EFFECTIVE VISCOSITY OF He II IN NARROW CAPILLARIES

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A mechanism of interaction between a phonon gas and an absolutely rigid surface is considered. The transition probability for scattering of quasi-particles by the He II—solid interface is found in explicit form for a number of cases. The effective viscosity in narrow He II-filled capillaries is calculated by applying the boundary conditions derived. It is assume' ' ~re that the tube diameter is much smaller than the mean free path of elementary excitations. The effective viscosity depends significantly on the relation between the phonon wavelength and the characteristic size of roughness on the surface.

ONE of the methods for the determination of the coefficient of first viscosity and the density of the normal component of He II is the measurement of the frictional force arising in the relative motion of two coaxial cylinders, the space between which is filled with the superfluid.[1-3] If the distance between the cylinders (2d) is much greater than the mean free path of the elementary excitations, then the macroscopic approach is valid.[1,2] The free path lengths relative to phonon-phonon and phonon-roton collisions, in accord with[4], are respectively equal to

$$l_{ph} \approx 10^{-3}T^{-9}\ \text{cm}, \quad l_{phr} \approx 10^{-8}T^{-9/2}e^{\Delta/T}\ \text{cm}, \qquad (1)$$

where $\Delta = 8.6^\circ$K is the parameter of the energy spectrum. Moreover, there is also a free path length relative to the scattering of phonons at small angles $\tilde{l}_{phph}$ ($\tilde{l}_{phph} < l_{phph}$).

In the region of sufficiently low temperatures, where the effect of rotons can be neglected ($T < 0.6^\circ$K), the free path length of the phonons can appreciably exceed the distance between the cylinders and the hydrodynamic approximation becomes inapplicable.[1,2] Here the collisions between quasi-particles, moving relative to one another at sufficiently great angles, take place comparatively rarely. Under these conditions, the quasi-particles collide freely with each of the two cylinders, transferring the tangential component of the momentum from one cylinder to the other (we note that $\tilde{l}_{phph}$ can be of the order of or even less than d). It is clear that the transfer of momentum in collisions with the wall depends significantly on the scattering law of the quasi-particles incident on the surface of the cylinders. The latter is, to a significant degree, determined by the detailed structure of the surface.[1)]

The situations described above ($d \ll l_{phph}$) arose in experiments carried out in[3]. It was shown there that the observed force acting on the moving cylinder is about one half the calculated value obtained under the assumption of complete diffusivity of the scattering of the phonons on the surface of the cylinders. This result is evidently connected with the fact that in the region of sufficiently low temperatures, the wavelength of the phonons λ becomes comparatively large (for $T = 1^\circ$K, $\lambda \approx 10^{-6}$ cm) and it is necessary to consider the particular specularity of the reflection of the quasi-particles. Another region of phenomena where the law of interaction of the quasi-particles with the surface is important is that of the thermal conductivity in small slits filled with He II (see, for example,[5] and the literature cited there). In this connection, the calculations which take into account the specularity of the reflection of phonons are of interest.

1. If the radius of the cylinder $R_c \gg d$, then one can restrict the considerations to the case of two infinite plates $z = \pm d$, one of which (for example, the lower) moves with the velocity u in a plane $\{x, y\}$. We shall consider the velocity u to be sufficiently small ($u < v_{cr}$, where v_{cr} is the critical velocity). We shall further assume that the distance between the plates satisfies the following inequality:

$$\lambda \ll d \ll \alpha l_{phr} \qquad (2)$$

Here α is the coefficient of diffusivity, the definition of which will be given below.

Since $d \ll l_{phph}$, then it is convenient to divide the phonons in the established regime into two subsystems: phonons flying upward (from the moving plate to the fixed one) and phonons flying downward. For $u = 0$, the distribution functions of both subsystems coincide ($n_\uparrow = n_\downarrow$) and the equilibrium Bose functions $n_0 = n(\epsilon)$ are the same ($\epsilon = cp$, c is the speed of sound, p the momentum of the phonon). If one of the plates moves ($u \neq 0$, but $u < v_{cr} \ll c$), then the distribution functions differ from equilibrium by a small addition which is different for each of the subsystems and depends essentially on the character of the interaction of the phonons with the wall. The answer can be written down immediately in the limiting case of purely diffuse scattering; then $n_\uparrow = n(\epsilon - \mathbf{p}\cdot\mathbf{u})$, and $n_\downarrow = n(\epsilon)$. In the general case of an arbitrary law of interaction of the phonons with the wall, the functions $n_\uparrow$ and $n_\downarrow$ should be found with the help of the corresponding boundary conditions.

By analogy with purely diffuse scattering, which does not violate the generality, it is convenient to seek the functions $n_\uparrow$ and $n_\downarrow$ in the form

[1)]We consider the walls to be absolutely rigid, which is entirely valid, since the product ρc (ρ is the density, c the speed of sound) in He II is much less than in the solid.

$$n_{\uparrow}=\left[\exp\left(\frac{\varepsilon-\mathbf{pu}_{\uparrow}}{T}\right)-1\right]^{-1},\qquad n_{\downarrow}=\left[\exp\left(\frac{\varepsilon-\mathbf{pu}_{\downarrow}}{T}\right)-1\right]^{-1}.\quad(3)$$

Here $\mathbf{u}_{\uparrow}(\mathbf{p})$ and $\mathbf{u}_{\downarrow}(\mathbf{p})$ are unknown functions of the momentum, determined by the law of interaction of the phonons with the wall. On the boundary, the functions $n_{\uparrow}$ and $n_{\downarrow}$ are connected by some integral relation which (for example, on the upper wall) can be written in the form (see Appendix)

$$n_{\downarrow}(\mathbf{p})=\int\limits_{p_z'>0} S(\mathbf{p},\mathbf{p}')\,n_{\uparrow}(\mathbf{p}')\,d\tau_{\mathbf{p}'},\qquad(4)$$

where $S(\mathbf{p}, \mathbf{p}')$ is the probability of scattering of a phonon from the state with momentum $\mathbf{p}$ into the state with $\mathbf{p}'$, referred to unit volume of momentum space $d\tau_{\mathbf{p}'} = d^3p'/2\pi\hbar)^3$.

2. Equation (4) and the relation analogous to it written for the lower (moving) plate form a system of integral equations for the determination of the distribution functions $n_{\uparrow}$ and $n_{\downarrow}$, the solution of which is generally a very complicated problem. Therefore, simple phenomenological theories are customarily used in the calculations, with the interaction of the quasi-particles with the wall described in terms of the "coefficient of diffusivity" or the "specularity" of the scattering. It is assumed here that some part $\alpha(\mathbf{p})$ of the quasi-particles incident on the surface with momentum $\mathbf{p}$ is scattered in a completely diffuse manner (i.e., uniformly in all directions); while the remainder $(1 - \alpha)$ part is reflected specularly. This leads to the following relation between $n_{\uparrow}$ and $n_{\downarrow}$ on the upper plate:

$$n_{\downarrow}=(1-\alpha)\,n_{\uparrow}+\alpha n_0.\qquad(5a)$$

The analogous relation on the moving surface has the form

$$n_{\uparrow}=(1-\alpha)\,n_{\downarrow}+\alpha n_{\mathrm{u}},\qquad n_{\mathrm{u}}=\left[\exp\left(\frac{\varepsilon-\mathbf{pu}}{T}\right)-1\right]^{-1}.\qquad(5b)$$

Substituting (3) in (5a) and (5b), with accuracy up to terms of first order of smallness in $\mathbf{u}$, we get

$$\mathbf{u}_{\downarrow}=(1-\alpha)\,\mathbf{u}_{\uparrow},\qquad \mathbf{u}_{\uparrow}=\mathbf{u}_{\downarrow}+\alpha(\mathbf{u}-\mathbf{u}_{\downarrow})\qquad(6)$$

or, finally,

$$\mathbf{u}_{\uparrow}=\frac{1}{2-\alpha}\mathbf{u},\qquad \mathbf{u}_{\downarrow}=\frac{1-\alpha}{2-\alpha}\mathbf{u}.\qquad(7)$$

It should be noted that the relation (7) is valid for $\alpha \neq 0$. For α that is sufficiently small but still different from zero, the stationary values of $\mathbf{u}_{\uparrow}$ and $\mathbf{u}_{\downarrow}$ are of the order of $u/2$. The time during which the stationary regime is established is $\sim d/c\alpha$.

It is shown that such a simple phenomenological approach, for the cases considered below, is equivalent in accuracy to a solution of the system of integral equations shown above. The proof of this is given in the Appendix, Sec. 4. Here the phenomenological coefficient of diffusivity $\alpha(\mathbf{p})$ is expressed in terms of the "microscopic" characteristic of the scattering rough surface $S(\mathbf{p}, \mathbf{p}')$ (the kernel of the integral equations):

$$\alpha(\mathbf{p})\,\mathbf{p}_{\perp}=\int\limits_{p_z'>0}(\mathbf{p}_{\perp}-\mathbf{p}_{\perp}')\,S(\mathbf{p},\mathbf{p}')\,d\tau_{\mathbf{p}'},\qquad(8)$$

where $\mathbf{p}_{\perp}$ is the projection of the vector $\mathbf{p}$ on the plane $z = 0$. Thus, in the relations (5)–(7), we mean by $\alpha(\mathbf{p})$ the accommodation coefficient of the tangential momentum—the fraction of the tangential momentum $\mathbf{p}_{\perp}$ transferred by the quasi-particles of momentum $\mathbf{p}$ incident on the wall. It follows from (8) that in completely diffuse scattering, when $S(\mathbf{p}, \mathbf{p}') = S(\mathbf{p})$, we have $\alpha = 1$, while for completely specular reflection, when $S(\mathbf{p}, \mathbf{p}') = (2\pi\hbar)^3\delta(\mathbf{p}_{\perp} - \mathbf{p}_{\perp}')\delta(p_z + p_z')$ will be $\alpha = 0$. In the general case of an arbitrary shape of the scattering probability from $\mathbf{p}$ to $\mathbf{p}'$, the inequality $0 \le \alpha(\mathbf{p}) \le 1$ is satisfied.

3. Knowing the distribution function, it is easy to compute the frictional force F_f, referred to a unit of surface of the plate. According to the definition, it is equal to the flux density of the tangential momentum through the fixed plate:

$$F_{\text{тр}}=\int\limits_{p_z>0}\mathbf{p}_{\perp}v_z(n_{\uparrow}-n_{\downarrow})\,d\tau_{\mathbf{p}},\qquad(9)$$

where $v_z = \partial\epsilon/\partial p_z$.

The effective coefficient of first viscosity in narrow capillaries filled with He II is easily obtained as $\eta_e = 2dF_f/u$. Now, making use of the relations (3) and (9), with accuracy up to terms independent of u, we finally get for the effective coefficient of first viscosity,

$$\eta_{\text{э}}=\frac{2d}{T}\int\limits_{p_z>0}\frac{\alpha}{2-\alpha}\,n_0(n_0+1)\,p_x^2v_z\,d\tau_{\mathbf{p}}.\qquad(10)$$

Thus, the entire problem reduces to the determination of the diffusivity coefficient $\alpha(\mathbf{p})$ in explicit form and the calculation of the integral (10). Here we shall start out from the definition of $\alpha(\mathbf{p})$ given by Eq. (8), and the explicit form of the kernel in those limiting cases which are considered in the Appendix, Secs. 2 and 3.

a) We begin our study with the case of limitingly low temperatures, when the wavelength of the phonons λ is much greater not only than the characteristic height of the roughness σ, but also its linear dimensions l in the (x, y) plane, i.e., when the inequalities $(k\sigma)^2 \ll 1$, $kl \ll 1$ are satisfied.[2)] For $l \approx 10^{-6}$–10^{-5} cm, this region of temperatures $T < \hbar c/l \approx 0.2$–$0.02$°K. The distribution of incoherently scattered phonons over the directions here is close to diffuse,[3)], to which corresponds a smooth dependence of the second component in (A.6) on the direction of the vectors $\mathbf{k}$ and κ (or the momenta $\mathbf{p}$ and $\mathbf{p}'$ corresponding to them). This circumstance permits us to compute the integral (8) without specifying the explicit form of the roughness correlation function $W(\rho)$ (while we are as a rule deprived of this possibility):

$$\alpha(\mathbf{k})=\frac{8}{3}\pi(k\sigma)^2k^2\tilde{W}(0)\frac{5+\sin^2\theta}{\cos\theta},\qquad(11a)$$

where θ is the angle of incidence, $\pi/2 - \theta \gg (k\sigma)^2(kl)^2$. For a Gaussian correlation function $W(\rho) = e^{-\rho^2/l^2}$, we

[2)]Here and below, we make use of the notation introduced in the Appendix for the calculation of the explicit form of $S(\mathbf{p},\mathbf{p}')$: $k = 2\pi/\lambda$ is the wave number, l the correlation radius, W the Fourier transform of the roughness correlation function, and so forth.

[3)]To avoid misunderstanding, we note that, in addition to the incoherent, almost diffuse, component we have in this case a strong coherent component (mean field), reflected in the specular direction. The reflection coefficient V (see (A.6)) increases for decrease in the parameters $(k\sigma)^2$ and (kl), approaching unity, so that the total diffusivity coefficient $\alpha(\mathbf{p})$ here tends to zero, as is seen from (11a), (11b).

then get

$$\alpha(\mathbf{k}) = \frac{2}{3}(k\sigma)^2(kl)^2\frac{5+\sin^2\theta}{\cos\theta}. \quad (11b)$$

Calculating the viscosity coefficient η_0 from Eq. (10) with $\alpha(\mathbf{k})$ in the form of (11b), we finally obtain

$$\eta_e = \frac{29\cdot 2^5\pi^6}{3\cdot 15^2}\frac{\sigma^2 l^2 T^8}{\hbar^7 c^8}d. \quad (12)$$

As follows from (11), the diffusivity coefficient is proportional to λ^{-4}; this limiting case is quite analogous to the volume Rayleigh scattering of waves by small particles. We note that although this result applies strictly only to the slanting roughness $\sigma^2 \ll l^2$ (this inequality was put forth in obtaining (A.6)), the noted analogy allows us to assume that the dependence $\alpha \sim \lambda^{-4}$ and, consequently, $\eta_e \sim T^8$ is preserved even for steep ($\sigma^2 \sim l^2$) but small (σ, $l \ll \lambda$) roughness.

b) For much higher temperatures, the wavelength of the phonons can be less than the roughness correlation radius, remaining, however, large in comparison with their height: $(k\sigma)^2 \ll 1$ and $(kl)^2 \gg 1$. In this case, $\widetilde{W}(\mathbf{k}_\perp - \boldsymbol{\kappa}_\perp)$ has a sharp maximum for $\boldsymbol{\kappa}_\perp = \mathbf{k}_\perp$ and all the phonons, according to (A.6), are reflected in directions close to specular. The diffusion coefficient here is not dependent either on the heights of the roughnesses or on the wavelength, but is determined only by the mean square deviations of the roughness, $\overline{\gamma^2}$:

$$\alpha = 4\overline{(\nabla_{\mathbf{k}_\perp/k}\zeta)^2} = 4\overline{\gamma^2}. \quad (13)$$

For a Gaussian isotropic correlation function $W(\rho) = e^{-\rho^2/l^2}$, $\gamma^2 = 2\sigma^2/l^2$, so that $\alpha(\mathbf{k}) = 8\sigma^2/l^2$. Substituting (13) in (10), we get for the viscosity coefficient

$$\eta_e = \frac{3}{4}\overline{\gamma^2}\rho_{ne}\, cd, \qquad \rho_{n\mathrm{ph}} = \frac{2\pi^2}{45}\frac{T^4}{\hbar^3 c^5}, \quad (14)$$

where $\rho_{n\mathrm{ph}}$ is the phonon part of the normal density. It is then seen that for scattering on a surface with slanting roughnesses of the effective length of the free path we have the quantity $\sim \overline{\gamma^2}d$.

c) Upon further increase in the temperature, the inequality $(k\sigma)^2 \ll 1$ is also violated and it is not possible to use Eq. (A.6) for S, obtained by the perturbation method, for the determination of the diffusivity coefficient $\sigma(\mathbf{k})$ and the viscosity η_e. Therefore, we return to the opposite limiting case, in which the scattering takes place according to the laws of geometric optics (Appendix, Sec. 3). The $\alpha(\mathbf{p})$ calculation from Eq. (8) with $S(\mathbf{p}, \mathbf{p}')$ in the form of (A.9) leads to the same result (13) as in the case of low $[(k\sigma)^2 \ll 1]$ and slanting $[(kl \gg 1]$ roughnesses. This result is not unexpected, for the reason that the limiting case considered in b) can be studied by the method of geometric optics as well as by perturbation theory, in spite of the fact that $\sigma \ll \lambda$. Actually, the inequalities $kl \gg 1$ and $\sigma^2 \ll l^2$ are known to guarantee the smallness of the wavelength of the phonon in comparison with the characteristic radii of curvature of the surface. Thus, Eq. (14) for η_e is seen to be valid even in the limiting case $(k\sigma)^2 \gg 1$, if only the surface is sufficiently slanting.

4. Upon increase in the curvature of the roughnesses, the scattering becomes more diffuse ($\alpha \to 1$) but, unfortunately, the initial assumptions for which Eq. (A.7) holds (absence of shadows and multiple reflections) do not strictly allow us to consider the limiting case of completely diffuse scattering, when $\alpha \approx 1$. However, we note that α will not depend on the height of the roughnesses (for $(k\sigma)^2 \gg 1$ and constant mean square deviation $\overline{\gamma^2}$), on the value of the momentum of the incident phonon, and for scattering on a surface with steep roughnesses ($\gamma^2 \gtrsim 1$). Thus, the temperature dependence of $\eta_e(T)$, given by Eq. (14), remains valid for $\alpha \approx 1$.

The results obtained completely agree with the experimental data of[3], according to which, at $\alpha \approx 0.7$, $\eta_e \sim T^{4\pm 0.1}$ over the entire experimental temperature range ($0.375°K \le T \le 0.6°K$). In this connection, it should be noted that the observed temperature dependence will differ sharply from the computed if we use in the considered case the results given in[6] ($1 - \alpha = \exp(-16\pi^2\sigma^2/\lambda^2)$).

At lower temperatures $T < \hbar c/l$, the dependence $\eta_e \sim T^4$ should change materially, approaching $\eta_e \sim T^8$, as follows from Eq. (12). The exact value of the transition temperature $T \sim \hbar c/l$ from $\eta_e \sim T^4$ to $\eta_e \sim T^8$ depends strongly on the state of the surface of the walls and changes in the limits $T = 0.02$–$0.2°K$ for change in the roughness correlation radius l from 10^{-5}–10^{-6} cm.

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APPENDIX

1. The roughness of the surface of a wall on which phonons are incident usually has a random character and should be described statistically. The random departures of the surfaces of the plates from the mean plane $z = \pm d$ we put in the form $z = \zeta(\mathbf{r})$, where $\zeta(\mathbf{r})$ is the random statistically homogeneous function of two variables $\mathbf{r} = \{x, y\}$, while $\overline{\zeta(\mathbf{r})} = 0$. Here and below the bar indicates averaging over the ensemble of possible values of the random function $\zeta(\mathbf{r})$. The velocity potential $\Phi(\mathbf{R}, t)$ of a sound wave satisfies the wave equation and the boundary condition $\partial\Phi/\partial m|_{z=\zeta(\mathbf{r})} = 0$ (n is the normal to the surface).

Thus we arrive at the problem of the diffraction at an absolutely rigid, statistically rough surface which arises in the solution of a wide circle of problems of radiophysics, hydroacoustics, etc. (for example, see[7-9] and the bibliography given there). This analogy permits us to use some results of the theory of wave propagation in the presence of roughnesses of the separation boundary (natural roughnesses of the relief, the wavy surface of the sea, and so forth) on the problem considered here of the scattering of phonons by a rough wall.

2. The potential $\Phi(\mathbf{R}, t)$ for waves (phonons) scattered by a rough surface is a functional of $\zeta(\mathbf{r})$ and is consequently a random variable itself. We represent Φ in the form $\Phi = \overline{\Phi} + \varphi$ where Φ is the mean field (the so-called coherent component) and φ is the fluctuating field. If the surface $z = \zeta(\mathbf{r})$ is sufficiently slanting ($\overline{\gamma^2} \ll 1$, where $\gamma = \nabla_\perp\zeta$, $\nabla_\perp = \{\partial/\partial x, \partial/\partial y\}$) and the characteristic heights of the roughnesses are much less than the wavelength of the phonons, then the scattering problem can be solved by the perturbation method. Then, in linear approximation in γ and ζ one

can obtain a set of boundary conditions for Φ and φ on the plane $z = 0$, from the boundary condition $\partial\Phi/\partial n = 0$ on the surface $z = \zeta(\mathbf{r})$:[10]

$$\frac{\partial\overline{\Phi}}{\partial z}=\overline{(\gamma\nabla_\perp)\varphi}-\overline{\zeta\frac{\partial^2\varphi}{\partial z^2}},\qquad \frac{\partial\varphi}{\partial z}=(\gamma\nabla_\perp)\overline{\Phi}-\zeta\frac{\partial^2\overline{\Phi}}{\partial z^2}. \tag{A.1}$$

Since the diffraction problem is linear (and, consequently, the superposition principle holds), it suffices to limit ourselves to the study of the scattering of a plane monochromatic wave $\Phi_0(\mathbf{R}, t) = e^{i(\mathbf{k}\cdot\mathbf{R}-\omega t)}$. (For the following, it is convenient to transform from the momentum $\mathbf{p}$ and energy ϵ of the phonons to the quantities usually required in diffraction theory—the wave vector $\mathbf{k} = \mathbf{p}/\hbar$ and the frequency $\omega = \epsilon/\hbar$.) The mean field Φ in this case is different from zero only in the direction of specular reflection from the mean surface $z = 0$:

$$\overline{\Phi}(\mathbf{R}) = e^{i\mathbf{k}_\perp\mathbf{r}}\left[e^{ik_z z}+V(\mathbf{k})e^{-ik_z z}\right],\quad \mathbf{R}=\{\mathbf{r}, z\}, \tag{A.2}$$

where $\mathbf{k}_\perp$ is the projection of $\mathbf{k}$ on the plane $z = 0$, and $V(\mathbf{k})$ is the effective reflection coefficient of the mean field, which depends on the wavelength $\lambda = 2\pi/k$, the angle of incidence and the statistical parameters of the roughnesses on the surface. The formulas for $V(\mathbf{k})$ can be found in[10] for certain limiting cases.

From the boundary condition (A.1), one can express the field $\overline{\Phi}(\mathbf{R})$ in terms of the derivatives of $\varphi(\mathbf{R})$ on the plane $z = 0$, by means of the Green's theorem:

$$\varphi(\mathbf{R})=\frac{1}{2\pi}\int\limits_{z'=0}\frac{e^{ik|\mathbf{R}-\mathbf{r}'|}}{|\mathbf{R}-\mathbf{r}'|}\left\{\gamma(\mathbf{r}')\nabla_{\mathbf{r}'}-\zeta(\mathbf{r}')\frac{\partial^2}{\partial z'^2}\right\}\overline{\Phi}(\mathbf{r}')\,d^2\mathbf{r}' \tag{A.3}$$

Then, in the linear approximation in the parameters of perturbation theory ($\overline{\gamma^2} \ll 1$ and $(k\sigma)^2 \ll 1$, where $\sigma^2 = \overline{\zeta^2(\mathbf{r})}$ is the dispersion of the roughness heights) we obtain for the spatial spectral intensity of the radiation of the fluctuating component in the direction κ ($\kappa^2 = k^2$) in the element of solid angle the following expression (see[8], Eq. (53.14)):

$$\overline{|\varphi_\varkappa|^2}d\Omega_\varkappa \sim (k^2-\varkappa_\perp\mathbf{k}_\perp)^2\sigma^2\widetilde{W}(\mathbf{k}_\perp-\varkappa_\perp)\,d\Omega_\varkappa, \tag{A.4}$$

where

$$\widetilde{W}(\mathbf{k}_\perp-\varkappa_\perp)=(2\pi)^{-2}\iint\limits_{-\infty}^{\infty}W(\rho)\,e^{-i(\mathbf{k}_\perp-\varkappa_\perp)\rho}\,d^2\rho$$

is the spatial spectrum of the roughnesses—the Fourier transform of the correlation function of the roughnesses of the surface $W(\rho) = \sigma^{-2}\overline{\zeta(\mathbf{r})\zeta(\mathbf{r}+\rho)}$.

The scattering probability density $S(\mathbf{k}, \kappa)$ from a state with wave vector $\mathbf{k}$ to a state with wave vector κ with account of the coherent component of the reflected field $\overline{\Phi}^{\text{refl}} = \overline{\Phi} - \Phi_0$ is proportional to the radiation spectral density (in the wave vectors κ)

$$S(\mathbf{k},\varkappa)\sim\overline{|\varphi_\varkappa|^2}+|\overline{\Phi}_\varkappa^{\text{refl}}|^2$$

and, by definition, is normalized by the condition

$$\int\limits_{\varkappa_z>0}S(\mathbf{k},\varkappa)\,d^3\varkappa=1.$$

Using Eqs. (A.2) and (A.4), we achieve the following formula for the probability density:

$$S(\mathbf{k},\varkappa)=\overline{|\Phi_\varkappa^{\text{refl}}|^2}\Big/\int\limits_{\varkappa_z>0}\overline{|\Phi_\varkappa^{\text{refl}}|^2}d^3\varkappa \tag{A.5}$$

$$=|V(\mathbf{k})|^2\delta(\varkappa_\perp-\mathbf{k}_\perp)\,\delta(k_z+\varkappa_z)+4\frac{(k^2-\varkappa_\perp\mathbf{k}_\perp)^2}{k|k_z|}\sigma^2\widetilde{W}(\mathbf{k}_\perp-\varkappa_\perp)\,\delta(k-\varkappa).$$

The first component on the right side of this formula corresponds to the coherently reflected field $\overline{\Phi}^{\text{refl}}$ and is proportional to the probability of specular reflection from a state with wave vector $\mathbf{k} = \{\mathbf{k}_\perp, k_z\}$ to a state $\kappa = \{\mathbf{k}_\perp, -k_z\}$. As is seen from (A.2), this component of the field ($\overline{\Phi}^{\text{refl}}$) has such a phase as if the reflection took place not from the rough surface but from the plane $z = 0$ (with reflection coefficient V). The second component in (A.5) describes the scattering of phonons into the fluctuating (incoherent or diffuse) component φ with random phase, so that $\overline{\Phi} = 0$.

It follows from (A.5) that the distribution of incoherently scattered phonons over the directions (i.e., diffuseness of the scattering) is determined not only by the parameter $k\sigma$, as is rather frequently the case in phenomenological theories (see, for example,[6], Ch. XI, and[11]) but also depends essentially on the "sharpness" of the correlation function $W(\rho)$, i.e., in the final analysis on the relation between the wavelength and the correlation radius l.[4)]

3. The previous section referred to the region of sufficiently low temperatures, when $(k\sigma)^2 < 1$ (for $\sigma \approx 5\times10^{-7}$ cm, $T < 0.4$°K). At much higher temperatures, the wavelength of the phonons decreases and the perturbation method becomes inapplicable. In the opposite limiting case, when the wavelength becomes much less than the characteristic radii of curvature of the surface roughnesses ($\sim l^2/\sigma$), the scattering "in the small" obeys the laws of geometric optics. Here the reflection at each point of the surface takes place in the same way as from the tangential plane—the so-called "Kirchhoff method" is based on this (see[12] and [8], Par. 54). The spatial Fourier component of the reflected field in this case is determined by the integral

$$\Phi_\varkappa^{\text{refl}}=\overline{\Phi}_\varkappa^{\text{refl}}+\varphi_\varkappa\sim\iint\limits_{-\infty}^{\infty}e^{-i[\mathbf{q}_\perp\mathbf{r}+q_z\zeta(\mathbf{r})]}\,[q_z-\mathbf{q}_\perp\gamma(\mathbf{r})]\,d^2\mathbf{r}, \tag{A.6}$$

where $\mathbf{q} = \kappa - \mathbf{k}$. The reflection coefficient of the coherent component is equal to $V(\mathbf{k}) = \overline{e^{-2ik_z\zeta(\mathbf{r})}}$ which, for the normal random surface (when the heights ζ are distributed according to a Gaussian curve) leads to the following result (cf.[6] Eq. (11.3.8)): $V(\mathbf{k}) = e^{-(2k\sigma\cos\theta)^2}$. In the limiting case $(k\sigma\cos\theta)^2 \gg 1$ (for the opposite sign of the inequality, all the results of the previous section are valid) the coherent component in the reflected field can be neglected and we get the following formula from (A.6) for $S(\mathbf{k}, \kappa)$:

$$S(\mathbf{k},\varkappa)=\frac{q^4}{4q_z^4|k_z|k}w_\gamma\left(-\frac{\mathbf{q}_\perp}{q_z}\right)\delta(k-\varkappa), \tag{A.7}$$

where w_γ is the distribution of the probability density of the derivatives ($\gamma = \{\partial\zeta/\partial x, \partial\zeta/\partial y\}$, i.e., of the slopes of the random surface $z = \zeta(\mathbf{r})$ relative to the mean density. For an isotropic Gaussian surface,

$$w_\gamma\left(\frac{\mathbf{q}_\perp}{q_z}\right)=\frac{1}{2\pi\overline{\gamma^2}}\exp\left(-\frac{\mathbf{q}_\perp^2}{2\overline{\gamma^2}q_z^2}\right). \tag{A.8}$$

We note that, although the coherent ("specular") component $\overline{\Phi}^{\text{refl}}$ is exponentially small in the considered limiting case of heights in comparison with the wavelength of the roughnesses, the diffuse scattering, i.e., the width of the function $S(\mathbf{k}, \kappa)$ around the specular

4)By l we understand the distance at which the function $W(\rho)$ falls off appreciably (by definition, $W(0) = 1$). It is obvious that l is the characteristic linear dimension of the roughnesses in the plane $z = 0$.

direction $\kappa_\perp = \mathbf{k}_\perp$, can be suitably small if only the surface is sufficiently sloping $(\overline{\theta^2} \ll 1)$.

4. We shall now show that in all the cases considered by us, the simple phenomenological approach, in which the interaction of the quasi-particles with the wall is obtained with the help of just the single function $\alpha(\mathbf{p})$, is equivalent in accuracy to the integral relation of type (4).

The number of quasi-particles with momentum $\mathbf{p}'$ in the interval $d\mathbf{p}'$ incident per unit time per unit area of the upper plate is equal to $n_\uparrow(\mathbf{p}')v_z' d\tau_{\mathbf{p}'}$. The total number of quasi-particles reflected from the unit surface with momentum $\mathbf{p}$ (in a unit range of momenta) is equal, by definition, to

$$n_\downarrow(\mathbf{p})v_z = -\int_{p_z'>0} S(\mathbf{p}', \mathbf{p})\, n_\uparrow(\mathbf{p}')\, v_z' d\tau_{\mathbf{p}'}.$$

Taking into account the fact that the probability of transition $S(\mathbf{p}', \mathbf{p})$ satisfies the principle of detailed balancing, the latter relation can be rewritten in the form (4). Substituting (3) in (4), we obtain, with accuracy up to terms of the first order of smallness,

$$\mathbf{p}_\perp \mathbf{u}_\downarrow = \int_{p_z'>0} S(\mathbf{p}, \mathbf{p}')\, \mathbf{p}_\perp' \mathbf{u}_\uparrow(\mathbf{p}')\, d\tau_{\mathbf{p}'}. \qquad (A.9)$$

In completely analogous fashion, we have at the lower (moving) plate

$$\mathbf{p}_\perp \{\mathbf{u} - \mathbf{u}_\uparrow(\mathbf{p})\} = \int_{p_z'>0} S(\mathbf{p}, \mathbf{p}')\, \mathbf{p}_\perp' \{\mathbf{u} - \mathbf{u}_\downarrow(\mathbf{p}')\}\, d\tau_{\mathbf{p}'}. \qquad (A.10)$$

For fine-grained $(pl/\hbar \ll 1)$ and coarse-grained $(pl/\hbar \gg 1)$ but gently sloping $(\sigma \ll l)$ roughnesses, the transition $S(\mathbf{p}, \mathbf{p}')$ has a sharp maximum in the direction of specular scattering—see Eqs. (A.4) and (A.6). Therefore, with the same degree of accuracy with which the probability $S(\mathbf{p}, \mathbf{p}')$ is calculated, the functions $\mathbf{u}_\uparrow(\mathbf{p}')$ and $\mathbf{u}_\downarrow(\mathbf{p}')$ in (A.9) and (A.10) can be taken out from under the integral sign with the replacement of $\mathbf{p}'$ by $\mathbf{p}$. Then, taking into consideration the definition of $\alpha(\mathbf{p})$ (8), we get the algebraic set of equations (7).

In the case of coarse-grained and curved roughnesses, starting out from (A.9) and (A.10), we shall prove, similar to what was done in the calculation of α), that the functions $\mathbf{u}_\uparrow$ and $\mathbf{u}_\downarrow$ do not depend on the value of the momentum of the incident phonon. The latter circumstance, as in the phenomenological approach, give us the possibility of determining the temperature dependence of the effective viscosity coefficient essentially without limitations on the degree of sloping of the surface.

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