SOVIET PHYSICS JETP VOLUME 31, NUMBER 2 AUGUST, 1970

THE PARABOLIC EQUATION APPROXIMATION FOR PROPAGATION OF WAVES IN A MEDIUM WITH RANDOM INHOMOGENEITIES

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Submitted June 20, 1969

Zh. Eksp. Teor. Fiz. 58, 624–634 (February, 1970)

We consider the propagation of monochromatic light in a medium with inhomogeneities of the dielectric constant, which are large scale (compared to the wavelength), for the case when the depolarization is small and we can use the scalar wave equation. Using Fradkin's method, we can write the solution in operator form or in the form of a path integral. If the dielectric constant probability distribution is Gaussian it turns out to be possible to average explicitly and we can obtain expressions for the average field, the mutual coherence function, and so on. Further we consider the parabolic equation approximation and the approximation of inhomogeneities delta-function correlated along the direction of the wave propagation (Markovian model), and we evaluate then the path-integrals for the average field and the mutual coherence function and find explicit expressions for these quantities. Using the solution of the complete scalar equation in the form of a path integral we were able to find the corrections to the parabolic equation solution for the average field, and using them to formulate the condition that the scattering per wave length is small and that the radius of the first Fresnel zone is small compared to the extinction length.

1. INTRODUCTION

THE problem of the propagation of electromagnetic radiation in a medium with random inhomogeneities has recently become the subject of considerable interest in connection with a whole set of different problems in astrophysics, plasma physics, and geophysics. Although there exist a rather wide class of problems where the solution obtained in the first Born approximation is c completely satisfactory, in a great number of cases w when the field fluctuations become large it is necessary to go beyond that approximation. We have shown in[1] that if the light propagation is described in the parabolic equation approximation, and if the fluctuations in the dielectric constant are Gaussian and delta-function correlated along the direction of the wave propagation, the problem of determining the statistical characteristics of the field can be solved exactly. The equation for the characteristic functional of the electromagnetic field has then the form of the Fokker-Planck equation in functional space and because of this the model considered in[1] corresponds to the approximation of a Markovian random process. We obtained in[2] corrections to a Markovian type solution caused by the fact that the longitudinal correlation radius is finite; in the important region these corrections turn out to be of the order of the ratio of the correlation radius to the length of the path traversed by the wave in the inhomogeneous medium. In the same paper a method was indicated to obtain equations for the moments of the electromagnetic field for the case of non-Gaussian fluctuations in the dielectric constant. We left alone, however, the problem of the applicability of the parabolic approximation itself in a given problem and of the possibility to use the model of delta-function correlated inhomogeneities in the complete scalar equation.

In the present paper we use the method proposed by Fradkin[3,4] to construct solutions of differential equations in the form of path integrals. It then turns out to be possible to write down explicitly expressions for the average field, the mutual coherence function, and so on, and to make clear the conditions under which the path integrals expressing these functions go over into expressions corresponding to the Markovian approximation and can be evaluated.

2. WRITING THE SOLUTION OF THE SCALAR WAVE EQUATION IN A RANDOM MEDIUM IN THE FORM OF A PATH INTEGRAL

In the scalar approximation the field of a light wave $\psi(\mathbf{r})e^{-i\omega t}$ satisfies the equation

$$[\Delta + k^2 + k^2\varepsilon_1(\mathbf{r})]\psi(\mathbf{r}) = 0, \qquad (1)^*$$

where k is the average wave number and $\epsilon_1(\mathbf{r})$ the relative deviation of the dielectric constant from its average value. The dependence of ψ and ϵ_1 on the time enters parametrically into (1) (quasi-stationary approximation).

Bearing in mind that in the following we want to compare the solution of Eq. (1) with the solution of the corresponding parabolic equation we consider the field of an emitter situated in the x = 0 plane (we assume that $\epsilon_1(\mathbf{r}) = 0$ for $x < 0$) which has a complex amplitude distribution $u_0(\rho)$ (where $\mathbf{r} = (x, \rho)$). We must then add to (1) the boundary condition

$$\psi(0, \rho) = u_0(\rho) \qquad (2)$$

and the condition of emission at infinity for $x > 0$.

We consider the field of a point source $G_0(x, \rho; x', \rho')$ in an unbounded space in which $\epsilon_1(-x, \rho) = \epsilon_1(x, \rho)$. The function G_0 satisfies the equation

$$[\Delta + k^2 + k^2\varepsilon_1(|x|, \rho)]G_0(x, \rho; x', \rho) = \delta(x - x')\delta(\rho - \rho') \qquad (3)$$

*$\Delta \equiv \nabla^2$.

and the condition of emission at infinity. We introduce the function

$$G(x, \rho; x', \rho') = G_0(x, \rho; x', \rho') - G_0(x, \rho; -x', \rho'). \qquad (4)$$

One can then easily show that the solution of the problem given by (1) and (2) can be expressed by the formula

$$\psi(x, \rho) = \int dx'\, d\rho' G(x, \rho; x', \rho')\delta'(x')u_0(\rho'). \qquad (5)$$

Following Fradkin's method we construct an explicit expression for G_0 and G. To do this we write G_0 in the form

$$G_0(\mathbf{r}, \mathbf{r}') = \int d\mathbf{p} G_0(\mathbf{r}, \mathbf{p}) \exp\{i\mathbf{p}(\mathbf{r}-\mathbf{r}')\}. \qquad (6)$$

For $G_0(\mathbf{r}, \mathbf{p})$ we get from (3) the equation

$$[\Delta + 2i\mathbf{p}\nabla + k^2\varepsilon_1(|x|, \rho) - p^2 + k^2]G_0(\mathbf{r}, \mathbf{p}) = 1. \qquad (7)$$

We can write the solution of Eq. (7) which satisfies the radiation condition in the form

$$G_0(\mathbf{r}, \mathbf{p}) = -i\int_0^\infty d\xi \exp\{i(k^2 - p^2)\xi - \xi\cdot 0\}\Phi(\xi, \mathbf{r}, \mathbf{p}), \qquad (8)$$

where $\Phi(\xi, \mathbf{r}, \mathbf{p})$ satisfies the equation

$$\frac{\partial\Phi}{\partial\xi} = [i\Delta - 2\mathbf{p}\nabla + ik^2\varepsilon_1(|x|, \rho)]\Phi; \quad \Phi(0, \mathbf{r}, \mathbf{p}) = 1. \qquad (9)$$

(The signs in front of the factors i in (8) are chosen in such a way that the term $\xi\cdot 0$ corresponds to an infinitesimal absorption: $k \to k + i0$.)

We consider instead of (9) a more complicated equation which contains an arbitrary function $\tau(\xi)$:

$$\frac{\partial\tilde{\Phi}}{d\xi} = [i\Delta - 2\mathbf{p}\nabla + ik^2\varepsilon_1(|x|, \rho) + \tau(\xi)\nabla]\tilde{\Phi}(\tau; \xi, \mathbf{r}, \mathbf{p}), \qquad (10)$$

$$\tilde{\Phi}(0; \xi, \mathbf{r}, \mathbf{p}) = \Phi(\xi, \mathbf{r}, \mathbf{p}), \quad \tilde{\Phi}(\tau; 0, \mathbf{r}, \mathbf{p}) = 1.$$

We integrate (10) over ξ from 0 to ξ_1 and then apply the operator $\delta/\delta\tau(\xi_2)$ where $0 < \xi_2 < \xi_1$. As a result we get

$$\frac{\delta\tilde{\Phi}(\tau; \xi_1, \mathbf{r}, \mathbf{p})}{\delta\tau(\xi_2)} = \int_0^{\xi_1} d\xi[i\Delta - 2\mathbf{p}\nabla + ik^2\varepsilon_1 + \tau(\xi)\nabla] \qquad (11)$$

$$\times\frac{\delta\tilde{\Phi}(\tau; \xi, \mathbf{r}, \mathbf{p})}{\delta\tau(\xi_2)} + \nabla\tilde{\Phi}(\tau; \xi_2, \mathbf{r}, \mathbf{p}).$$

As (10) is a first order equation in ξ with boundary conditions at $\xi = 0$, $\tilde{\Phi}(\tau; \xi, \mathbf{r}, \mathbf{p})$ depends only on values of $\tau(\xi')$ for $\xi' < \xi$ and is independent of values $\tau(\xi')$ with $\xi' > \xi$. The variational derivative in (11) is thus equal under the integral sign to zero for $\xi_2 > \xi$, and hence the lower limit of integration must be replaced by ξ_2. Putting after that $\xi_2 = \xi_1$ we get the formula[1)]

$$\frac{\delta\tilde{\Phi}(\tau; \xi, \mathbf{r}, \mathbf{p})}{\delta\tau(\xi)} = \nabla\tilde{\Phi}(\tau; \xi, \mathbf{r}, \mathbf{p}). \qquad (12)$$

Exactly in the same way

$$\frac{\delta^2\tilde{\Phi}}{\delta\tau^2(\xi)} = \Delta\tilde{\Phi} \qquad (13)$$

and we can rewrite Eq. (10) in the form

$$\frac{\partial\tilde{\Phi}}{\partial\xi} = i\frac{\delta^2\tilde{\Phi}}{\delta\tau^2(\xi)} + [\tau(\xi)\nabla - 2\mathbf{p}\nabla + ik^2\varepsilon_1]\tilde{\Phi}. \qquad (14)$$

We shall look for a solution of (14) in the form

$$\tilde{\Phi}(\tau; \xi, \mathbf{r}, \mathbf{p}) = \exp\left\{i\int_0^\xi d\eta \frac{\delta^2}{\delta\tau^2(\eta)}\right\}\varphi(\tau; \xi, \mathbf{r}, \mathbf{p}). \qquad (15)$$

The operators

$$\exp\left\{-i\int_0^\xi d\eta\frac{\delta^2}{\delta\tau^2(\eta)}\right\} \text{ and } [(\tau(\xi) - 2\mathbf{p})\nabla + ik^2\varepsilon_1]$$

commute[2)] so that we get for φ the first-order equation

$$\frac{\partial\varphi}{\partial\xi} = \{[\tau(\xi) - 2\mathbf{p}]\nabla + ik^2\varepsilon_1(|x|, \rho)\}\varphi, \qquad (16)$$

$$\varphi(\tau; 0, \mathbf{r}, \mathbf{p}) = 1,$$

which has a solution in the form

$$\varphi(\tau; \xi, \mathbf{r}, \mathbf{p}) = \exp\left\{ik^2\int_0^\xi d\xi'\varepsilon_1\left(\mathbf{r} - 2\mathbf{p}(\xi - \xi') + \int_{\xi'}^\xi d\eta\,\tau(\eta)\right)\right\}. \qquad (17)$$

Using (17), (15), and (10) we get

$$\Phi(\xi, \mathbf{r}, \mathbf{p}) = \exp\left\{i\int_0^\xi d\eta\frac{\delta^2}{\delta\tau^2(\eta)}\right\}\cdot\exp\left\{ik^2\int_0^\xi d\xi'\varepsilon_1(\mathbf{r} - 2\mathbf{p}(\xi - \xi') + \int_{\xi'}^\xi d\eta\,\tau(\eta))\right\}\Bigg|_{\tau=0}. \qquad (18)$$

We now change from the operator form of the solution to a representation in the form of a path integral. To do this we must write the functional

$$Q[\tau] = \exp\left\{ik^2\int_0^\xi d\xi'\varepsilon_1\left(\mathbf{r} - 2\mathbf{p}(\xi - \xi') + \int_{\xi'}^\xi d\eta\,\tau(\eta)\right)\right\}$$

as a path Fourier integral after which we let the operator occurring in (18) act and put $\tau = 0$. Using the inverse Fourier transformation after this to eliminate the Fourier transform Q and evaluating a Gaussian type path integral we can obtain the equation

$$\Phi(\xi, \mathbf{r}, \mathbf{p}) = \int D\mathbf{v}\exp\left\{\frac{i}{4}\int_0^\xi d\xi'\mathbf{v}^2(\xi') + ik^2\int_0^\xi d\xi'\varepsilon_1\left(\mathbf{r} - 2\mathbf{p}(\xi - \xi') + \int_{\xi'}^\xi d\eta\,\mathbf{v}(\eta)\right)\right\}, \qquad (19)$$

where

$$D\mathbf{v} = \prod_{\xi'=0}^{\xi} d\mathbf{v}(\xi') \Big/ \int\cdots\int\prod_{\xi'=0}^{\xi} d\mathbf{v}(\xi')\exp\left\{\frac{i}{4}\int_0^\xi d\xi'\mathbf{v}^2(\xi')\right\}.$$

We note that the measure which we have introduced is normalized by the condition

$$\int D\mathbf{v}\exp\left\{\frac{i}{4}\int_0^\xi d\eta\,\mathbf{v}^2(\eta)\right\} = 1,$$

which can conveniently be used for actual calculations.

We introduce in (19) a new integration variable through the substitution $\mathbf{v}(\xi) \to 2\mathbf{p} + \mathbf{v}(\xi)$ and substitute it into (8) and (6). We can perform the integration over $\mathbf{p}$ (it leads to a δ-function) and as a result we get

$$G_0(\mathbf{r}, \mathbf{r}') = -i\int_0^\infty d\xi\exp\{ik^2\xi - \xi\cdot 0\}\int D\mathbf{v}\,\delta\left(\mathbf{r} - \mathbf{r}' + \int_0^\xi d\eta\,\mathbf{v}(\eta)\right)$$

[1)]When taking the limit it is necessary to study separately the contribution from the point $\xi_2 = \xi_1$ in the $\Delta\tilde{\Phi}$. There may in general appear here a numerical coefficient which differs from unity (see, for instance, [5,1]). The rule for evaluating the contribution from this point which is established from similar problems which have been solved exactly can be formulated as follows: at $\xi_2 = \xi_1$ we must assume that $\tau(\xi) = \tau(\xi + 0)$.

[2)]To establish this we must also take into account that $\tau = \tau(\xi + 0)$ (see preceding footnote).

$$\times \exp\left\{\frac{i}{4}\int_0^\xi d\eta \mathbf{v}^2(\eta) + ik^2\int_0^\xi d\xi'\varepsilon_1\left(\mathbf{r}+\int_{\xi'}^{\xi} d\eta \mathbf{v}(\eta)\right)\right\}. \quad (20)$$

Using Eqs. (4) and (5) and bearing in mind that $\epsilon_1(x) = \epsilon_1(|x|)$ we can write down an expression for $\psi(x, \rho)$:

$$\psi(x,\rho) = -i\int_0^\infty d\xi \exp\{ik^2\xi - \xi\cdot 0\}\int D\mathbf{v}\exp\left\{\frac{i}{4}\int_0^\xi d\eta \mathbf{v}^2(\eta)\right.$$

$$\left. + ik^2\int_0^\xi d\xi'\varepsilon_1\left(\left|x+\int_{\xi'}^{\xi} d\eta v_x(\eta)\right|,\ \rho+\int_{\xi'}^{\xi} d\eta \mathbf{v}_\perp(\eta)\right)\right\}$$

$$\times\int dx'\left\{\delta\left(x-x'+\int_0^\xi d\eta v_x(\eta)\right)-\delta\left(x+x'+\int_0^\xi d\eta v_x(\eta)\right)\right\}$$

$$\times\delta'(x')u_0\left(\rho+\int_0^\xi d\eta \mathbf{v}_\perp(\eta)\right). \quad (21)$$

To facilitate the comparison of this expression with the solution of the parabolic equation which has the form $\psi = ue^{ikx}$ we change integration variables in (21), $v_x \to v_x - 2k$ and afterwards use the δ-functions

$$\delta\left(x \pm x' + \int_0^\xi d\eta v_x(\eta)\right).$$

to eliminate the factor proportional to an integral of v_x which occurs in the index of the exponent. As a result we get

$$\psi(x,\rho) = -2i\int_0^\infty d\xi\int dx'\delta'(x')\exp\{ik(x-x')-\xi\cdot 0\}$$

$$\times\int D\mathbf{v}\delta\left(x-x'-2k\xi+\int_0^\xi d\eta v_x(\eta)\right)\exp\left\{\frac{i}{4}\int_0^\xi d\eta \mathbf{v}^2(\eta)\right.$$

$$+ ik^2\int_0^\xi d\xi'\varepsilon_1\left(\left|x-2k(\xi-\xi')+\int_{\xi'}^{\xi} d\eta v_x(\eta)\right|,\right.$$

$$\left.\left.\rho+\int_{\xi'}^{\xi} d\eta \mathbf{v}_\perp(\eta)\right)\right\}u_0\left(\rho+\int_0^\xi d\eta \mathbf{v}_\perp(\eta)\right). \quad (22)$$

Equations (21) and (22) give the field of an emitter with a complex amplitude distribution $u_0(\rho)$ in an inhomogeneous medium.

If the point of observation lies in the wave zone of the emitter, i.e., $kx \gg 1$, we can drop in (21), (22) the absolute magnitude sign in the first argument of ϵ_1. Indeed, in the region which is important for the integration over v_x, $v_x^2\xi \lesssim 1$, i.e., $|v_x| \lesssim \xi^{-1/2}$. Bearing this in mind, we deduce from the δ-functions occurring in (22) that $\xi \sim x/k$. If we take into account the presence of δ-functions we can write the first argument of ϵ_1 in the form

$$2k\xi' - \int_0^{\xi'} d\eta v_x(\eta) \sim (2k - v_x)\xi',$$

and as $v_x \sim \xi^{-1/2} \sim (x/k)^{-1/2}$, we have $v_x/k \sim (kx)^{-1/2} \ll 1$. As $\xi' > 0$, the integral term which is small compared to the main term can not change the sign of the argument and we can thus for $kx \gg 1$ drop the absolute magnitude sign in (21) and (22). (We note that this approximation corresponds to neglecting the term $1/ikx$ in comparison with unity in the Green function of the same problem in free space.

3. PARABOLIC EQUATION APPROXIMATION

In the parabolic equation approximation we look for the solution of (1) in the form $\psi(\mathbf{r}) = u(\mathbf{r})e^{ikx}$ and we omit in the equation for u the term $\partial^2 u/\partial x^2$:

$$2ik\frac{\partial u}{\partial x} + \Delta_\perp u + k^2\varepsilon_1(\mathbf{r})u = 0. \quad (23)$$

The emission condition is automatically satisfied and the second boundary condition is the same as (2):

$$u(0,\rho) = u_0(\rho). \quad (2a)$$

We can apply to Eq. (23) the same procedure as was used above (in actual fact it turns out to be much simpler as (23) is a first order equation in x and we can introduce the "source" τ_1 as $\tau_1(x)$). As a result we easily get the following form for u:

$$u(x,\rho) = \exp\left\{\frac{i}{2k}\int_0^x d\xi_1\frac{\delta^2}{\delta\tau_1^2(\xi_1)}\right\}u_0\left(\rho+\int_0^x d\xi_1\tau_1(\xi_1)\right)$$

$$\times\exp\left\{\frac{ik}{2}\int_0^x d\xi_1\varepsilon_1\left(\xi_1, \rho+\int_{\xi_1}^x d\eta_1\tau_1(\eta_1)\right)\right\}\Bigg|_{\tau_1=0}. \quad (24)$$

Here $\tau_1 = (\tau_{1y}, \tau_{1z})$ while the variable ξ_1 has now another meaning and is connected with the old variable ξ by the relation $\xi = 2k\xi_1$.

Using a Fourier transformation we can write $u(x, \rho)$ as a path integral:

$$u(x,\rho) = \int D\mathbf{v}_1\exp\left\{\frac{ik}{2}\int_0^x d\xi_1\mathbf{v}_1^2(\xi_1)+\frac{ik}{2}\int_0^x d\xi_1\varepsilon_1\left(\xi_1, \rho\right.\right.$$

$$\left.\left.+\int_{\xi_1}^x d\eta_1\mathbf{v}_1(\eta_1)\right)\right\}u_0\left(\rho+\int_0^x d\xi_1\mathbf{v}_1(\xi_1)\right), \quad (25)$$

$$D\mathbf{v}_1 = \prod_{\xi=0}^x d\mathbf{v}_1(\xi_1)\Big/\int\cdots\int\prod_{\xi=0}^x d\mathbf{v}_1(\xi_1)\exp\left\{\frac{ik}{2}\int_0^x d\xi_1\mathbf{v}_1^2(\xi_1)\right\}.$$

Using the integral representations of (22) and (25) and assuming that $\epsilon(\mathbf{r})$ is a Gaussian random field we can easily write down expressions for the averages of the ψ or for the averages of their bilinear combinations. We then use the formula (the angle brackets denote the averaging over the ensemble of possible realizations of $\epsilon_1(\mathbf{r})$)

$$\left\langle\exp\left\{ia\int d\xi\varepsilon_1(\mathbf{r})\right\}\right\rangle = \exp\left\{-\frac{a^2}{2}\iint d\xi_1 d\xi_2\langle\varepsilon_1(\mathbf{r}_1)\varepsilon_1(\mathbf{r}_2)\rangle\right\},$$

which is valid in the case of a Gaussian distribution for ϵ_1 when $\langle\epsilon_1\rangle = 0$. We shall denote the correlation function $\langle\epsilon_1(\mathbf{r}_1)\epsilon_1(\mathbf{r}_2)\rangle$ by $B_\epsilon(\mathbf{r}_1, \mathbf{r}_2) = B(\mathbf{r}_1 - \mathbf{r}_2)$ (uniformity).

The average of u is equal to

$$\langle u(x,\rho)\rangle = \int D\mathbf{v}_1\exp\left\{\frac{ik}{2}\int_0^x d\xi_1\mathbf{v}_1^2(\xi_1)\right.$$

$$\left.-\frac{k^2}{8}\iint_0^x d\xi_1 d\xi_1' B\left(\xi_1-\xi_1', \int_{\xi_1}^{\xi_1'} d\eta_1\mathbf{v}_1(\eta_1)\right)\right\}u_0\left(\rho+\int_0^x d\xi_1\mathbf{v}_1(\xi_1)\right). \quad (26)$$

One can not evaluate the integral (26) in the general case. However, it can be found for the model with inhomogeneities which are delta-function correlated along the direction of propagation which was introduced in[1]. In that case the true correlation function $B(\xi, \rho)$ is replaced by an effective correlation function

$$B^{\text{eff}}(\xi,\rho) = \delta(\xi)A(\rho), \quad A(\rho) = \int_{-\infty}^{\infty} d\xi B(\xi,\rho). \quad (27)$$

In this approximation

$$\langle u(x,\rho)\rangle = \exp\{-k^2A(0)x/8\}$$

$$\times\int D\mathbf{v}_1\exp\left\{\frac{ik}{2}\int_0^x d\xi_1\mathbf{v}_1^2(\xi_1)\right\}u_0\left(\rho+\int_0^x d\xi_1\mathbf{v}_1(\xi_1)\right). \quad (28)$$

One can easily evaluate the last integral if we substitute instead of the function $u_0(\boldsymbol{\rho})$ its Fourier transform

$$u_0(\rho)=\int d\varkappa\, u_0(\varkappa)\, e^{i\varkappa\rho}, \quad (29)$$

and afterwards perform in the path integral the shift $\mathbf{v}_1 = \mathbf{u} - \boldsymbol{\kappa}/\kappa$.

As a result we get

$$\langle u(x,\rho)\rangle=\exp\{-k^2A(0)x/8\}\int d\varkappa u_0(\varkappa)\exp\{i\varkappa\rho-i\varkappa^2x/2k\}. \quad (30)$$

We obtained this equation in[1] by a different method. We note also that one can easily obtain (30) also directly from the averaged Eq. (24). We apply the operator formula for writing out the solution to evaluate the coherence function

$$\Gamma(x,\rho_1,\rho_2)=\langle u(x,\rho_1)u^*(x,\rho_2)\rangle$$

for the model (27). Using (24) to form the requisite combination uu* and averaging we get

$$\Gamma(x,\rho_1,\rho_2)=\exp\left\{\frac{i}{2k}\int_0^x d\xi_1\left[\frac{\delta^2}{\delta\tau_1^2(\xi_1)}-\frac{\delta^2}{\delta\tau_2^2(\xi_1)}\right]\right\}u_0\left(\rho_1+\int_0^x d\xi_1\tau_1(\xi_1)\right)u_0^*\left(\rho_2+\int_0^x d\xi_1\tau_2(\xi_1)\right)$$
$$\times\exp\left\{-\frac{\pi k^2}{4}\int_0^x d\xi_1 D\left(\rho_1-\rho_2+\int_{\xi_1}^x d\eta_1[\tau_1(\eta_1)-\tau_2(\eta_1)]\right)\right\}\Bigg|_{\tau_{1,2}=0}, \quad (31)$$

where $\pi D(\boldsymbol{\rho}) = A(0) - A(\boldsymbol{\rho})$. We now again use the Fourier transform (29) after which we get

$$\Gamma(x,\rho_1,\rho_2)=\int d\varkappa_1 d\varkappa_2\exp\{i(\varkappa_1\rho_1-\varkappa_2\rho_2)\}u_0(\varkappa_1)u_0^*(\varkappa_2)$$
$$\times\exp\left\{\frac{i}{2k}\int_0^x d\xi_1\left[\frac{\delta^2}{\delta\tau_1^2(\xi_1)}-\frac{\delta^2}{\delta\tau_2^2(\xi_1)}\right]\right\}\exp\left\{i\int_0^x d\xi_1[\varkappa_1\tau_1(\xi_1)-\varkappa_2\tau_2(\xi_1)]-\frac{\pi k^2}{4}\int_0^x d\xi_1 D\left(\rho_1-\rho_2+\int_{\xi_1}^x d\eta_1[\tau_1(\eta_1)-\tau_2(\eta_1)]\right)\right\}\Bigg|_{\tau_{1,2}=0}. \quad (32)$$

We put

$$\tau_1(\eta)-\tau_2(\eta)=\mathbf{a}(\eta),\quad \tau_1(\eta)+\tau_2(\eta)=\mathbf{b}(\eta).$$

and then

$$\delta^2/\delta\tau_1^2(\xi)-\delta^2/\delta\tau_2^2(\xi)=4\delta^2/\delta\mathbf{a}(\xi)\delta\mathbf{b}(\xi)$$

and we can perform the operation $\delta/\delta\mathbf{b}(\xi)$ as $\mathbf{b}$ occurs only in the exponent:

$$\Gamma(x,\rho_1,\rho_2)=\int d\varkappa_1 d\varkappa_2\exp\{i(\varkappa_1\rho_1-\varkappa_2\rho_2)\}u_0(\varkappa_1)u_0^*(\varkappa_2)\exp\left\{-\frac{\varkappa_1-\varkappa_2}{k}\int_0^x d\eta_1\frac{\delta}{\delta\mathbf{a}(\eta_1)}\right\}\exp\left\{i\frac{\varkappa_1+\varkappa_2}{2}\int_0^x d\xi_1\mathbf{a}(\xi_1)-\frac{\pi k^2}{4}\int_0^x d\xi_1 D\left(\rho_1-\rho_2+\int_{\xi_1}^x d\eta\mathbf{a}(\eta)\right)\right\}\Bigg|_{a=0}.$$

The action of the remaining operator is reduced to a shift and we get finally

$$\Gamma(x,\rho_1,\rho_2)=\int d\varkappa_1 d\varkappa_2\exp\left\{i(\varkappa_1\rho_1-\varkappa_2\rho_2)-i\frac{(\varkappa_1^2-\varkappa_2^2)x}{2k}-\frac{\pi k^2}{4}\int_0^x d\xi D\left(\rho_1-\rho_2-\frac{\varkappa_1-\varkappa_2}{k}\xi\right)\right\}u_0(\varkappa_1)u_0^*(\varkappa_2). \quad (33)$$

One checks easily that (33) is a solution of the equation

$$\frac{\partial\Gamma}{\partial x}-\frac{i}{2k}(\Delta_{\rho_1}-\Delta_{\rho_2})\Gamma+\frac{\pi k^2}{4}D(\rho_1-\rho_2)\Gamma=0,\qquad \Gamma(0,\rho_1,\rho_2)=u_0(\rho_1)u_0^*(\rho_2), \quad (34)$$

which was obtained in[6,1] and which is equivalent, as was shown by Dolin,[6] to the small-angle approximation of the radiative transfer equation. An expression equivalent to (33) was obtained by Dolin[7] when solving the radiative transfer equation.

In the case of a plane incident wave $u_0(\boldsymbol{\kappa}) = u_0\delta(\boldsymbol{\kappa})$ and it follows from (33) that

$$\Gamma(x,\rho_1,\rho_2)=\Gamma_0\exp\{-\pi k^2D(\rho_1-\rho_2)x/4\},$$

which is the same as the result given in[1].

4. REGION OF APPLICABILITY OF THE PARABOLIC APPROXIMATION AND THE MARKOVIAN MODEL

We compare the expressions (22) and (25) for the solutions of the complete scalar equation and the parabolic approximation. If we put $v_x = 0$ in the integral in (22) in the arguments of the δ-function and of ϵ_1 and only differentiate the factor $\exp\{ik(x - x')\}$ with respect to x', we can reduce the expression thus obtained after a change of variables to the form $\psi = u(x, \boldsymbol{\rho})e^{ikx}$ where $u(x, \boldsymbol{\rho})$ is given by Eq. (25).

We study the error arising when changing to the parabolic approximation in the expression for the average field. We average (22) and right away use the asymptotic expression for $kx \gg 1$. Differentiation with respect to x' can then be changed to differentiation with respect to x and then

$$\langle\psi(x,\rho)\rangle=-2i\frac{\partial}{\partial x}\int_0^\infty d\xi\exp\{ikx-\xi\cdot 0\}\int Dv\delta\left(x-2k\xi+\int_0^\xi d\eta v_x(\eta)\right)\exp\left\{\frac{i}{4}\int_0^\xi d\eta v^2(\eta)-\frac{k^4}{2}\iint_0^\xi d\eta_1 d\eta_2\right.$$
$$\left.\times B\left(2k(\eta_1-\eta_2)+\int_{\eta_1}^{\eta_2}d\eta v_x(\eta),\ \int_{\eta_1}^{\eta_2}d\eta\mathbf{v}_\perp(\eta)\right)\right\}u_0\left(\rho+\int_0^\xi d\eta\mathbf{v}_\perp(\eta)\right). \quad (35)$$

The change to the parabolic equation is equivalent to neglecting in (35) the quantity v_x (everywhere except the term v_x^2) and differentiating only the factor e^{ikx} with respect to x. To find the corrections to the solution of the parabolic equation we take in (35) terms with v_x into account in the linear approximation (i.e., exactly in the δ-function sign and to first approximation in the expansion $B(x + \xi, \boldsymbol{\rho}) = B(x, \boldsymbol{\rho}) + \xi B'_x(x, \boldsymbol{\rho}) + \ldots$). Moreover, we use the model (27) of delta-function correlated inhomogeneities and we shall assume that u_0 = constant. The integration over η_1, η_2, and $\mathbf{v}_\perp$ in (35) is elementary (the last one reduces to the normalization integral) and we get

$$\langle\psi(x,\rho)\rangle=\frac{u_0}{ik}\frac{\partial}{\partial x}\exp\{ikx\}\int_0^\infty d\xi\exp\left\{-\frac{k^2A(0)}{8}(4k\xi-x)\right\}\int Dv_x\exp\left\{\frac{i}{4}\int_0^\xi d\eta v_x^2(\eta)\right\}\delta\left(\xi-\frac{x}{2k}-\frac{1}{2k}\int_0^\xi d\eta v_x(\eta)\right). \quad (36)$$

We can evaluate the integral over v_x (it reduces to the normalization integral) by substituting

$$\delta(x)=(2\pi)^{-1}\int d\alpha e^{i\alpha x}$$

and after that making the shift $v_x = u + \alpha/k$. The final expression for $\langle\psi\rangle$ has the form

$$\langle\psi(x, \rho)\rangle = u_0[1 - ikA(0)/4\sqrt{1 + ikA(0)/2}] \times \exp\{ikx\sqrt{1 + ikA(0)/2} + k^2A(0)x/8\}. \quad (37)$$

The solution (37) goes over into the corresponding solution of the parabolic equation

$$\langle\psi(x)\rangle = u_0 \exp\{ikx - k^2A(0)x/8\} = u_0 \exp\{ikx - \gamma x/2\},$$

where $\gamma = k^2A(0)/4$ is the scattering coefficient, if the condition $kA(0) \ll 1$ is satisfied, or

$$\gamma \ll k, \qquad \lambda\gamma \ll 1 \quad (38)$$

(attenuation of the average field over a wavelength small) as well as the condition $k^3A^2(0)x \ll 1$, or

$$\gamma^2 x \ll k. \quad (39)$$

This last condition imposes a restriction upon the distance over which the parabolic equation has the correct value of phase. One can interpret condition (39) as the requirement that the radius of the first Fresnel zone is small compared to the extinction length:

$$\sqrt{\lambda x} \ll 1/\gamma = d.$$

If we bear in mind that $A(0) \sim \sigma_\epsilon^2 l_0$, where σ_ϵ^2 is the mean square of the dielectric constant fluctuations and l_0 their correlation radius, we can write (38) in the form

$$kl_0\sigma_\varepsilon^2 \ll 1. \quad (38a)$$

We obtained in[2] a condition under which the delta-function correlatedness of the inhomogeneities was applicable to the parabolic equation. This condition is the same as (38a).[3)]

The condition formulated here imposes a restriction upon the magnitude of the fluctuations σ_ϵ^2. Apart from these restrictions we also have the purely "geometrical" conditions for the applicability of the parabolic equation: $k l_0 \gg 1$, $x \ll k^3 l_0^4$ (see, for instance,[8]) which, in contrast to condition (38a) could be obtained relatively simply.

As the condition for the applicability of the parabolic equation and the Markovian approximation $\lambda\gamma \ll 1$ turn out to be the same, if it is violated both the parabolic equation and the Markovian approximation lose their validity simultaneously. Because of this the Markovian approximation can not be used in the case when the parabolic approximation is inapplicable. Physically this is explained by the fact that if backward scattering is taken into account delta-function correlated inhomogeneities cause total reflection of the wave and thus, if backward scattering is important in a given problem, it is impossible to neglect the longitudinal dimensions of the inhomogeneities.

In conclusion we note that if we consider the corrections to the mutual coherence function $\Gamma(x, \rho_1, \rho_2) = \langle u(x, \rho_1)u^*(x, \rho_2)\rangle$ we can obtain yet another restriction:

$$\sigma_\varepsilon^2 kx \ll 1,$$

the meaning of which lies in the requirement that the backward scattering intensity of the wave is small.

[3)] As the result of the inexactness allowed in [2] we derived instead of the condition $\gamma\lambda \ll 1$ the condition $\gamma l_0 \ll 1$.

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Translated by D. ter Haar
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