At the same time, at a vessel revolution frequency $\Omega \approx 1$ rps the distance between vortices is $d \approx 5 \times 10^{-3}$ cm and exceeds $\lambda_{1s\to 1p}$ slightly.

Thus, the presence of an absorption line with wavelength $\approx 10^{-3}$ cm is a favorable circumstance for revealing the lattice by observing the diffraction of light by bubbles captured by vortices.

2. Inasmuch as the charged homogeneous system of bubbles and vortices is unstable, the use of only negative ions is insufficient. We therefore consider the behavior of quasineutral mixtures of ions in rotating HeII. We confine ourselves only to the temperature interval 1.7–1.5°K. Because the behavior of the ions in this region has been sufficiently well investigated and is well described by the existing models.

Ions of both signs, upon colliding with rotons, execute Brownian motion. As the ion wanders, it can fall into the zone of action of the vortex and be captured by the latter. After a certain time, owing to the thermal fluctuations, the ion can leave the vortex. The time of emergence is large in the case of a bubble, on the order of several seconds. To the contrary, positive ions are hardly concentrated on the vortices.

This is connected with the fact that a positive ion in liquid helium is a piece of solid helium of radius ~ 8 Å with a positive charge at the center[9]. Just like a bubble, a positive ion is attracted to the vortex filament, but the depth of the potential well is in this case estimated only at 20°K[6] (the difference in the case of a bubble, for which the well depth is $\approx 45°$K, is due to the difference between the radii of the ions). In the case of thermal equilibrium, the potential well produced by the vortex will contain $(1/4)n\pi a^2 l \exp(u_0/T)$ particles (n—concentration of the particles in the volume; a—width of the well, approximately equal to the diameter of the ions; l—length of the vortex; U_0—depth of well). It turns out that when $T > 1.5°$K, up to a density

$n = 10^9\ cm^{-3}$, only several positive ions can be located on a vortex length of 1 cm. Therefore only negative ions will be concentrated on the vortices.

Besides the Brownian motion and the interaction with the vortices, the ions of unequal signs are gradually attracted to one another and recombine.

We note the following circumstance, which is important for the determination of the average number of bubbles captured by the vortices. In the considered temperature interval $T < 2°K$, at ion densities n exceeding $10^6\ cm^{-3}$ (we are interested in sufficiently large ion concentrations, in order that the scattering by the bubbles be noticeable), the Debye-Huckel formula, which determines the screening radius r_D, is no longer valid, since r_D obtained from this formula turns out to be smaller than the average inter-ion distance $r_i = n^{-1/3}$. In this case the screening occurs at a distance on the order of r_i. Therefore, if the mean distance between ions captured by the vortex exceeds $n^{-1/3}$ (the final result confirms this assumption), then the ions on the vortex filament will be screened from one another (i.e., the vortex does not produce anything resembling a charged wire).

The average number of bubbles captured by the vortex filament is determined by the equality of the rates of departure and capture of particles by the vortex. The rate of departure of particles from the vortex filament is determined by the rate of recombination of the captured bubbles, since at the ion concentrations of interest to us ($n > 10^6\ cm^{-3}$) the recombination time is much shorter than the time of departure of the particles as a result of the thermal motion.

Since, as already noted, the bubbles captured by the vortex are screened from one another, recombination occurs on each of them independently. The recombination rate, and consequently also the rate of departure of bubbles from the vortex, is therefore equal to the rate of recombination on one bubble, multiplied by the number of bubbles captured by the vortex. Analogously, the rate of capture of bubbles equals the rate of capture of the wandering particles by the vortex, multiplied with the coefficient $(1 - \alpha)$, which takes into account the fact that part of the vortex length is already occupied by ions.

Thus, the number of bubbles captured by the vortex filament per unit length (which we denote by N) is determined from the equation

$$NJ = (1 - \alpha) j, \tag{1}$$

where J is the rate of recombination of one ion; j is the rate of capture of the wandering bubbles by a vortex filament of length of 1 cm.

3. To find the recombination rate J in liquid helium, let us consider the motion of an ion in the field of an immobile Coulomb center. The period of revolution of such a field, in the absence of viscosity, is $t_c = \pi e^2 (m/2|E|^3)^{1/2}$; expressing the energy of the particle in degrees and substituting for m the ion mass ≈ 100 m He^4, we obtain $t_c \approx 10^{-5}|E|^{-3/2}$ sec. We note that at the temperatures of interest to us, the coefficients of dynamic friction of the ions is $\beta \approx 10^9 - 10^{10}\ sec^{-1}$ (β—the coefficient entering in the Langevin equation $\dot{v} = -\beta v + f_{pot} + f_{rand}$, describing the motion of the wandering particle). Therefore, up to energies $|E| \approx 100°K$, where the motion of an ion in a Coulomb potential the inequality $t_c \gg 1/\beta$ is satisfied; this leads to conditions for the applicability of the Smoluchowski equation for the description of motion with allowance for viscosity. This is a consequence of the fact that the frequency of collision of the ion with the rotons at $T = 1.6°$ is $\approx 10^{12}\ sec^{-1}$.

The Smoluchowski equation has the following form:

$$\frac{\partial w}{\partial t} = \operatorname{div}\left(\frac{kT}{m\beta} \operatorname{grad} w + \frac{\operatorname{grad} U}{m\beta} w\right), \tag{2}$$

where w is the density of distribution of the ions relative to the Coulomb center.

Let us find the stationary solution of this equation at the boundary condition $w = 0|_{r=r_0}$ and $w = n^{(+)}|_{r\to\infty}$, where r_0 is the certain small distance, which when reached causes the ion to recombine with certainty (the result does not depend on the exact choice of the value of r_0). We obtain

$$w = n^{(+)} \exp\left(-\frac{U}{kT}\right) \int_{r_0}^{r} \frac{1}{r^2} \exp\left(\frac{U}{kT}\right) dr \left(\int_{r_0}^{\infty} \frac{1}{r^2} \exp\left(\frac{U}{kT}\right) dr\right)^{-1}, \tag{3}$$

from which we get for the recombination rate

$$J = 4\pi n^{(+)} e^2 / m\beta. \tag{4}$$

In order to take into account the mutual screening of the ions, we obtain J in the case when $U = -(e^2/r)\exp\{-r/r_D\}$.

We get

$$J = \frac{4\pi n^{(+)} e^2}{m\beta} \left[\int_0^{\infty} \exp\left(-z \exp\left\{-\frac{e^2}{kTr_D z}\right\}\right) dz\right]^{-1}. \tag{5}$$

If $e^2/kTr_D \lesssim 1$, which is satisfied up to an ion concentration $n \approx 10^9\ cm^{-3}$, then

$$\int_0^{\infty} \exp\left[-z \exp\left(-\frac{e^2}{kTr_D z}\right)\right] dz \approx 1,$$

the current is $J \approx 4\pi n^{(+)} e^2/m$, and consequently, the recombination time is $t_{rec} = 1/J = m\beta/4\pi n^{(+)} e^2$. No account is taken here of the time required for the final recombination of the ions. It is clear only that after at time $t = t_{rec}$ the ion ceases to play a role in the kinetics and in the electrostatics, since it is transformed into a dipole with a small moment.

We proceed to calculate j—the rate of capture of the wandering bubbles by a vortex filament of length 1 cm.

According to[6], the far tail of the interaction potential of the particle with the vortex U(r) is proportional to r^{-2}; at the point $r = r^{(-)}$ ($r^{(-)}$ is the radius of the bubble), we have $|U| \sim 0.1°K$ and increases rapidly with decreasing r, assuming values $U_0 \approx 45°K$ at $r = 0$. In view of the smallness of the potential at $r > r^{(-)}$ compared with the temperature, it can be assumed that the ion wanders freely, until it falls into the region $r < r^{(-)}$, after which it is captured by the vortex.

The ion is captured by the vortex if the loss of energy δE on passing through the well exceeds its initial energy $\sim T$. Let us estimate the value of δE. We assume that when $r < r^{(-)}$ the potential of the interaction of the ion with the vortex has the form of a parabola, i.e., $U(r) = (1/2) m\omega^2 r^2$, where the frequency of the oscillation of the particle in the well is $\omega \approx 2 \times 10^{10}\ sec^{-1}$[10]. Then the equation of motion with allowance for the viscosity will be

$$\frac{d^2\mathbf{r}}{dt^2} = -\beta\frac{d\mathbf{r}}{dt} - m\omega^2\mathbf{r}. \quad (6)$$

From this, recognizing that $\beta/2 \ll \omega$, we obtain

$$r = r^0 \cos(\omega t + \alpha) e^{-\beta t/2}. \quad (7)$$

The loss of particle energy δE on passing through the well turns out to be therefore $\sim U_0(1 - e^{-\beta T}) \approx U_0\beta\pi/\omega$, which greatly exceeds T at the temperatures considered by us. Thus, the rate of capture of the bubble by the vortex is the rate of falling of the particle into a cylinder of radius $r^{(-)}$. If it is assumed that the ions move in the helium in analogy with molecules in a gas, then we obtain for the rate of entry

$$j = \left(2\pi\frac{T}{m}\right)^{1/2} r^{(-)} n^{(-)}, \quad (8)$$

where $n^{(-)}$ is the density of the bubbles in the vessel after subtracting the ions captured by the vortices.

The same is obtained if it is assumed that all the particles move in a plane perpendicular to the vortex with a velocity $v_T = (wT/m)^{1/2}$, and after passing through a distance v_T/β they turn in an arbitrary direction.

In order to be able to use formula (8), it is necessary that the density of the ions have time to become equalized sufficiently rapidly. Let us consider how the particle density becomes equalized. We isolate around the vortex a cylinder of area s and assume that only ions in this cylinder are captured. After a time $\Delta t \approx n^{(-)}s/j$, the cylinder "becomes depleted." According to the theory of Brownian motion, the mean square of the deviation of the wandering particle is proportional to the wandering time

$$\langle |r - r_0|^2 \rangle_{av} = \frac{6T}{m\beta} t. \quad (9)$$

Consequently, after a time Δt, the particle covers an area

$$S \approx \frac{4\pi T}{m\beta}\Delta t \approx \frac{4\pi T}{m\beta}\frac{n^{(-)}s}{j}.$$

In order for the density to have time to become equalized, it is necessary to have S exceed s.

Expressing β in terms of the mobility of the bubbles $\mu^{(-)}$ ($\beta = e/m\mu$), we obtain

$$S \approx \frac{2\sqrt{2}}{r^{(-)}}(Tm)^{1/2}\frac{300\mu^{(-)}}{e}s \quad (10)$$

(the mobility is measured here in cm^2/sec-V). Substituting the numerical values of the quantities, we get

$$S \approx 10\,\mu^{(-)}s. \quad (11)$$

According to measurements by Reif and Meyer[11], the mobility of the bubbles is $\mu^{(-)} \approx 0.3$ at T = 1.5°K, so that it is possible to use formula (8) to determine the rate of capture of the bubbles by the vortices.

Substituting the above-obtained expressions for J and j in (1), we get N, the number of bubbles on 1 cm of the vortex filament. If the number of negative ions captured by the vortices is small compared with the total number of bubbles in the volume, then the density of the non-captured ions is $n^{(-)} = n/2$, and we have, substituting (4) and (8) in (1)

$$(1-\alpha)\left(\frac{2\pi T}{m}\right)^{1/2} r^{(-)}\frac{n}{2} = 4\pi n^{(+)} e(300\mu^{(+)})N, \quad (12)$$

$\mu^{(+)}$ denotes here the mobility of the positive ion ($\mu^{(+)}$ exceeds $\mu^{(-)}$ somewhat[11] and $n^{(+)}$ is the concentration of the positive ions ($n^{(+)} = n/2$).

Substituting the numerical values in (12), we obtain

$$N \approx \frac{(1-\alpha)}{\mu^{(+)}}100, \quad (13)$$

i.e., at temperatures 1.7—1.5°K we have $N \approx 100$.

When the vessel rotates at several revolutions per second, the density of the number of vortices is $\sim 10^5$ cm^{-2}, and therefore if the ion density is $n \gtrsim 10^8$ cm^{-3}, then the concentration of the bubbles falling on the vortices will be $\approx 10^7$ cm^{-3}. Thus, the best concentration of the ions is $n \approx 10^7$—10^9 cm^{-3}. With further increase of the density n, the fraction of the bubbles captured by the vortices decreases.

4. The energy spectrum of the negative ions in liquid helium has been described in detail by Fomin[7]. In its structure, the spectrum of the bubble recalls the spectrum of a molecule. The distances between the electron terms are 0.1—0.01 eV. Each electron term splits into vibrational levels with intervals 10^{-4}—10^{-5} eV. The vibrational levels are connected with the possible deformation motions of the bubbles. The mass corresponding to the vibrational motion is $\sim\rho(r^{(-)})^3$ (ρ—density of helium, $r^{(-)}$—bubble radius). This exceeds by many orders of magnitude the mass of the electron, and therefore we can use the adiabatic approximation to describe the system, i.e., we can assume that the state of the electron in the well corresponds to the instantaneous values of the form of the bubble $\mathbf{R}(\theta, \varphi)$, and the wave function of the system can be represented in the form

$$\psi\{\mathbf{r}, \mathbf{R}(\theta, \varphi)\} = U_{\mathbf{R}}(\mathbf{r})\, w\{\mathbf{R}(\theta, \varphi)\}, \quad (14)$$

where $U_{\mathbf{R}}(\mathbf{r})$ satisfies the Schrödinger equation for an electron in a well having the form $\mathbf{R}(\theta, \varphi)$.

In the dipole approximation, the cross section for coherent scattering has the following form (see[12], part I, Ch. VI):

$$d\sigma_{coh} = \left|\sum_n \left\{\frac{(\mathbf{d}_{1n}\mathbf{e}'^*)(\mathbf{d}_{n1}\mathbf{e})}{\omega_{n1} - \omega} + \frac{(\mathbf{d}_{1n}\mathbf{e})(\mathbf{d}_{n1}\mathbf{e}'^*)}{\omega_{n1} + \omega}\right\}\right|^2 \frac{\omega^4}{\hbar^2 c^4} do'. \quad (15)$$

In the case of resonant fluorescence (we are interested precisely in this case) it is necessary to retain only the first term, and the summation will already be carried out only over the vibrational states of the resonant term. Taking (14) into account, we obtain

$$d\sigma_{coh} = \left|\sum_n \frac{\langle w_{(n)2} | w_{(0)1}\rangle^2}{\omega_{n1} - \omega}\right|^2 \left|\sum_{M_z}(\mathbf{d}_{12}\mathbf{e}'^*)(\mathbf{d}_{21}\mathbf{e})\right|^2 \omega^4\, do'. \quad (16)$$

The indices 1 and 2 correspond here to the ground and to the excited electron terms of the bubble, while the indices (0) and (n) pertain to the vibrational levels.

We shall separate among the vibrational levels the region of levels for which the quantity $\langle w_{(n)2} | w_{(0)1}\rangle$ differs significantly from zero. Since the system is described by the adiabatic approximation, these levels are determined from the Franck-Condon principle (see, for example,[12], Ch. V).

Let the work of the region of these levels be δE. The frequency of the incident light ω is chosen such that

$$\delta E \ll |\omega - \omega_{n'1}| \ll \omega, \quad (17)$$

where n′ belongs to the selected levels (the right-hand inequality is necessary in order for the scattering to remain resonant). Such a frequency ω can always be chosen, since the transition occurs to vibrational states with large quantum numbers. In this case the quantity $(\omega_{n'1} - \omega)^{-1}$ remains constant for the selected level; we shall denote it henceforth by $(\omega_{FC1} - \omega)^{-1}$. Therefore, recognizing that $\sum_n \langle w_{(n)2}|w_{(0)1}\rangle^2 = 1$, we obtain

$$d\sigma_{coh} = (\omega_{FC1} - \omega)^2 \left| \sum_{M_2} (\mathbf{d}_{12}\mathbf{e}'^*)(\mathbf{d}_{21}\mathbf{e}) \right|^2 \omega^4 do' \tag{18}$$

Analogously, using the optical theorem, we obtain the total cross section σ_t of all the possible scattering processes

$$\sigma_t = 2\pi \sum_{M_2} |\mathbf{d}_{12}\mathbf{e}|^2 \omega \frac{\Gamma}{(\omega_{FC1} - \omega)^2}, \tag{19}$$

where Γ is the total probability (per second) of the ''decay'' of the level in the region separated by the Franck-Condon principle.

Thus,

$$\sigma_{coh}/\sigma_t \approx \Gamma_{sp}/\Gamma, \tag{20}$$

where Γ_{sp} denotes the probability of spontaneous emission of an electron in a rigid well having the form of a bubble.

According to the estimate in[7], $\Gamma_{sp} \approx 0.001\ \Gamma$. This is due to the fact that the vibrational motions of the shape of the bubble are accompanied by emission of sound, which leads to a strong broadening of the vibrational levels.

In scattering of light by the bubbles, the intensity of the coherent scattering in the direction determined from the Bragg condition is proportional to the square of the number of ordered scattering centers, i.e., to the square of the number of captured ions. The probability of satisfaction of the Bragg condition in the case of rotation is $6\delta\varphi_M/2\pi \approx 10^{-3}$ ($\delta\varphi_M$ is the width of the diffraction maximum, and the number 6 is due to the fact that the lattice is triangular). Incoherent scattering (as shown above, is stronger than the coherent scattering by a factor 10^3) is proportional to the total number of bubbles. Thus, if the scattering occurs on 1 cm^3 and an ion concentration $n \approx 10^8\ cm^{-3}$ and a concentration of the captured bubbles $n_3 \approx 10^7\ cm^{-3}$, the intensity of the current scattering at the center of the diffraction maximum exceeds the intensity of the incoherent background, since

$$\frac{n_3^2}{n}\frac{6\delta\varphi_M}{2\pi}\frac{\Gamma_{sp}}{\Gamma} > 1.$$

This makes it possible to observe a spot. In addition, the situation improves with increasing volume.

In conclusion we note that besides ionization with β radiation, a quasineutral mixture of ions can be obtained by drawing out ions of opposite signs from two reservoirs by means of an electric field.

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