SOVIET PHYSICS JETP VOLUME 31, NUMBER 2 AUGUST, 1970

RAREFIED IMPERFECT BOSE GAS IN A FIELD OF RANDOMLY DISTRIBUTED FIXED IMPURITIES

V. A. GERGEL', R. F. KAZARINOV, and R. A. SURIS

Submitted July 31, 1969

Zh. Eksp. Teor. Fiz. **58**, 686–698 (February, 1970)

The problem of the behavior of a slightly imperfect Bose gas in a field of randomly distributed impurities is considered. The effect of the impurity atoms on the ground state of the system is determined and found to reduce to a local change in the density of the condensate near the impurity atoms. The spectrum and attenuation of one-particle excitations are calculated. The spectrum of the long wavelength excitations turns out to be acoustic, and their scattering by fluctuations of the sound velocity induced by the impurity atoms turns out to be Rayleigh scattering, i.e., the scattering cross section turns out to be proportional to the fourth power of the wave vector of the excitation. The obtained results are used to estimate the linewidths for the absorption and amplification of light by a Bose-Einstein condensate of excitons in semiconductors. It is found that for a sufficiently high concentration of excitons this width is much smaller than the width of the usual exciton absorption line, which is due to the scattering of individual excitons by lattice defects.

INTRODUCTION

IN an article[1] by the authors at one time it was shown that the appearance of a Bose-Einstein condensate of excitons leads to a number of interesting features in the optical properties of semiconductors with allowed direct transitions. The major effect stipulated by the presence of the condensate is the formation of a line of negative light absorption.

The results of article [1] were obtained by neglecting the scattering of excitons by defects in the crystal lattice, and the estimates cited there for the linewidths did not take into account the specific properties of the scattering of excitons which are associated with their Bose condensation. The object of the present article is an investigation of the properties of a Bose-Einstein condensate of excitons in the presence of randomly distributed crystal defects. The results of article [1] were obtained for the case of a small density of excitons under the assumption of their effective repulsion. In this connection the problem in fact reduced to an investigation of an imperfect Bose gas of small density. In the present article we shall not complicate the discussion by a systematic account of the internal structure of the excitons, but we shall treat the problem of a slightly imperfect Bose gas in a field of randomly distributed impurities. It should be noted that this problem is of interest by itself, irrespective of its connection with the optical properties of semiconductors containing an exciton condensate.

Those difficulties which arise in connection with an investigation of the scattering of particles with large wavelengths in a random field of defects are well known. These difficulties are associated with the fact that when the wavelength of the particle becomes comparable with its mean free path, it is necessary to take into account simultaneous scattering by different impurities. This complicated problem has not been solved up to the present time. Since the basic features of a slightly imperfect Bose gas appear precisely in the region of small momenta, at first glance it may seem that this markedly complicates the case, if it does not make it impossible in general to solve our problem. The situation, however, is not so hopeless. It is well known[2] that the long-wave one-particle excitations of a slightly imperfect Bose gas are sound quanta. Therefore one can anticipate that their scattering will obey the Rayleigh law, i.e., the cross section turns out to be proportional to the fourth power of the wave vector. Therefore the mean free path will increase more rapidly than the wavelength, and the nonacceptable situation mentioned above will never be realized.

The calculation presented below confirms these simple qualitative considerations. In the following section the spectrum and attenuation of the one-particle excitations of a slightly imperfect Bose gas in a field of randomly distributed impurities will be calculated, starting from hydrodynamical equations. Then the same results will be obtained with the aid of the formalism of Belyaev, with additional account of the scattering by impurities. The first semi-phenomenological approach, being less rigorous, gives a physically more intuitive picture of the phenomenon under consideration. The microscopic theory, in which a rather complicated diagram technique is applied, is much less intuitive. In return it eliminates doubt as to the correctness of the obtained results, since it does not contain any unproved assumptions. The agreement of the results of the semiphenomenological investigation with the results obtained in the microscopic theory to a certain extent confirm the correctness of the starting equation of the semiphenomenological theory. For simplicity we shall everywhere consider the temperature equal to zero.

SEMIPHENOMENOLOGICAL INVESTIGATION OF THE PROBLEM

In this section we shall start from the equation for the so-called "macroscopic" wave function Ψ:[1)]

$$i\frac{\partial\Psi}{\partial t} = -\frac{\Delta}{2m}\Psi + \tau|\Psi|^2\Psi + V\Psi. \tag{1}$$

[1)] We shall use the system of units in which $\hbar = 1$.

Here m denotes the mass of the particle, and V $= \sum_i u(\mathbf{r} - \mathbf{R}_i)$ is the potential of the randomly distributed impurity atoms. The term $\tau|\Psi|^2\Psi$ takes into account the interaction energy of the particles in the low-density approximation. The quantity τ is the matrix for the forward scattering of particles by each other, and its sign corresponds to a repulsion between the particles. Equation (1) appears to be quite plausible, recalling the equation for the superfluid component in the two-fluid model of a superfluid liquid.[2] In the absence of impurities Eq. (1) gives exactly the same dispersion law for the elementary excitations as the microscopic theory of Belyaev.[3] One can have confidence that Eq. (1) correctly describes the phenomena of interest to us.

It is convenient to seek the wave function in the form $\Psi = \sqrt{n}\, e^{i\varphi}$, where n denotes the concentration and $m^{-1}\nabla\varphi = \mathbf{v}$ denotes the velocity of the particles. According to Eq. (1) n and **v** satisfy the system of equations

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{m}\nabla\left(-\frac{\Delta\sqrt{n}}{2m\sqrt{n}} + \frac{mv^2}{2} + \tau n + V\right) = 0. \qquad (2)$$

Equations (2) describe both the ground state of the system and the propagation in it of excitations whose spectrum and attenuation we must determine.

In the ground state the velocity is equal to zero, and the concentration does not depend on the time. It is determined from the second of Eqs. (2) in which $\partial\mathbf{v}/\partial t$ and **v** are set equal to zero:

$$-\frac{\Delta\sqrt{n}}{2m\sqrt{n}} + \tau n + V = \mu, \qquad (3)$$

where μ is an integration constant which is the chemical potential of the system. It is easy to see that in the ground state the concentration of particles varies from point to point due to the action of the potential of the impurities. We must find this dependence of n on the coordinates because the excitations are scattered both by the potential V and by the inhomogeneities of n.

If the potential u (**r**) of an individual impurity varies slightly over distances of the order of the characteristic length $k_0^{-1} = (m\tau n_0)^{-1/2}$ (n_0 is the average concentration of bosons), then the concentration repeats the profile of the potential V. In the present article we consider the opposite limiting case, when the potential u (**r**) of an individual impurity corresponds to a short-range interaction such that $\varphi k_0 \ll 1$ (φ is the amplitude for the scattering of a free particle by the potential u (**r**)), as of more interest physically. We note that the results which we obtain will also be valid for impurities with a slowly varying potential. In this case it will only be necessary to replace in all formulas the amplitude for the scattering by an individual impurity by the corresponding Fourier component of the potential u (**r**).

Let us assume that the solution of Eq. (3) differs slightly from the homogeneous solution, i.e., we shall seek it in the form $n = n_0 + \chi$, where $n_0 \equiv \bar{n}$ is the average concentration, and χ, the correction which depends on the coordinates, is small in comparison with n_0. In fact, in the immediate vicinity of an impurity atom (at distances of the order of φ) χ is not small. But for the problem of interest to us concerning the scattering of long-wave excitations, this region turns out to be unimportant. Linearizing Eq. (3) with respect to χ, we obtain the equation

$$\left[-\frac{\Delta}{2m} + 3\tau n_0 - \mu + V\right]\frac{\chi}{2n_0} = \mu - \tau n_0 - V, \qquad (4)$$

whose exact solution, written in the momentum representation, is given by

$$\frac{\chi(\mathbf{p})}{2n_0} = g(\mathbf{p})\{(\tau n_0 - \mu)\delta(\mathbf{p}) + T_{\mathbf{p},0}[1 + (\tau n_0 - \mu)g(0)]\}, \qquad (5)$$

where $T_{\mathbf{p},0}$ is the matrix for the scattering of a particle with energy $\mu - 3\tau n_0$ by the potential V, and g (p) $= -[(p^2/2m) + 3\tau n_0 - \mu]^{-1}$ is the free Green's function. It is easy to verify that the matrix for scattering by the potential V of many impurities is represented by the sum

$$T_{\mathbf{p},\mathbf{p}'} = \sum_i \overset{\mathbf{R}_i}{t}_{\mathbf{p},\mathbf{p}'} + \sum_{i\neq j}\int \frac{d^3p_1}{(2\pi)^3}\overset{\mathbf{R}_i}{t}_{\mathbf{p},\mathbf{p}_1} g(\mathbf{p}_1)\overset{\mathbf{R}_j}{t}_{\mathbf{p}_1,\mathbf{p}'}$$
$$+ \sum_{\substack{i\neq j\\ j\neq k}}\int \frac{d^3p_1\, d^3p_2}{(2\pi)^6}\overset{\mathbf{R}_i}{t}_{\mathbf{p},\mathbf{p}_1} g(\mathbf{p}_1)\overset{\mathbf{R}_j}{t}_{\mathbf{p}_1,\mathbf{p}_2} g(\mathbf{p}_2)\overset{\mathbf{R}_k}{t}_{\mathbf{p}_2,\mathbf{p}'} + \ldots, \qquad (6)$$

where $\overset{\mathbf{R}_i}{t}_{\mathbf{p},\mathbf{p}'} = t_{\mathbf{p},\mathbf{p}'} \exp\{i(\mathbf{p} - \mathbf{p}')\ \mathbf{R}_i\}$ is the matrix for scattering by an impurity located at the point $\mathbf{R}_i$. According to the definition of n_0, the average of χ with respect to the position of the impurities is equal to zero. This determines the value of the chemical potential μ. In order to find μ, it is necessary to average expression (6) over all possible configurations of the impurity atoms. If the impurity atoms are randomly distributed over the crystal, the averaging means integration over the coordinates of each impurity atom with subsequent division by the corresponding power of the volume. In connection with such averaging the terms of (6), in which all impurity atoms are different, form the series

$$\delta(\mathbf{p} - \mathbf{p}')\nu t_{\mathbf{p},\mathbf{p}}[1 + g(\mathbf{p})\nu t_{\mathbf{p},\mathbf{p}} + g(\mathbf{p})\nu t_{\mathbf{p},\mathbf{p}} g(\mathbf{p})\nu t_{\mathbf{p},\mathbf{p}} + \ldots]$$
$$= \delta(\mathbf{p} - \mathbf{p}')\nu t_{\mathbf{p},\mathbf{p}} / [1 - g(\mathbf{p})\nu t_{\mathbf{p},\mathbf{p}}], \qquad (7)$$

corresponding to successive independent scattering. Here ν denotes the concentration of impurity atoms. Having substituted (7) into expression (5) and having set it equal to zero, we find that

$$\mu = \tau n_0 + t_{0,0}\nu. \qquad (8)$$

But in addition to the series (7) the averaging of those terms of (6) in which the same impurity is encountered more than once gives an expression corresponding to simultaneous scattering. The diagrams shown in Fig. 1 correspond to the simplest of these. Here a solid line represents the free Green's function, a cross represents the scattering matrix $\overset{\mathbf{R}_i}{t}_{\mathbf{p},\mathbf{p}'}$, and the dashed lines join crosses pertaining to a single atom.

Calculation of the so-called "imbedded" graphs of the type shown in Fig. 1a is rather simple. Inside the dashed lines two or more crosses corresponding to in-

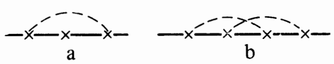

FIG. 1

dependent scattering may be found. The summation of all such graphs leads to the result that inside the dashed lines a Green's function is found in which the energy is shifted by an amount $-t\nu$, i.e., equal to $\mu - 3\tau n_0 - t\nu = -2\tau n_0$. Thus, taking account of the imbedded graphs only leads to the replacement of $t^{t\nu-2\tau n_0}_{\mathbf{p},\mathbf{p}}$ by $t^{-2\tau n_0}_{\mathbf{p},\mathbf{p}}$. In the general case it is impossible to take "overlapping" graphs of the type shown in Fig. 1b into account exactly, but we may neglect them if the impurity concentration satisfies the inequality $\nu\varphi^2/k_0 \ll 1$ where $\varphi = mt/2\pi$. We assume everywhere in what follows that this rather weak condition on the impurity concentration is fulfilled. In fact, since with the shift due to successive scattering taken into account the energy in the internal Green's functions of diagram 1b is negative and equal to $-2\tau n_0$, its value is proportional to

$$t\nu\frac{\nu\varphi^2}{k_0}(\varphi k_0)\ln\left(\frac{1}{\varphi k_0}\right) \ll t\nu.$$

In order to illustrate the nature of the inhomogeneities in the concentration of bosons, which are generated by the impurity atoms, in Eq. (5) we fix the coordinate of the i-th atom and average over the position of all remaining atoms. In the coordinate representation we shall have

$$\chi^{\mathbf{R}_i}(\mathbf{r}) = -n_0\frac{\varphi}{|\mathbf{r}-\mathbf{R}_i|}\exp\{-2k_0|\mathbf{r}-\mathbf{R}_i|\}. \tag{9}$$

Here it is appropriate to make a comment with regard to the assumption $\chi \ll n_0$, which we used in order to solve Eq. (3). As a consequence of this assumption formula (9) in general is not valid for $|\mathbf{r} - \mathbf{R}_i| \lesssim \varphi$. However, as will be shown below the Fourier transform of expression (9) appears in the cross section for the scattering of excitations. It is quite clear that for long waves, $k \ll k_0$, the region $r \lesssim \varphi$ in which the nonlinearity of Eq. (3) is important and where $\chi \sim n_0$ gives a negligible contribution (of the order of $(\varphi k_0)^2$) to this Fourier transform.

Having determined the ground state of the system, we proceed to determine the spectrum and attenuation of its elementary excitations. For this purpose, having returned to Eqs. (2), let us consider a state differing slightly from the ground state, i.e., such that $\delta n \equiv n - n_0 - \chi \ll n_0$, $v^2 \ll \tau n_0 m^{-1}$. Linearizing Eqs. (2) and eliminating $\mathbf{v}$, we obtain

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta + \frac{\Delta^2}{4m^2}\right)\delta n = \nabla\left\{\frac{\chi}{n_0}\nabla\left[\left(c^2 - \frac{\Delta}{4m^2}\right)\delta n\right\}\right.$$
$$+\frac{\Delta}{4m^2}\left[\Delta\left(\delta n\frac{\chi}{n_0}\right) - \nabla\delta n\cdot\nabla\frac{\chi}{n_0}\right] \equiv \hat{F}[\chi]\,\delta n, \tag{10}$$

where $c^2 = \tau n_0 m^{-1}$. Equation (3), in which $n = n_0 + \chi$, was used in the derivation of Eq. (10), and also the nonlinear terms in χ are omitted on the basis of the remark made above. The spectrum and attenuation of the elementary excitations of the system are determined by the poles of the Green's function $G(\omega, \mathbf{p}, \mathbf{p}')$ of this equation. According to Eq. (10)

$$G(\omega,\mathbf{p},\mathbf{p}') = G_0(\omega,\mathbf{p})\delta(\mathbf{p}-\mathbf{p}') + \int\frac{d^3p''}{(2\pi)^3}G_0(\omega,\mathbf{p})\langle\mathbf{p}|\hat{F}[\chi]|\mathbf{p}''\rangle G(\omega,\mathbf{p}'',\mathbf{p}'). \tag{11}$$

Here $G_0(\omega, \mathbf{p}) = (\omega^2 - c^2p^2 - (p^4/4m^2) + i\delta)^{-1}$ is the Green's function in the absence of an impurity, and

$$\langle\mathbf{p}|\hat{F}[\chi]|\mathbf{p}'\rangle = \left(\frac{\chi}{n_0}\right)_{p-p'}\left[\frac{p'^2(p^2+p'^2)}{4m^2} - (\mathbf{p}\mathbf{p}')\left(\frac{p^2+p'^2}{4m^2}+c^2\right)\right]. \tag{12}$$

We are interested not in an exact solution of Eq. (11), but in a solution $\overline{G}(\omega, \mathbf{p})$ averaged over all impurity configurations. One can find it by iterating (11) and averaging all expressions which arise in this connection. The averaging procedure was explained by us in considerable detail in connection with the calculation of the chemical potential; therefore we shall not concentrate attention on it just now. As usual the averaged Green's function has the form

$$\overline{G}(\omega,\mathbf{p}) = [G_0^{-1}(\omega,\mathbf{p}) - \Sigma(\omega,\mathbf{p})]^{-1}. \tag{13}$$

The simplest term of the mass operator, $\overline{\Sigma}_1 = \overline{\langle\mathbf{p}\,|\,F[\chi]\,|\mathbf{p}\rangle}$ vanishes since $\overline{\chi} = 0$. The major term turns out to be the operator

$$\Sigma_2 = \int\frac{d^3p_1}{(2\pi)^3}\overline{\langle\mathbf{p}|F[\chi]|\mathbf{p}_1\rangle G_0(\omega,\mathbf{p}_1)\langle\mathbf{p}_1|F[\chi]|\mathbf{p}\rangle}, \tag{14}$$

which corresponds to the graph shown in Fig. 2a, where a cross means $\langle\mathbf{p}\,|\,\hat{F}[\chi]\,|\mathbf{p}'\rangle$. For $p \ll mc$ (long wavelength excitations)

$$\Sigma_2 = -\frac{\pi}{3}\frac{\nu\varphi^2}{m}cp^2\left[7 + i\left(\frac{\omega}{mc^2}\right)^3\right]. \tag{15}$$

As simple but rather tedious calculations show, the expressions corresponding to the graphs for simultaneous scattering shown in Figs. 1b and 2b are $\nu\varphi^2/k_0$ times smaller than (15). Having substituted (15) into (13) we finally find that for momenta smaller than mc the pole of the Green's function $\overline{G}(\omega, \mathbf{p})$ is located at the point

$$\omega = cp - \frac{7\pi}{6}\frac{\nu\varphi^2}{m}p - i\frac{\pi}{6}\nu\varphi^2c\left(\frac{p}{mc}\right)^4. \tag{16}$$

The quantity $-7\pi\nu\varphi^2/6m$ represents the change in the velocity of sound due to scattering by impurities, and $\frac{1}{6}\pi\nu\varphi^2c\,(p/mc)^4$ describes the attenuation of the excitations. Thus, the long-wave excitations of our system of bosons are sound quanta which, just as is assumed for a sound wave, are scattered by impurities according to the Rayleigh law.

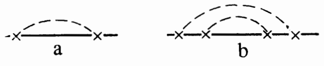

FIG. 2

Here it should be noted that the Rayleigh character of the scattering of sound excitations in a superfluid liquid was obtained by Khalatnikov and Zharkov[4] within the framework of a phenomenological theory of superfluidity. For an infinitely heavy impurity the interaction Hamiltonian used by them takes the form

$$H_{int} = \frac{\partial\varepsilon_0}{\partial\rho}\rho' + \frac{1}{2}\frac{\partial^2\varepsilon_0}{\partial\rho^2}\rho'^2,$$

where $\epsilon_0(\rho)$ is the energy associated with the introduction of an impurity into the superfluid liquid and ρ' denotes the correction to the density. In application to a system of small density ($c^2 \sim \rho \equiv mn$) the differential cross section for the scattering of a phonon by an impurity at an angle θ is given by

$$d\sigma = \left(\frac{p^2}{4\pi\hbar^2c^2}\right)^2\left[\frac{\partial\varepsilon_0}{\partial\rho}\cos\theta - \rho\frac{\partial^2\varepsilon_0}{\partial\rho^2}\right]^2, \tag{17}$$

and one can easily determine the phenomenological constants $\partial\epsilon_0/\partial\rho$ and $\partial^2\epsilon_0/\partial\rho^2$ by knowing the distribution of the condensate concentration near an impurity atom, which is given by Eq. (9):

$$\frac{\partial\varepsilon_0}{\partial\rho}=2\pi\frac{\hbar^2}{m^2}\varphi, \qquad \frac{\partial^2\varepsilon_0}{\partial\rho^2}=\frac{3}{2}\pi\frac{\hbar^2}{m^2}\frac{\varphi}{\rho}\sqrt{\varphi k_0}.$$

Thus $\rho(\partial^2\epsilon_0/\partial\rho^2)/(\partial\epsilon_0/\partial\rho) \approx \varphi k_0 \ll 1$ and, considering the scattering of acoustic excitations of a system of small density by impurities with a short-range potential, one can restrict one's attention to only the first term inside the square brackets of (17) for all angles θ with the exception of a small region near $\theta = \pi/2$, which is unimportant to us since we are interested in the integral cross section, which determines the damping of the excitations. In this connection expression (17) exactly agrees with the formula for the cross section for scattering by an individual impurity, which in our formalism can be obtained directly from Eq. (10) by substituting formula (9) into it.

We may regard our problem as solved provided there is no "element of doubt" associated with the starting equation of the semiphenomenological theory. Therefore below we consider the problem of a slightly imperfect Bose gas in a field of fixed impurities at T = 0 starting from, so to speak, first principles.

THE MICROSCOPIC THEORY

It is well known that at T = 0 all particles of a perfect Bose gas are found in the state of minimum energy, in other words, in the condensate. Therefore Wick's theorem is not valid for the field operators of the free bosons and, in considering the interacting Bose particles we may not use the usual diagram technique. A formalism for calculating the properties of a Bose system was developed by Belyaev.[3] Its distinctive feature consists in the fact that the operators $\xi^+(\xi)$ of creation (annihilation) of particles in the condensate are separated out in the field operators of the bosons. Here the one-particle Green's function is divided into two parts: the condensate $-i\langle T[\xi(t)\xi(t')]\rangle$ and the Green's function $G(x, x') = -i\langle T[\psi(x), \psi(x')]\rangle$ for the excitations of the uncondensed phase, where $\psi(x)$ and $\psi^+(x)$ are the field operators of the particles after subtraction of the condensate operators ξ and ξ^+. Since Wick's theorem is satisfied for the free operators of the uncondensed phase, one can represent the Green's function of the uncondensed phase in the form of a sum of graphs in which the exact operators ξ for the condensate play the role of a classical external field, equal to $\sqrt{n_0}$, where n_0 denotes the at present unknown exact concentration of particles in the condensate with the interaction taken into account. The summation of these diagrams leads to a system of equations relating the function $G(x, x')$ and the "anomalous" Green's function $F^+(x, x') = -i\langle T[\psi^+(x)\psi^+(x')]\rangle$, corresponding to the propagation of a pair of particles which are excited out of the condensate. In these equations the concentration n_0 of particles in the condensate is a parameter which is later determined from the conditions that the total number of particles in the system is constant and minimization of its energy.

In spite of the fact that in our problem the homogeneous nature of space is violated by the potential V of the impurities, as the condensate operators we distinguish operators for the creation and annihilation of particles in the state with zero momentum. Since the concentration of impurity atoms satisfies the condition $\nu\varphi^2/k_0 \ll 1$, as will be seen below, the interaction with the impurities turns out to be small in the sense that a macroscopic occupation of the state with p = 0 is not destroyed.

The Green's functions of the uncondensed phase, averaged with respect to the position of the impurity atoms, are expressed in terms of normal $\Sigma^{11}(\omega, p)$ and anomalous $\Sigma^{02}(\omega, p)$, $\Sigma^{20}(\omega, p)$ self-energy parts in the following way:[5]

$$G(\omega, \mathbf{p}) = \frac{\omega - A(\omega, \mathbf{p}) + p^2/2m + S(\omega, \mathbf{p}) - \mu}{(\omega - A(\omega, \mathbf{p}))^2 - (p^2/2m + S(\omega, \mathbf{p}) - \mu)^2 + \Sigma^{02}(\omega, \mathbf{p})\Sigma^{20}(\omega, \mathbf{p}) + i\delta},$$

$$F(\omega, \mathbf{p}) = \frac{-\Sigma^{02}(\omega, \mathbf{p})}{(\omega - A(\omega, \mathbf{p}))^2 - (p^2/2m + S(\omega, \mathbf{p}) - \mu)^2 + \Sigma^{02}(\omega, \mathbf{p})\Sigma^{20}(\omega, \mathbf{p}) + i\delta} \tag{18}$$

where

$$A(\omega, \mathbf{p}) = {}^1\!/_2[\Sigma^{11}(\omega, \mathbf{p}) - \Sigma^{11}(-\omega, -\mathbf{p})],$$

$$S(\omega, \mathbf{p}) = {}^1\!/_2[\Sigma^{11}(\omega, \mathbf{p}) + \Sigma^{11}(-\omega, -\mathbf{p})], \qquad \mu = \Sigma^{11}(0, 0) - \Sigma^{02}(0, 0). \tag{19}$$

The spectrum and attenuation of the elementary excitations are determined by the positions of the poles of these Green's functions.

In the absence of impurities and in the approximation of small density

$$\Sigma^{11} = 2\tau n_0, \qquad \Sigma^{02} = \Sigma^{20} = \tau n_0, \tag{20}$$

where n_0 denotes the concentration of the condensate, and τ is the matrix describing the forward scattering of the particles against each other. We must determine the corrections to expressions (20) due to the interaction of the particles with the impurity atoms, which we denote by $\Delta\Sigma^{11}(\omega, p)$ and $\Delta\Sigma^{02}(\omega, p)$. In this connection we shall use the usual technique of averaging over the position of the impurity atoms, which is used, for example, in the theory of superconducting alloys.[5] However, in our problem the situation is complicated somewhat by the fact that the interaction of the particles with the impurity atoms is significantly renormalized due to the interaction between bosons. In fact, as we saw in the previous section, the impurity atoms create around themselves inhomogeneities in the density of the condensate, and the excitations are scattered by both the potential of the impurity atoms and by these inhomogeneities. In this connection, together with the renormalized interaction with an individual impurity atom (we denote the effective potential for such a process by Γ_1 and we shall represent it by a clear triangle), expressions appear corresponding to an interaction with inhomogeneities in the density of the condensate which are "nonadditive" with respect to the impurity atoms. We shall represent them in the form of the graphs shown in Fig. 3. By definition these graphs cannot be represented as a sequence of triangles joined by a single line.

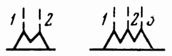

FIG. 3

It turns out that multiple scattering by the potential of a single impurity atom and by the inhomogeneities of the condensate created by it give the major contribution to $\Delta\Sigma$. A sequence of diagrams corresponding to this process is shown in Fig. 4. Since the scattering by one impurity atom occurs in the background of the interactions with all remaining impurity atoms, the solid lines in these graphs denote the exact, averaged Green's functions given by (18). Thereby all possible "imbedded" diagrams are taken into consideration. As to the scattering by inhomogeneities in the concentration of condensate which are nonadditive with respect to the impurity atoms, which corresponds, for example, to the diagram shown in Fig. 5c, we show that upon fulfilment of the condition $\nu\varphi^2/k_0 \ll 1$ one can neglect this process, and one can equally well neglect simultaneous scattering by additive inhomogeneities for different impurity atoms (for example, the diagram shown in Fig. 5a where the black triangle denotes the sum of the diagrams in Fig. 4).

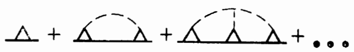

FIG. 4

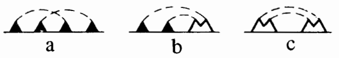
a b c

FIG. 5

Because of the fact that there are several types of Green's functions, and also normal and anomalous functions among the effective potentials Γ (with two incoming or outgoing lines), it is convenient to carry out calculations in matrix form. In this connection the solid line shown in Figs. 4 and 5 is associated with the matrix

$$\widetilde{G}(\omega, \mathbf{p}) = \begin{pmatrix} G(\omega,\mathbf{p}) & F^{+}(\omega,\mathbf{p}) \\ F(\omega,\mathbf{p}) & G(-\omega,-\mathbf{p}) \end{pmatrix}. \tag{21}$$

The effective potentials Γ are also represented by matrices. The convenience of the matrix formulation consists in the fact that matrix multiplication automatically sums all topologically equivalent diagrams, differing only by the type of internal Green's functions and effective vertices.

In order to fulfil this program we must first determine the effective potentials Γ. The difference between Γ_1 and the potential of an individual impurity is due to the existence of graphs of the type shown in Fig. 6.

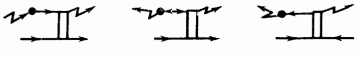

FIG. 6

Here a small circle denotes the potential u (q) of an individual impurity, the solid lines represent the functions (18) for the uncondensed phase, in which $\omega = 0$, since the impurity atoms are assumed to be fixed, a rectangle represents the matrix τ for the scattering of particles against each other, and the zig-zag lines correspond to the condensate operators. At small momentum transfers these diagrams are as important as the "hole" potential u (q). In this case the smallness of the density of the condensate is cancelled by the small denominator of the functions G (0, q) and F (0, q) as $q \to 0$.

Having combined all graphs of the type shown in Fig. 6 in which the circles of the potential u (q) are distributed in different ways with regard to the rectangle τ, we obtain the following result to first order in the impurity potential:

$$\Gamma_{1,\,\mathbf{p},\,\mathbf{p}+\mathbf{q}} - u(\mathbf{q}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = u(\mathbf{q}) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot 2\tau n_0 [G(0,\mathbf{q}) + F(0,\mathbf{q})]. \tag{22}$$

The quantity $(\Gamma_1 - u)^{11}$ is twice as large as Γ_1^{02} because of the presence of the exchange diagrams. We at once note that expression (22) vanishes at zero momentum transfer q because of the properties which the Green's functions for the uncondensed phase possess at zero momentum. In fact, formulas (18) are valid only for momenta different from zero.[3] At zero momentum the functions G and F vanish by definition since they are constructed out of operators with nonvanishing momentum. Of course, this remark pertains to any diagram in which there are unintegrated Green's functions of the uncondensed phase.

Since we are considering impurity atoms with a large short-range potential, it is impossible for us to confine our attention to the graphs of Fig. 6, but all diagrams of this type should be taken into account, in which every kind of number of circles corresponding to the impurity potential is distributed on the line with zero frequency. It is not difficult to verify that the summation of such diagrams leads to a replacement of the potential u (q) in the right hand side of expression (22) by $t_{0,\mathbf{q}}^{-2\Sigma^{02}}$, which is the matrix for the scattering of a free particle with energy $-2\Sigma^{02}$ by a single impurity. We shall represent this quantity by a cross. Now we may write in explicit form an integral equation for the effective scattering matrix for scattering by a single impurity, corresponding to a summation of the graphs in Fig. 4:

$$\begin{aligned} T^{\omega}_{\mathbf{p},\,\mathbf{p}'} = {} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(\mathbf{p}-\mathbf{p}') + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot 2\tau n_0 [G(0,\mathbf{p}-\mathbf{p}') \\ & + F(0,\mathbf{p}-\mathbf{p}')]\, t^{-2\Sigma^{02}}_{0,\,\mathbf{p}-\mathbf{p}'} + \int \frac{d^3p''}{(2\pi)^3} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(\mathbf{p}-\mathbf{p}') + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right. \\ & \left. \times 2\tau n_0 [G(0,\mathbf{p}-\mathbf{p}'') + F(0,\mathbf{p}-\mathbf{p}'')]\, \tilde{t}^{-2\Sigma^{02}}_{0,\,\mathbf{p}-\mathbf{p}''} \right\} \widetilde{G}(\omega,\mathbf{p}'')\, T^{\omega}_{\mathbf{p}'',\,\mathbf{p}'}. \end{aligned} \tag{23}$$

The change $\Delta\Sigma$ of the mass operators due to multiple scattering by a single impurity is given by

$$\Delta\Sigma(\omega,\mathbf{p}) \equiv \begin{pmatrix} \Delta\Sigma''(\omega,\mathbf{p}) & \Delta\Sigma^{02}(\omega,\mathbf{p}) \\ \Delta\Sigma^{20}(\omega,\mathbf{p}) & \Delta\Sigma(-\omega,-\mathbf{p}) \end{pmatrix} = \nu T^{\omega}_{\mathbf{p},\,\mathbf{p}}, \tag{24}$$

where ν denotes the concentration of impurity atoms.

The interaction of excitations with inhomogeneities in the density of the condensate which are nonadditive with respect to the impurity atoms correspond, for example, to the diagrams shown in Fig. 7, in which the crosses pertain to different impurity atoms. If in these diagrams both crosses belong to one and the same impurity atom, they represent nonlinear corrections of

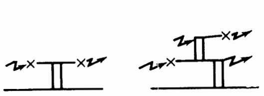

FIG. 7

order $\varphi k_0 \ll 1$ to the additive part of the inhomogeneity in the density of the condensate which appears in Eq. (23), and we may neglect them. On the other hand, from here it follows that we must solve Eq. (23) with the same degree of accuracy. The exact Green's functions averaged over the position of the impurity atoms, which enter into Eq. (23), in turn are expressed in terms of the matrix elements $T^{\omega}_{p,p'}$. Therefore, Eq. (23) should in general be solved self-consistently. But, as we see below, the attenuation of elementary excitations turns out to be so small in comparison with their energy that one can neglect it in the solution of Eq. (23). The change in the energy of the excitations due to scattering by impurities only leads to a shift of the chemical potential. In this connection the value of the chemical potential μ, which is determined from condition (19), which is equivalent to the requirement of minimizing the system's energy with respect to the concentration of particles in the condensate,[5] turns out to be such that in Eq. (23) it is necessary to substitute the Green's functions (18) in which $\Sigma^{11} - \mu = \Sigma^{02} = \tau n_0$. Strictly speaking, the interaction with impurity atoms leads to a renormalization of the velocity of sound. However, as was shown in the preceding section, $\Delta c^2/c^2 \sim \nu\varphi^2/k_0 \ll 1$, where $c^2 = m^{-1}\tau n_0$, and to take this renormalization into account in Eq. (23) would require an excess of accuracy.

Now, having used the definition of $t^{-2\tau n_0}_{p,p'}$ which is the matrix for the scattering of a free particle by a single impurity:

$$t^{-2\tau n_0}_{\mathbf{p},\mathbf{p}'} = u(\mathbf{p}-\mathbf{p}') - \int \frac{d^3p''}{(2\pi)^3} \frac{t^{-2\tau n_0}_{\mathbf{p},\mathbf{p}''} u(\mathbf{p}''-\mathbf{p}')}{p''^2/2m + 2\tau n_0}$$
$$\equiv [1 + tg(-2\tau n_0)]u, \tag{25}$$

we operate with the operator $[1 + \tan(-2\tau n_0)]$ on Eq. (23). The equation which is obtained as a result admits a solution by the method of iteration with respect to the parameter $\varphi k_0 \ll 1$. Neglecting the dispersion of t, to the first approximation we obtain

$$T^{\omega}_{\mathbf{p},\mathbf{p}'} - t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{cases} -\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \dfrac{2\tau n_0}{2\tau n_0 + (\mathbf{p}-\mathbf{p}')^2/2m}\, t, & \mathbf{p} \neq \mathbf{p}' \\ 0 & \mathbf{p} = \mathbf{p}' \end{cases} \tag{26}$$

Having substituted (25) into (23) and (19) we find that actually $\Sigma^{02} = \tau n_0$, and $\mu = \tau n_0 + t\nu$, i.e., the chemical potential is shifted by an amount $t\nu$ in comparison with its value in the system without impurities. An imaginary part of $T^{\omega}_{p,p'}$ appears in the second approximation:

$$\operatorname{Im} T^{\omega}_{\mathbf{p},\mathbf{p}} = \operatorname{Im} \int \frac{d^3p'}{(2\pi)^3} t^2 \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \frac{2\tau n_0}{2\tau n_0 + (\mathbf{p}-\mathbf{p}')^2/2m}\right] \tilde{G}(\omega, \mathbf{p}')$$
$$\times \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \frac{2\tau n_0}{2\tau n_0 + (\mathbf{p}-\mathbf{p}')^2/2m}\right]. \tag{27}$$

Since the singularity of the functions $G(\omega, p)$ and $F(\omega, p)$ at zero momentum is not essential inside the integral, we have used formulas (18) in Eq. (27) without any restrictions. In the acoustic portion of the spectrum, which is the region of interest to us, i.e., for $\omega \ll \tau n_0 \equiv mc^2$, $p \ll mc \equiv k_0$

$$\operatorname{Im} \Delta\Sigma(\omega, \mathbf{p}) = \pi \frac{\nu\varphi^2|\omega|}{m^3c^5} \left\{ \omega^2 \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - \omega\left(\frac{\omega^2}{2mc^2} + \frac{p^2}{2m}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right.$$
$$\left. + \frac{1}{16}\left(\frac{p^4}{m^2} + \frac{\omega^4}{m^2c^2} + \frac{10}{3}\frac{p^2\omega^2}{m^2c^2}\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right\}. \tag{28}$$

The following approximations give small corrections in the parameter $\varphi k_0 \ll 1$ to expressions (28) and (26). In the opposite limiting case $p \gg k_0$, $\omega \gg \tau n_0$, formula (27) gives the usual expression $2\pi\nu\varphi^2 p/m$ for the attenuation.

Now let us clarify the role of simultaneous scattering of the excitations by several impurities. As we already said above this is, in the first place, correlated scattering by the potential of several impurities and the additive part of the inhomogeneity of the density of the condensate, created by those same impurity atoms and, in the second place, scattering by inhomogeneities in the concentration of the condensate, which are nonadditive in the impurity atoms.

Some of the simplest diagrams corresponding to these processes are shown in Fig. 5. We omit the rather complicated calculations and immediately present the result of an estimate of the magnitude of these graphs. It turns out that any of the diagrams of this type, notwithstanding the factor $\nu\varphi^2/k_0$ which we regard as small, contains terms proportional to a smaller power of the frequency than (28). In the region of the acoustic part of the spectrum these terms are individually large, but all together they cancel satisfactorily. The remaining terms are of the same order in frequency as (28), but $\nu\varphi^2/k_0$ times smaller than this expression. Therefore, if the concentration of impurity atoms satisfies the inequality $\nu\varphi^2/k_0 \ll 1$, one can neglect the processes of simultaneous scattering.

One can easily understand the physical meaning of the condition $\nu\varphi^2/k_0 \ll 1$ by having noted that $1/\nu\varphi^2$ is the mean free path of a "hole" particle, and k_0^{-1} is that characteristic length over which "dressing" of the particles occurs, owing to the interaction with the condensate. Therefore the condition $\nu\varphi^2/k_0 \ll 1$ means that on the path between collisions a particle is able, by interacting with the condensate, to change into a sound quantum, and only upon fulfilment of this condition can one expect Rayleigh scattering of the one-particle excitations in a Bose gas.

Having determined $A(\omega, p)$, $S(\omega, p)$, and $\Sigma^{02}(\omega, p)$ from formulas (28), (26), (24), and (20), and having substituted them into (18), we obtain the result that in the acoustic part of the spectrum the attenuation of the excitations is given by

$$\gamma = \frac{\pi}{6}\nu\varphi^2 c\left(\frac{p}{mc}\right)^4,$$

i.e., it agrees with the semi-phenomenological calculation. In the absence of a condensate the attenuation would be equal to $2\pi\nu\varphi^2 p/m$. Thus, in a system containing a condensate the scattering of the long wavelength excitations turns out to be strongly suppressed. This effect is due to both the change in the spectrum of the excitations and the distinctive screening of the impurity potential due to perturbation of the density of the condensate near the impurity atoms.

CONCLUSION

Now let us go on to the question of the width of the lines for the absorption and emission of light by a

Bose-Einstein condensate of excitons in semiconductors with allowed direct transitions. This width is determined by the scattering of single-exciton excitations of the system with momentum $2\pi\hbar\lambda^{-1}$, where λ is the wavelength of light in the crystal, because the emission and absorption of light by the condensate of excitons are accompanied by the creation of a single-particle excitation with such momentum.[1] If the concentration of excitons is sufficiently high so that an excitation with momentum $2\pi\hbar\lambda^{-1}$ belongs to the linear part of the spectrum, i.e., if the inequality

$$\lambda^{-1} \ll mc / 2\pi\hbar = \sqrt{nf / \pi},$$

is fulfilled, where f denotes the amplitude for the scattering of the excitons against each other, this width turns out to be $(1/12)\pi^{3/2}(nf\lambda^2)^{-3/2}$ times smaller than the linewidth for absorption by free excitons. Thus, for example, in GaAs for a concentration of excitons of 10^{+16} cm^{-3} the line would be narrowed by roughly 100 times. Such a strong suppression of the scattering might lead to an interesting property for a laser utilizing the line of exciton amplification. Such a laser might be achieved in the form of a p-i-n structure.[6] For resonator dimensions of the order of 10^{-1} cm, the distance between the modes 10^{11} sec^{-1} may turn out to be larger than the linewidth. In fact, the width of the exciton line in GaAs usually amounts to 10^{11} to 10^{12} sec^{-1}. The presence of the exciton condensate should reduce it to 10^9 to 10^{10} sec^{-1}. On the other hand, as was indicated in [1] the frequency of the exciton amplification line varies with the concentration of excitons, and consequently with the level of injection. Therefore the frequency of the amplification line will coincide with the eigenfrequency of the resonator only for certain values of the injection level, and the dependence of the intensity of light generation on the injection current must have the form of abrupt spikes. The indicated feature would permit one to determine whether generation is a consequence of ordinary interband transitions, or whether it is associated with the appearance of a Bose-Einstein condensate of excitons.

---

[1] V. A. Gergel', R. F. Kazarinov, and R. A. Suris, Zh. Eksp. Teor. Fiz. **53**, 544 (1967([Sov. Phys.-JETP **26**, 354 (1968)].

[2] N. N. Bogolyubov, Concerning the Hydrodynamics of a Superfluid Liquid, JINR Preprint R-1395, Dubna, 1963.

[3] S. T. Belyaev, Zh. Eksp. Teor. Fiz. **34**, 417 (1958) [Sov. Phys.-JETP **7**, 289 (1958)].

[4] I. M. Khalatnikov and V. N. Zharkov, Zh. Eksp. Teor. Fiz. **32**, 1108 (1957) [Sov. Phys.-JETP **5**, 905 (1957)].

[5] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1962 (English Transl., Prentice-Hall, Inc., 1963).

[6] V. A. Gergel', R. F. Kazarinov, and R. A. Suris, Zh. Eksp. Teor. Fiz. **54** 298 (1968) [Sov. Phys.-JETP **27**, 159 (1968)].

Translated by H. H. Nickle
84