CONCERNING THE SYNCHRONIZATION OF CLOCKS IN THE GENERAL THEORY OF RELATIVITY

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The possibility of a global description of temporal and spatial relations in the four-dimensional continuum of the theory of relativity is investigated. To this end a function of time and an equation of space are introduced as hypersurfaces of simultaneous points.

As is well known, the ordering in time of events occurring at one and the same spatial point is the result of cause-effect relations which are assumed as the basis of any physical measurement or physical discussion. In the theory of relativity the time, reckoned by a sequence of such events, is called the proper time. It is related to the coordinates of the events with the aid of the equation

$$\tau = \int_a^{x^0} [-g_{00}(x^0, x^i = \text{const})]^{1/2} dx^0 \tag{1}$$

(Latin indices take the values 1, 2, and 3). But in order to determine the temporal relations of arbitrary events and for a global geometrical description of their mutual positions in physical three-dimensional space, we require a special postulate. To this end we use Einstein's definition[1] of the simultaneity of two arbitrary events which are separated by a nonvanishing spatial distance.

We assume that the space-time continuum is a four-dimensional Riemann space with fundamental form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad g_{00} < 0, \quad g < 0,$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$. In other respects the metric of the space is arbitrary; if the curvature tensor $R_{\mu\nu\lambda\sigma}$ vanishes then we are dealing with inertial or noninertial reference frames in the special theory of relativity; however if $R_{\mu\nu\lambda\sigma} \neq 0$ then we are dealing with the general theory of relativity.

Let us consider two neighboring points P and P′. Let two events take place at the point P: A_1 denotes the sending of the light signal to P′ and A_3 denotes the return of the signal, which was reflected at the point P′ (event A_2'). The moments of proper time $\tau(A_1)$, $\tau(A_2')$, and $\tau(A_3)$ correspond to the events A_1, A_2', and A_3. According to Einstein's definition event A_2 at P, whose moment of proper time $\tau(A_2)$ satisfies the condition

$$\tau(A_2) = {}^1/_2[\tau(A_1) + \tau(A_3)]. \tag{2}$$

is simultaneous with event A_2' at P′. Using this definition and the vanishing of the interval ds along a light ray, one can obtain[2,3] expressions for the time interval $d\tau$ between arbitrarily close events, measured with respect to the clock of one of them, and for the distance dl between them:

$$d\tau = \sqrt{-g_{00}}\left(dx^0 + \frac{g_{0i}}{g_{00}} dx^i\right), \tag{3a}$$

$$dl^2 = h_{ik} dx^i dx^k; \tag{3b}$$

here

$$h_{ik} = g_{ik} - g_{0i} g_{0k} / g_{00}. \tag{4}$$

Equations (3) are invariant, but the functions h_{ik} are the components of a symmetric tensor of the second rank with respect to the group of coordinate transformations

$$x'^0 = x'^0(x^0, x^i), \quad x'^k = x'^k(x^i), \quad \partial x'^k / \partial x^0 = 0. \tag{5}$$

Following Zel'manov[4] we shall associate the set of all coordinate systems connected by the transformations (5) with one and the same reference frame.

We define a vector ρ_i and an antisymmetric tensor χ_{ik} by the following equations:

$$\rho_i = \frac{1}{2g_{00}}[2g_{0i,0} - g_{00,i} - g_{0i}(\ln g_{00})_{,0}],$$

$$\chi_{ik} = \frac{1}{2g_{00}\sqrt{-g_{00}}}[g_{0i}g_{0k,0} - g_{0k}g_{0i,0} + g_{00}g_{0i,k} - g_{00}g_{0k,i} + g_{0k}g_{00,i} - g_{0i}g_{00,k}] \tag{6}$$

(a comma in front of a subscript means the partial derivative with respect to the corresponding coordinate).

The properties of the linear Pfaffian form (3a) are closely related to the values of the quantities ρ_i and χ_{ik} which have been introduced.[1)] Three cases are possible.

1. The case $\rho_i = \chi_{ik} = 0$. The form (3a) is a total differential. A one-parameter function $t(x^0, x^i) + t_0$ exists for which the relation $dt = d\tau$ is valid. Therefore t is a function whose variation determines the flow of time at any arbitrary point of space. The constant t_0 plays the role of an origin of time reference common to all points:

$$t = \tau(x^0, x^i) - t_0, \tag{7a}$$

$$x^0 = x^0(t, x^i; t_0). \tag{7b}$$

The hypersurface $x^0 = x^0(b, x^i; t_0)$, where b is an arbitrary constant, is the set of simultaneous points of space corresponding to the moment of time t = b. The metric tensor, given on this hypersurface by its equation, coincides with the tensor h_{ik} given by Eq. (4). Thus, physical space is the hypersurface whose equation is given in the explicit form (7b) or in the implicit form (7a).

[1)] As far back as 1944, Zel'manov [5] presented in his candidate's dissertation a similar analysis of the form (3a). However, the present article contains a number of new propositions both in the formulation of the problem and in its solution.

The coefficients ω_{ik} of the second quadratic form are determined by the equations

$$\omega_{ik} = -{}^1/_2\,\partial h_{ik}/\partial t. \tag{8a}$$

Equations (3), (4), and (8a) or (7b) completely describe the internal and external (in the enveloping four-dimensional Riemann space) geometry of physical space.

2. The case $\chi_{ik} = 0$, $\rho_i \neq 0$. The form (3a) is holonomic; therefore it may be represented in the form of a product $d\tau = v\,du$ containing two scalar functions $v = v(x^0, x^i)$ and $u = u(x^0, x^i)$. Simultaneous events are distributed on the hypersurface

$$u(x^0, x^i) = C = \mathrm{const} \tag{9a}$$

or, if this equation is solved with respect to x^0, on the hypersurface

$$x^0 = \tilde{u}(x^i; C). \tag{9b}$$

The first quadratic form of the hypersurface, given by Eqs. (9), coincides with the quadratic form (3b); hence it follows that physical space in this case is also represented by the hypersurface of simultaneous points. And although the problem of the possibility of synchronization of the clocks in a finite region of space is solved positively, now it is still impossible to introduce a unique function of a single point which might play the role of the time. This is associated with the fact that if there is one integrating factor v for the Pfaffian (3a), then the most general integrating factor is equal to the product $vF(u)$, where F is an arbitrary function. Then

$$d\tau = v\,du = vF(u)\,dw, \qquad w = \int \frac{du}{F(u)}.$$

The set of functions u and w determines one and the same family of hypersurfaces of simultaneous points, but they parametrize it differently.

One can introduce the time in the following way. Let us choose a certain point as the support point, and let us denote its spatial coordinates by x_0^i. Let an event take place at another point x_1^i, which has the value of its time coordinate equal to x_1^0. The coordinates of all events, which are simultaneous with this event, satisfy the equation

$$x^0 = \tilde{u}(x^i; u(x_1^0, x_1^i)),$$

which turns into an identity if $x^i = x_1^i$ and $x^0 = x_1^0$. In particular, $\tilde{u}$ determines the value of the time coordinate of the event at the support point, which is simultaneous with the event under consideration. Therefore the integral

$$t = \int_a^{\bar{u}} \sqrt{-g_{00}(x^0, x_0^i)}\,dx^0, \qquad \bar{u} \equiv \tilde{u}(x_0^i, u(x_1^0, x_1^i))$$

represents the proper time measured on a clock at the support point from a certain arbitrary moment a up to the event (x_1^0, x_1^i). Omitting the subscript 1 because of the arbitrariness of the event (x_1^0, x_1^i) and assuming $a = x_0^0$, we finally obtain an expression for the time t, measured at the support point, between an arbitrary event (x^0, x^i) and the event (x_0^0, x_0^i) which is chosen as the origin of reference:

$$t = \int_{x_0^0}^{\bar{u}} \sqrt{-g_{00}(\xi, x_0^i)}\,d\xi. \tag{10a}$$

Thus the time t is a function of two points

$$t = t(x_0^0, x_0^i; x^0, x^i) \tag{10b}$$

and satisfies the obvious relations

$$t(x_0^0, x_0^i; x^0, x_0^i) = \tau(x^0, x_0^i),$$
$$t(x_0^0, x^i; x^0, x^i) = \tau(x^0, x^i).$$

From the definition of the time (10) it follows that one can now rewrite the equation of the hypersurface of simultaneous events (9) in the form t = const.

For many applications it is convenient to use the following expression for the time differential:

$$dt = \sqrt{-g_{00}}\,e^{\Phi}\left(dx^0 + \frac{g_{0i}}{g_{00}}dx^i\right), \tag{11}$$

where the scalar function Φ is determined by the line integral

$$\Phi = \int_{(x_0^i)}^{(x^i)} \rho_k(t = \mathrm{const}, x^n)\,dx^k. \tag{12}$$

We introduce the following notation:

$$\partial_0 \equiv \frac{\partial}{\partial t\,|_{x^i=\mathrm{const}}} = \frac{e^{-\Phi}}{\sqrt{-g_{00}}}\frac{\partial}{\partial x^0\,|_{x^i=\mathrm{const}}},$$
$$\partial_i \equiv \frac{\partial}{\partial x^i\,|_{t=\mathrm{const}}} = \frac{\partial}{\partial x^i\,|_{x^0=\mathrm{const}}} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0\,|_{x^k=\mathrm{const}}}.$$

The meaning of the operations ∂_0 and ∂_i is obvious. Using them one can verify that the curl of the vector ρ_i is equal to zero; therefore the function Φ exists and is unique.

Taking Eqs. (3b), (4), and (11) into account, one can represent the interval ds in canonical form:

$$ds^2 = -e^{-2\Phi}dt^2 + dl^2.$$

The physical meaning of the function $\Phi(t, x^i; x_0^i)$ consists in the fact that it is equal to the logarithm, taken with the opposite sign, of the velocity of light v_c at the moment of time t at the point of space x^i, measured with respect to a clock located at the point x_0^i. The local velocity of light, measured with respect to a local clock, is always equal to unity. This follows from the property of the function Φ,

$$\Phi(t, x_0^i; x_0^i) = \Phi(t, x^i; x^i) = 0$$

and agrees with the result of an analysis of the characteristic surface of Maxwell's equations.[6] In general the velocity of light is different from unity and, just like the time t, is a function of two points—it depends on both the point of space for which it is measured and on the location of the measuring unit. However, at any arbitrary point the velocity of light remains the largest of all possible velocities in physical processes.

In contrast to the velocity of light, its gradient is determined by only the properties of space at a given point and at a given moment of time:

$$\partial_i \ln v_c = -\rho_i. \tag{13}$$

As is shown in[5], the vector ρ_i is equal to the acceleration vector relative to the reference frame of a body freely resting in it at a given instant of time. Therefore a body which is freely resting at a given moment experiences an acceleration relative to the reference frame in the direction of decreasing velocity of light.

The coefficients ω_{ik} of the second quadratic form of the spatial hypersurface t = const may be determined from the formula

$$\omega_{ik} = -{}^{1}/_{2}e^{\Phi}\partial_0 h_{ik}. \tag{8b}$$

The tensor of relative deformation of the reference frame, introduced by Zel'manov,[5] only differs by sign from the tensor ω_{ik} which determines the external geometry of physical space in the enveloping four-dimensional Riemann space.

We shall call reference frames in which $\chi_{ik} = 0$ synchronous.[3] It is obvious that the reference frames considered in Sec. 1 are related to synchronous frames and constitute a subclass of the class of synchronous reference frames. If $\rho_i = 0$ then $\Phi = 0$ and all formulas of Sec. 2 go over into the corresponding case $\rho_i = \chi_{ik} = 0$. In synchronous reference frames the flow of time depends on the spatial point if $\rho_i \neq 0$, and does not depend on the spatial point if $\rho_i = 0$. Both these and other reference frames exist in an arbitrary four-dimensional Riemann space independently of its curvature (only the fulfillment of general conditions imposed on the signature and continuity is required). In an arbitrary field of gravitation a set of synchronous reference frames can be found in which $\rho_i = 0$, and the time intervals between arbitrary events do not depend on the location of the observer (the rate of clocks is the same at all points of space). On the other hand, in the Euclidean space-time of the special theory of relativity there exists a set of synchronous reference frames (noninertial) in which $\rho_i \neq 0$, and the rate of clocks is different at different points of space. Thus, one can formulate the following statement: the rhythm of physical processes is a function of the state of motion of the reference frame and does not depend on the intensity of the gravitational field.

3. The case $\chi_{ik} \neq 0$. The Pfaffian form (3a) does not have an integrating factor. The equation $d\tau = 0$ has a solution only in the class of surfaces of dimension less than three. Thereby the question of the introduction of time and the global properties of physical space loses its foundation. Let us emphasize that nonsynchronous reference frames in which the tensor χ_{ik} vanishes occur in all four-dimensional Riemann spaces without exception, said spaces finding application both in the general and in the special theory of relativity (for example, rotating coordinate systems). Forms of time and physical space, which would satisfy Einstein's simultaneity condition in the small, do not exist in these reference frames. And together with this, all meaning associated with the change of a physical system in time and its motion in space is lost. Also the meaning of such fundamental concepts of physics as the total charge, mass, conservation laws (since they are associated with a change in time of quantities which occupy a finite volume of space), wave fronts, and others is lost. Is this a difficulty in principle, is it inherent in the nature of the space-time continuum, or does it appear as a result of an inflexible definition of simultaneity? The answer to this question requires a careful analysis of the concept of simultaneity, compatible with the other postulates of the theory of relativity.

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[3] L. Arifov and I. Gutman, Dokl. Akad. Nauk UzSSR 1, 15 (1965); Izv. Akad. Nauk UzSSR 1, 93 (1965). L. Arifov, Candidate's Dissertation, Tashkent, 1965.

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[5] A. L. Zel'manov, Candidate's Dissertation, Moscow, 1944.

[6] L. Arifov, Izv. vyssh. uch. zaved., Fizika 3, 32 (1967).

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