SOVIET PHYSICS JETP VOLUME 31, NUMBER 3 SEPTEMBER 1970

# THE EFFECT OF SPECULAR REFLECTION OF ELECTRONS ON THE SURFACE IMPEDANCE

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Submitted April 28, 1969

Zh. Eksp. Teor. Fiz. 58, 865–877 (March 1970)

We consider the impedance of a metal in a magnetic field parallel to the surface under anomalous skin-effect conditions. We show that the specularity coefficient appreciably affects the H- and ω-dependence of the impedance; this makes it possible to determine this coefficient. We show that taking specularity into account may also lead to resonance effects.

## 1. INTRODUCTION

In this paper we consider the influence of specular electrons on the surface impedance of a bulk metal specimen in a magnetic field parallel to the surface when the skin-effect is anomalous. It is well known (see, for instance, [1,2]) that under anomalous skin-effect conditions the main contribution to the current comes from electrons that glance along the metal surface or collide with it at a very small angle. It was noted in [3] that in that case the reflection of an electron from the surface is nearly specular so that the bad definition of the orbit along the surface may be large compared to the size of the inhomogeneities in the surface.

When there is a magnetic field parallel to the surface these electrons lead to a new kind of orbit in the surface layer: open orbits (see Fig. 1, orbits of type a, **H** is perpendicular to the plane of the figure) which appreciably affects the surface impedance.

Indeed, it is well known that $\delta_{eff}$, the effective skin depth is connected with the effective conductivity $\sigma_{eff}$ as follows:[1)]

$$\delta_{eff} \sim \frac{c}{\sqrt{\omega\sigma_{eff}}}. \qquad (1.1)$$

In the general case

$$\sigma_{eff} = \sigma_V + \sigma_S, \qquad (1.2)$$

where $\sigma_V$ is the effective conductivity connected with the electrons that do not hit the surface of the metal, while $\sigma_S$ is determined by the electrons that hit the


FIG. 1

metal surface (see Fig. 1, orbits b and a, respectively).

The main contribution to the effective conductivity comes from those electrons which gain the maximum energy from the external electromagnetic field and manage to lose it while they are in the skin-layer. Electrons moving along the tangent to the metal surface and therefore traversing a distance $(\delta_{eff}f)^{1/2}$ in the skin-layer gain most energy in the skin-layer. They are a fraction $\delta_{eff}/(\delta_{eff}r)^{1/2}$ of all electrons.

For orbits of the kind of Fib. 1b, not all those electrons will manage to collide and lose energy in the skin-layer, but only the fraction $(r\delta_{eff})^{1/2}/l$. Therefore

$$\sigma_V \sim \sigma \frac{\delta_{eff}}{(\delta_{eff}r)^{1/2}} \frac{(r\delta_{eff.})^{1/2}}{l} \frac{1}{1-e^{-2\pi\gamma}}. \qquad (1.3)$$

The last factor arises when we take into account the possibility of repeated rotations of electrons in the skin-layer.

For orbits of type a of Fig. 1 all electrons moving along the tangent to the metal surface (incident at angles of order $(\delta_{eff}/r)^{1/2}$) stay completely inside the skin-layer. Therefore $\sigma_S \sim \sigma\delta_{eff}/(r\delta_{eff})^{1/2}$. When writing down this last formula we assumed that the reflection is purely specular and $\omega t_0 \ll 1$, i.e., we did not take it into account that because of a change in the phase of the electromagnetic field an electron may be decelerated rather than accelerated. This occurs because in this formula σ is the static conductivity $(\sigma \sim ne^2 l/mv_0)$ which is determined only if the mean free path $l$ is finite. In the general case we must replace $l = v_0 t_0$ by the quantity $l_{eff}$ (see footnote 1).

The quantity $l_{eff}$ is obtained from the expansion of the "angular resonance" factor

$$\left(1 - q\exp\left\{-\int_0^T\left(i\omega + \frac{1}{t_0}\right)dt'\right\}\right)^{-1}, \quad T \sim 2\frac{2\pi}{\Omega}\left(\frac{\delta_{eff}}{r}\right)^{1/2}, \qquad 1.4)$$

which occurs when we take into account the motion of an electron along an orbit of type a of Fig. 1 with an angle of incidence which is close to zero (T is the time between two successive collisions of the electron with the surface).

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[1)]We use the following notation: σ is the static conductivity when there is no magnetic field, r the radius of the electron orbit, $v_0$ the electron velocity on the Fermi surface, n the elctron density, $\gamma = i\omega/\Omega + 1/\Omega t_0$, ω the frequency of the electromagnetic field, Ω the cyclotron frequency, $l$ and $t_0$ are the mean free path and flight time;

$$l_{eff} = v_0\frac{2\pi}{\Omega}\left(\frac{\delta_{eff}}{r}\right)^{1/2} \Big/ \left\{(1-q) - \tau_{eff}^{-1}\frac{2\pi}{\Omega}\left(\frac{\delta_{eff}}{r}\right)^{1/2}\right\},$$

$$\tau_{eff}^{-1} = e\frac{v_0}{c}H\frac{\partial q}{\partial p_\perp} + 2q\left(i\omega + \frac{1}{t_0}\right),$$

where q and $\partial q/\partial p_\perp$ are, respectively, the specularity coefficient and its derivative with respect to the quasi-momentum component at right angles to the surface when the electron angle of incidence at the surface is zero.

We note that $\sigma/l \sim c^2/\delta^2 v_0$, where $\delta^2 = mc^2/2\pi ne^2$, i.e., $\sigma/l$ depends only on what kind of metal we are dealing with.

Thus,

$$\sigma_S \sim (\sigma / l)(\delta_{eff}/r)^{1/2} l_{eff}. \quad (1.5)$$

Now $\delta_{eff}$ (see (1.1) and (1.2)) depends on whether $\sigma_V$ or $\sigma_S$ (see (1.3) and (1.5)) is the larger. If $\sigma_V \gg \sigma_S$ we have, using the relation $\sigma/l \sim c^2/\delta^2 v_0$

$$\delta_{eff} \sim \left(\delta^2 \frac{v_0}{\omega}(1 - e^{-2\pi\gamma})\right)^{1/3}. \quad (1.6)$$

We also use a well-known result (see [1]) that for sufficiently weak magnetic fields ($|2\pi\gamma| \gtrsim 1$) $\delta_{eff}$ is completely independent of H while in strong fields ($|2\pi\gamma| \ll 1$) we have $\delta_{eff} \sim H^{-1/2}$. We note that also for cyclotron resonance occurring when $|(\exp 2\pi\gamma) - 1| \ll 1$ ($\omega = n\Omega$, where n is an integer) we obtained the correct index ($1/3$) of the resonance factor $[1 - (\exp -2\pi\gamma)]$ in the case of a quadratic dispersion law when all electrons rotate with the same frequency $\Omega$. When the dispersion law is not quadratic we must take it into account that not all electrons take part in the resonance but only those with frequencies $\Omega(p_z)$ that are extrema with respect to $p_z$, in order to obtain the correct index ($1/6$).

If, however, $\sigma_{eff}$ is determined by $\sigma_S$ rather than by $\sigma_V$ (see (1.5)) two cases are possible.

If $1 - q \gg 2\pi\Omega^{-1}(\delta_{eff}/r)^{1/2}\tau_{eff}^{-1}$ we have $\sigma_S \sim \sigma(\delta_{eff}/l)(1-q)^{-1}$ and $\sigma_S \sim \sigma_V$ and the H- and $\omega$-dependence of $\delta_{eff}$ is practically the same as in the case (1.6).

In the opposite limiting case we get

$$\delta_{eff} \sim [\delta^2 r^{1/2} \omega^{-1} \tau_{eff}^{-1}]^{2/5}. \quad (1.7)$$

The $\omega$- and especially the H-dependence are essentially different in (1.6) and (1.7).

The introduction of the reflection coefficient, which is a steep function of the collision angle with the surface (see Fig. 2a; $\alpha$ is the angle of incidence to the surface of the electron), allows us thus to say something about the quality of the surface.

Moreover, taking the specularity coefficient into account for metals with a non-convex Fermi surface leads to a resonance as in that case there are electrons for which the orbit is geometrically separated (see Fig. 3). Electrons which collide with the metal surface in such a way that the sections I or II coincide with the metal surface are geometrically separated.

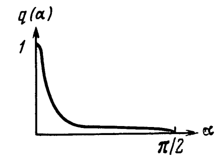


FIG. 2

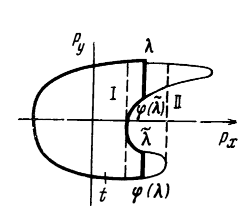


FIG. 3

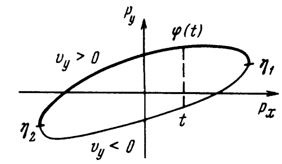


FIG. 4. Intersection of the Fermi surface with a plane perpendicular to the magnetic field.

## 2. MATHEMATICAL STATEMENT OF THE PROBLEM. SOLUTION OF THE KINETIC EQUATION IN TERMS OF QUADRATURES AND CONTRIBUTIONS OF DIFFERENT PARTS OF THE SOLUTION UNDER ANOMALOUS SKIN-EFFECT CONDITIONS

Let there be a high-frequency electromagnetic field in the half-space occupied by a metal in a magnetic field parallel to the surface and directed along the z-axis. In that case the complete set of equations consists (see [1,2]) of the Maxwell equations which in the one-dimensional case have the form

$$E_\mu''(y) = \frac{4\pi i\omega}{c^2} j_\mu(y), \quad \mu = x, z, \quad (2.1)$$

$$j_y = 0 \quad (2.1a)$$

(the y-axis is along the interior normal to the metal surface) and the kinetic equation for the non-equilibrium correction to the distribution function $f_1$, which in terms of the variables $\epsilon$, $p_z$, t (t is the time in which an electron revolves along its orbit in momentum space) has the form

$$v_y \frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial t} + i\omega f_1 + \frac{f_1}{t_0} = e\mathbf{v}(t)\mathbf{E}(y)\frac{\partial f_0}{\partial \varepsilon}. \quad (2.2)$$

(We have here introduced the free flight time $t_0(\epsilon, p_z, t)$ instead of the collision integral; this can always be done under anomalous skin-effect conditions.)

The current is connected with $f_1$ through

$$j = -\frac{2e}{h^3}\int \mathbf{v} f_1 \, d\mathbf{p}, \quad (2.3)$$

where $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$ while the integration is over the whole of momentum space. The periodicity of $f_1$ plays the role of a boundary condition in t:

$$f_1\left(t + \frac{2\pi}{\Omega}\right) = f_1(t), \quad \Omega = \frac{eH}{c}\left(\frac{1}{2\pi}\frac{\partial S}{\partial \varepsilon}\right)^{-1}. \quad (2.4)$$

The boundary condition in the coordinate y has the form (see [4])

$$f_1(y=0, t)\,|_{v_y(t)>0} = q(\varepsilon, p_z, t) f_1(y=0, \varphi(t)) + \rho, \quad (2.5)$$

$\varphi(t)$ is determined as follows: $p_x(t) = p_x(\varphi(t))$ where $t < \varphi(t) < t + 2\pi/\Omega$ (see Fig. 4). For a convex Fermi surface $\varphi(t)$ is uniquely determined. The quantity $\rho$ is a linear functional of $f_1$. The only requirement which $\rho$ must satisfy is that j(y) vanishes for y = 0.

In the case considered we can put $\rho \equiv 0$ in the basic approximation in $(\delta_{eff}/r)^{1/2}$. Physically this is connected with the fact that under anomalous skin-effect conditions the "population" of non-equilibrium states with $|v_y| \ll v_0$ is considerably larger than the "population" of states with other values of $v_y$ and during collisions with the surface transitions from states with $|v_y| \ll v_0$ are thus most probable. This is completely analogous to the possibility of introducing a free flight time $t_0(\epsilon, p_z, t)$ under anomalous skin-effect conditions (see [1]). We introduce $\Psi(y, \mathbf{p}) = f_1(y, \mathbf{p}) - f_1(y, -\mathbf{p})$. As $\epsilon(\mathbf{p}) = \epsilon(-\mathbf{p})$ we have the following expression for the current:

$$\mathbf{j} = -\frac{e}{h^3}\int \mathbf{v}\Psi \, d\mathbf{p}. \quad (2.6)$$

We put

$$\hat{L}(\mathbf{p}) = \frac{1}{v_y}\left\{\frac{\partial}{\partial t} + i\omega + \frac{1}{t_0(\mathbf{p})}\right\}.$$

Using the symmetry of the collision integral under the transformation $\mathbf{p} \to -\mathbf{p}$, we then get $\hat{L}(-\mathbf{p}) = -\hat{L}(\mathbf{p})$.

And thus we can write (2.2) for $\mathbf{p}$ and $-\mathbf{p}$ in the form

$$\frac{\partial f_1}{\partial y}(y,\mathbf{p})+\hat{L}(\mathbf{p})f_1(y,\mathbf{p})=e\frac{\mathbf{v}(\mathbf{p})}{v_y(\mathbf{p})}\mathbf{E}(y)\frac{\partial f_0}{\partial \varepsilon}, \tag{2.7}$$

$$\frac{\partial f_1}{\partial y}(y,-\mathbf{p})-\hat{L}(\mathbf{p})f_1(y,-\mathbf{p})=e\frac{\mathbf{v}(\mathbf{p})}{v_y(\mathbf{p})}\mathbf{E}(y)\frac{\partial f_0}{\partial \varepsilon}. \tag{2.7a}$$

Acting, respectively, upon (2.7) and (2.7a) with the operators $\partial/\partial y - \hat{L}$ and $\partial/\partial y + \hat{L}$ and subtracting the two results we get

$$\left(\frac{\partial^2}{\partial y^2}-\hat{L}^2\right)\Psi(y,\mathbf{p})=-2e\mathbf{E}(y)\hat{L}\left(\frac{\mathbf{v}}{v}\frac{\partial f_0}{\partial \varepsilon}\right). \tag{2.8}$$

From this it is clear that we can continue the functions $\Psi(y, \mathbf{p})$ and $\mathbf{E}(y)$ in an even manner into the region $y < 0$ and perform the Fourier transformation in Eqs. (2.1), (2.6), and (2.8):[2)]

$$(k^2+\hat{L}^2)\psi(k,\mathbf{p})+2\Psi'(0,\mathbf{p})=2e\mathscr{E}_j(k)\hat{L}\left(\frac{v_j}{v_y}\frac{\partial f_0}{\partial \varepsilon}\right), \tag{2.9a}$$

$$k^2\mathscr{E}_\mu(k)+2E_\mu'(0)=-\frac{4\pi i\omega}{c^2}j_\mu(k), \tag{2.9b}$$

$$j_y(k)=0, \tag{2.9c}$$

$$j_i(k)=-\frac{e}{h^3}\frac{eH}{c}\int v_i\psi(k;\,\varepsilon,p_z,t)\,d\varepsilon\,dp_z\,dt. \tag{2.9d}$$

For the integration we changed from the variables $p_x$, $p_y$, $p_z$ to $\epsilon$, $p_z$, $t$. We must still write down the boundary condition for $\Psi(y, \mathbf{p})$. Subtracting (2.7a) from (2.7) and putting $y = 0$, we get

$$\Psi'(0,\mathbf{p})=-\hat{L}[f_1(\mathbf{p})+f_1(-\mathbf{p})].$$

One shows easily that if in the change of variables $\mathbf{p}$ changes to $\epsilon$, $p_z$, $t$, then $-\mathbf{p}$ changes to $\epsilon$, $-p_z$, $t+\pi/\Omega$. Using (2.5) we then get

$$\Psi'(0;\varepsilon,p_z,t)=$$
$$=\hat{L}\left\{\frac{\operatorname{sgn} v_y}{1-q_1q_2}[(1+q_1q_2)\Psi(0;\varepsilon,p_z,t)-2q_1\Psi(0;\varepsilon,p_z,\varphi(t))]\right\}, \tag{2.10}$$

where $q_1 = q(\epsilon, p_z, t)$, $q_2 = q(\epsilon, p_z, \varphi(t))$.

In the calculations we used the fact that for a convex Fermi-surface $\varphi(t + \pi/\Omega) = \varphi(t) + \pi/\Omega$. We have here also introduced the following definition: $q(\mathbf{p}) = q(-\mathbf{p})$. This can be done because $q(\mathbf{p})$ was initially defined for those $\mathbf{p}$ for which $v_y(\mathbf{p}) > 0$ (see (2.5)). Similar to Eq. (5.2) of [1], we can get one solution of (2.9a) periodic in $t$, where $\Psi'(0;\epsilon, p_z, t)$ is given by (2.10):

$$\psi(k;\,\varepsilon,\,p_z,\,t)$$
$$=\frac{2}{e^{2\pi\gamma}-1}\int_t^{t+2\pi/\Omega}g(k;\varepsilon,p_z,t_1)\exp\left\{\int_t^{t_1}\gamma_0\,dt_2\right\}\cos\left\{k\int_t^{t_1}v_y\,dt_2\right\}dt_1, \tag{2.11}$$

where

$$\gamma_0=i\omega+\frac{1}{t_0(\varepsilon,p_z,t)},\quad \gamma=i\frac{\omega}{\Omega}+\frac{1}{2\pi}\int_0^{2\pi/\Omega}\frac{dt'}{t_0(\varepsilon,p_z,t')},$$

$$g(k;\varepsilon,p_z,t_1)=e\mathscr{E}_jv_j\frac{\partial f_0}{\partial\varepsilon}-\frac{|v_y|}{1-q_1q_2}\cdot$$
$$\times[(1+q_1q_2)\Psi(0;\varepsilon,p_z,t_1)-2q_1\Psi(0;\varepsilon,p_z,\varphi(t_1))].$$

Performing simple but cumbersome calculations we find

$$j_i(k)=\sum_{j=1}^{3}\left\{\mathscr{K}_{ij}(k)\mathscr{E}_j(k)-\int_0 Q_{ij}(k;k')\mathscr{E}_j(k')\,dk'\right\}, \tag{2.12}$$

$$\mathscr{K}_{ij}(k)=\frac{2e^3H}{ch^3}\int\left(-\frac{\partial f_0}{\partial\varepsilon}\right)\frac{d\varepsilon\,dp_z}{e^{2\pi\gamma}-1}\int_0^{2\pi/\Omega}v_i(t)\,dt\int_t^{t+2\pi/\Omega}v_j(t_1)$$
$$\times\exp\left\{\int_t^{t_1}\gamma_0\,dt_2\right\}\cos\left\{k\int_t^{t_1}v_y\,dt_2\right\}dt_1; \tag{2.13}$$

$$Q_{ij}(k;k')=\frac{2e^3H}{ch^3}\int\left(-\frac{\partial f_0}{\partial\varepsilon}\right)d\varepsilon\frac{dp_z}{e^{2\pi\gamma}-1}\int_0^{2\pi/\Omega}v_i(t)\,dt$$
$$\times\int_t^{t+2\pi/\Omega}|v_y(t_1)|\,dt_1\cos\left(k\int_t^{t_1}v_y\,dt_2\right)$$
$$\times\left\{\left(1-q_2(t_1)\exp\left\{-\int_{\varphi(t_1)}^{t_1+2\pi/\Omega}\gamma_0\,dt_2\right\}\right)^{-1}dt_1\right.$$
$$\times\int_{\varphi(t_1)-2\pi/\Omega}^{t_1}v_j(\xi)\exp\left\{\int_t^{\xi}\gamma_0\,dt_2\right\}\cos\left\{k'\int_{t_1}^{\xi}v_y\,dt_2\right\}d\xi$$
$$-q_1(t_1)\exp\left\{-\int_{t_1}^{\varphi(t_1)}\gamma_0\,dt_2\right\}\left(1-q_1(t_1)\exp\left\{-\int_{t_1}^{\varphi(t_1)}\gamma_0\,dt_2\right\}\right)^{-1}dt_1$$
$$\left.\times\int_{t_1}^{\varphi(t_1)}v_j(\xi)\exp\left\{\int_t^{\xi}\gamma_0\,dt_2\right\}\cos\left\{k'\int_{t_1}^{\xi}v\;\,dt_2\right\}d\xi\right\}. \tag{2.14}$$

In what follows we shall assume that $(\delta_{eff}/r)^2 \ll 1/\Omega t_0$. We can then put $\mathscr{E}_y(k)$ identically equal to zero in Eqs. (2.1) and (2.12) in the main approximation in $(\delta_{eff}/r)^{1/2}$ and we need not at all consider (2.1a) as it is used only to determine $E_y$ (for the reasons for this see [1, 2, 5]).

We note that the results obtained do not depend at all on this last approximation which was made merely to simplify the calculations (this is already clear from the qualitative consideration in the Introduction).

## 3. CALCULATION OF THE SURFACE IMPEDANCE

The surface impedance is introduced as follows:

$$-\frac{4\pi i\omega}{c^2}E_\mu(0)=\sum_\nu Z_{\mu\nu}E_\nu'(0). \tag{3.1}$$

First of all it is necessary to evaluate the current density $j_\alpha(k)$. Its magnitude will depend significantly on how close to unity the reflection coefficient is in the "angular resonance" factor (see (2.14)).

The following cases are possible:[3)]

I. The condition

$$1-q_1(\eta_\alpha-0)\gg|\rho_{eff}|. \tag{3.2}$$

holds. We shall see that experimentally this case cannot be distinguished from the purely diffusive case.

II. The inequality

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[2)]We have not taken into account here that $\Psi'(y, \mathbf{p})$ has a discontinuity at $y = 2r$, provided $q$ differs from unity for electrons which are incident to the surface at small angles. However, this is unimportant for us as we are interested in $q$ close to unity.

[3)]We introduce the notation

$$\rho_{eff}=\frac{2\pi}{\Omega}\left(\frac{\delta_{eff}}{r}\right)^{1/2}\left(\frac{\partial q_1}{\partial t}(\eta_\alpha-0)+2q_1(\eta_\alpha-0)\gamma_0(\eta_\alpha)\right),$$

where

$$\frac{\partial q}{\partial t}(\eta_\alpha)=\frac{\partial q}{\partial \mathbf{p}}\dot{\mathbf{p}}\cong\frac{\partial q}{\partial p_y}\dot{p}_y=\frac{eH}{c}v_x\frac{\partial q}{\partial p_y}=\frac{eH}{c}\frac{\partial\varepsilon}{\partial p_x}\frac{\partial q}{\partial p_y}.$$

We have used here the fact that $\dot{p}_z = 0$ and $\dot{p}_x(\eta_\alpha) \sim v_y(\eta_\alpha) = 0$, $\gamma_0(\eta_\alpha) = i\omega + 1/t_0(\eta_\alpha)$. The quantity $\eta_\alpha$ is determined from the condition $v_y(\eta_\alpha) = 0$ ($\alpha = 1, 2$) (see Fig. 4). We note that we can express $\rho_{eff}$ in terms of $\tau_{eff}$ (see footnote[1]).

$$1 - q_1(\eta_\alpha - 0) \ll |\rho_{\text{eff}}|. \tag{3.3}$$

is satisfied. One can then have

$$|\rho_{\text{eff}}| \ll 1, \tag{3.3a}$$

which corresponds to the case when all electrons which are important in the case of the anomalous skin-effect with $v_y \approx 0$ are specularly reflected from the surface and due to that the surface impedance is determined solely by those electrons, or one can have

$$|\rho_{\text{eff}}| \gg 1, \tag{3.3b}$$

when only an insignificant part of the electrons that are important from the point of view of the anomalous skin-effect are specularly reflected, which leads to the fact that the surface impedance is determined by electrons that do not collide with the surface.

We consider now consecutively all cases. We can then always evaluate the current density using the saddle point method as those k are important for which $kv_0/\Omega \gg 1$.

I. In the case (3.2) we can use (2.12) to (2.14) to evaluate $j_\mu(k)$. As the final answer is extremely complicated, we shall formulate the results of the investigation.

In the cyclotron resonance region, when $|(\exp 2\pi\gamma) - 1| \ll 1$, the surface impedance, or rather its derivatives with respect to H and $\omega$ will be practically independent of the specularity coefficient.

The smooth part of the impedance (the non-resonance region) will change when we change $q_1(\eta_\alpha \pm 0)$ to the quantity $\sim (1 - q_1(\eta_\alpha - 0))^{-1/3}$. It is important that in the main approximation in $(1 - q_1(\eta_\alpha - 0)^{-1/3}$ this change depends neither on the frequency nor on the magnetic field. In that case it is therefore practically impossible to determine experimentally the magnitude of the specularity coefficient.

II. When conditions (3.3) and (3.3a) are satisfied we can satisfy ourselves that the term with $Q_{\mu\nu}(k, k')$ (see (2.14)) will give the main contribution to the current density (2.12). The contribution from the term with $\mathcal{K}_{\mu\nu}(k)$ (see (2.13)) will be smaller by a factor $(\delta_{\text{eff}}/r)^{1/2}$. After rather complicated calculations of the integrals, using the saddle-point method, we get for the current density (compare [6])

$$j_\mu(k) = \frac{\sqrt{\pi} e^4 H^2}{c^2 h^3} r^{5/2} \sum_\nu A_{\mu\nu} \int_0^\infty dk' \mathscr{E}_\nu(k') \frac{1}{\sqrt{kk'}} \left\{ \frac{1}{|k-k'|^{1/2}} - \frac{1}{(k+k')^{1/2}} \right.$$

where (3.4)

$$A_{\mu\nu} = \frac{cr^{-5/2}}{eH} \sum_{\alpha=1}^{2} \int dp_z \frac{v_\mu(\eta_\alpha) v_\nu(\eta_\alpha)}{|v_y'(\eta_\alpha)|^{1/2}} \left[ \frac{\partial q_1}{\partial t}(\eta_\alpha - 0) + 2q_1(\eta_\alpha - 0)\gamma_0(\eta_\alpha) \right]^{-1}, \tag{3.4a}$$

r is a characteristic radius of the orbit, the determination of which we shall not give, since it does not appear in the final equations.

The problem has thus been reduced to determining the surface impedance $Z_{\mu\nu}$ (see (3.1)) from the set of integral equations

$$-k^2 \mathscr{E}_\mu(k) - 2E_\mu'(0) = \frac{4\pi i\omega}{c^2} \frac{\sqrt{\pi} e^4 H^2}{c^2 h^3} r^{5/2} \sum_\nu A_{\mu\nu} \cdot$$

$$\times \int_0^\infty dk' \mathscr{E}_\nu(k') \frac{1}{\sqrt{kk'}} \left( \frac{1}{|k-k'|^{1/2}} - \frac{1}{(k+k')^{1/2}} \right), \quad \mu, \nu \equiv x, z. \tag{3.5}$$

To solve the problem in the general case we can most simply proceed as follows. We introduce the quantities

$$\mathscr{E}_{x'}(k) = \mathscr{E}_x(k)\cos\varphi + \mathscr{E}_z(k)\sin\varphi,$$
$$\mathscr{E}_{z'}(k) = -\mathscr{E}_x(k)\sin\varphi + \mathscr{E}_z(k)\cos\varphi, \tag{3.6}$$

where we define $\varphi$ from

$$\operatorname{tg} 2\varphi = 2A_{xz}/(A_{xx} - A_{zz}). \tag{3.7}$$

We introduce $E'_{x'}(0)$ and $E'_{z'}(0)$ through similar formulae. (3.5) then becomes

$$-k^2 \mathscr{E}_{\mu'}(k) - 2E_{\mu'}'(0)$$

$$= iA_{\mu'\mu'} \frac{4\pi\omega}{c^2} \frac{\sqrt{\pi} e^4 H^2}{c^2 h^3} r^{5/2} \int_0^\infty dk' \mathscr{E}_{\mu'}(k') \frac{1}{\sqrt{kk'}} \left\{ \frac{1}{|k-k'|^{1/2}} - \frac{1}{(k+k')^{1/2}} \right\},$$
$$\mu' = x', z', \tag{3.8}$$

where

$$A_{x'x'} = A_{xx}\cos^2\varphi + 2A_{xz}\cos\varphi\sin\varphi + A_{zz}\sin^2\varphi,$$
$$A_{z'z'} = A_{xx}\sin^2\varphi - 2A_{xz}\cos\varphi\sin\varphi + A_{zz}\cos^2\varphi. \tag{3.9}$$

We write (3.8) in a more convenient form

$$-\xi^2 F_\sigma(\xi) + 1 = e^{i\sigma} \int_0^\infty d\xi' F_\sigma(\xi') \frac{1}{\sqrt{\xi\xi'}} \left\{ \frac{1}{|\xi - \xi'|^{1/2}} - \frac{1}{(\xi + \xi')^{1/2}} \right\} \tag{3.10}$$

where

$$\sigma = \pi/2 + \arg A_{\mu'\mu'}; \quad \xi = k/k_{\mu'},$$
$$k_{\mu'} = \left[ \frac{4\pi\omega}{c^2} \frac{\sqrt{\pi}}{c^2} \frac{e^4 H^2}{h^3} r^{5/2} |A_{\mu'\mu'}| \right]^{2/5} \sim \left( \frac{\omega}{\Omega} r^{-1/2} \delta^{-2} |A_{\mu'\mu'}| \right)^{2/5},$$
$$\delta^2 = \frac{mc^2}{2\pi n e^2}, \quad F_\sigma(\xi) = -\frac{k_{\mu'}^2}{2E_{\mu'}(0)} \mathscr{E}_{\mu'}(k). \tag{3.10a}$$

The surface impedance $Z_{\mu\nu}$ (see (3.1)) can be expressed in terms of the formulae:

$$Z_{xx} = Z_{x'x'}\cos^2\varphi + Z_{z'z'}\sin^2\varphi,$$
$$Z_{xz} = (Z_{x'x'} - Z_{z'z'})\cos\varphi\sin\varphi, \tag{3.11}$$
$$Z_{zz} = Z_{x'x'}\sin^2\varphi + Z_{z'z'}\cos^2\varphi,$$

where $Z_{\mu'\mu'}$ is determined from

$$Z_{\mu'\mu'} = -\frac{4\pi i\omega}{c^2} \frac{E_{\mu'}(0)}{E_{\mu'}'(0)} = \frac{8i\omega}{c^2 k_{\mu'}} \int_0^\infty F_\sigma(\xi)\, d\xi = \frac{8i\omega}{c^2 k_{\mu'}} I(\sigma). \tag{3.12}$$

The quantity $\varphi$ (see (3.7) and (3.4a)) is in general complex, i.e., we cannot diagonalize the surface impedance by rotating the axes in the xz-plane.

One can solve Eq. (3.10) exactly in the general case (see the Appendix). We give the result in two important limiting cases when $\varphi$ is a real quantity and the $Z_{\mu'\mu'}$ (see (3.12)) are the principal values of the impedance tensor.

If

$$\omega \ll \left| \frac{\partial q_1}{\partial t}(\eta_\alpha - 0) + \frac{1}{t_0(\eta_\alpha)} \right|,$$

we have $Z_{\mu'\mu'}(\omega) \sim \omega^{3/5}$

$$Z_{\mu'\mu'}(H) \sim \begin{cases} H^{-1/5}, \text{ when } \dfrac{1}{t_0(\eta_\alpha)} \gg \left| \dfrac{\partial q_1}{\partial t}(\eta_\alpha - 0) \right| \\ H^{1/5}, \text{ when } \dfrac{1}{t_0(\eta_\alpha)} \ll \left| \dfrac{\partial q_1}{\partial t}(\eta_\alpha - 0) \right|. \end{cases}$$

If

$$\omega \gg \left| \frac{\partial q_1}{\partial t}(\eta_\alpha - 0) + \frac{1}{t_0(\eta_\alpha)} \right|,$$

we have $Z_{\mu'\mu'}(\omega) \sim \omega$, $Z_{\mu'\mu'}(H) \approx H^{-1/5}$.

In the general case we can give an expression for the impedance using (3.11), (3.12), and (A.13)—see the Appendix.

In the case when conditions (3.3) and (3.3b) are satisfied, the main contribution to the current will come from the electrons that do not hit the metal surface and from the electrons which are diffusely scattered by the surface. The contribution from the specular electrons will be smaller by a factor $\rho_{eff}$ and this case is, in fact, practically purely the diffuse case. The H- and $\omega$-dependence of the impedance is in the main approximation the same as the one given in case I.

## 4. CONSIDERATION OF A NON-CONVEX FERMI SURFACE

If the intersection of the Fermi-surface with a plane at right angles to the magnetic field is non-convex, we have for the boundary condition for the kinetic equation instead of (2.5):[4]

$$f_1(y=0,t)\,|_{v_y(t)>0}=\sum_{s=1}^{\varkappa} q_s(\varepsilon, p_z, t) f_1(y=0;\ \varphi_s(t)), \quad (4.1)$$

where $\kappa$ is the number of those solutions of the equation $p_x(t) = p_x(\varphi_s(t))$ for which $\varphi_s(t)$ satisfies the condition $t \leq \varphi_s(t) \leq t + 2\pi/\Omega$ and $v_y(\varphi_s(t)) < 0$; finally, the quantity $\kappa$ depends on t.

We must solve Eq. (2.2) with the boundary conditions (2.4) and (4.1). The simplest method to obtain the solution (see, e.g., [2]) is the one in which one determines the energy acquired by an electron along its motion in the electromagnetic field, taking the possibility of collisions into account and the probability for a scattering from the surface under various angles. One can at once write down the answer for the case of a convex Fermi-surface and one obtains it relatively simply. However, for the case of a non-convex Fermi-surface even in the case $\kappa \leq 2$ (see (4.1) and Fig. 3) it is rather cumbersome to obtain the answer and we shall not write it down. We only discuss the results.

It is clear from the solution that for fixed $p_z$ the intersections I and II (see Fig. 3) are separate. This geometric separation of the orbit leads to an effect which is completely analogous to the cyclotron resonance which only occurs for electrons which collide with the metal surface.[5] The resonance will occur at frequencies which are extremal with respect to $p_z$ because there are relatively more of such electrons than with other values of $p_z$.

The resonance frequencies will be

$$\omega^{-1}=\frac{1}{2\pi}\{m^{-1}[\varphi(\lambda)-\lambda]+n^{-1}[\varphi(\tilde{\lambda})-\tilde{\lambda}]\}, \quad (4.2)$$

where m, n = 0, 1, 2 and $\lambda$ and $\tilde{\lambda}$ are the times when the electron collides with the metal surface which goes through the sections I or II (see Fig. 3) for extremal values of $p_z$. We write down the resonance part of the impedance $Z_{res}$ for the frequencies $\omega^{-1} = (1/2\pi)m^{-1} \times [\varphi(\lambda) - \lambda]$ for section II of Fig. 3. This can easily be done, using simple physical considerations. The amplitude of the resonance will be of the order of the cyclotron resonance amplitude so that the resonance electron approaches the surface along the tangent and acquires thus at each approach to the surface an amount of energy of the same order of magnitude as an electron that takes part in cyclotron resonance acquires from the electromagnetic field. The sharpness of the resonance curve will be less than in the case of cyclotron resonance because the mean free path is effectively diminished when non-specular reflection from the surface is taken into account. Thus[6]

$$Z_{\rm res} \sim Z(0)\left(1-q_1(\lambda)\exp\left\{-\int_{\lambda}^{\varphi(\lambda)}\gamma_0\,dt'\right\}\right)^{-1/6},$$

where $\lambda$ corresponds to intersection II of Fig. 3 for an extremal value of $p_z$; Z(0) is the impedance when there is no magnetic field. We get a similar form for the impedance for the frequencies of (4.2).

An experimental study of the resonance of electrons colliding with the surface might make it possible to use the height and steepness of the resonance curve to determine the specularity coefficients near geometrically separated intersections for extremal values of $p_z$.

## 5. CONCLUSIONS

The introduction of the coefficient for the reflection of electrons from the surface of a metal in a magnetic field parallel to the surface under anomalous skin-effect conditions leads thus to a number of observed effects, and one can use the dependence of the specularity coefficient on the electron angle of incidence to study those effects experimentally.

First of all, the specularity coefficient leads to a change in the smooth part of the magnetic field and frequency dependence of the surface impedance.

The following cases are then possible:

I. $1 - q(\eta_\alpha - 0) \gg \rho_{eff}$. In that case there is practically no effect whatever of the specularity coefficient on the H- and $\omega$-dependence of the impedance and in rather weak magnetic fields ($|2\pi\gamma| \gtrsim 1$) the impedance is completely independent of H and equal to the impedance for H = 0; however, when $|2\pi\gamma| \ll 1$, $Z(H) \propto H^{-1/3}$ and is independent of the dispersion law. At cyclotron resonance which occurs when $\omega = m\Omega$ (m an integer) the specularity coefficient has then no influence whatever.

II. $1 - q(\eta_\alpha - 0) \ll \rho_{eff}$. We must consider here two possibilities:

1) If $\rho_{eff} \ll 1$, the H- and $\omega$-dependence of the impedance depends significantly on the relation between the quantities $\partial q(\eta_\alpha - 0)/\partial t$, $\omega$, and $1/t_0$ and differs from the dependences in case I. The ratio of the real part of the surface impedance to the imaginary one also depends on H and $\omega$. An experimental study of the H- and $\omega$-dependence of the impedance can thus determine q and $\partial q/\partial p_y$ for $v_y = 0$.

We note a very strong effect which occurs when we take specularity into account. The asymptotic behavior in H of the impedance in strong magnetic fields in the

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[4]Under anomalous skin-effect conditions the contribution from diffusely reflected electrons which correspond to the $\rho$ in (2.5) is small and we can therefore put $\rho$ equal to zero in (4.1).

[5]We assume then that the specularity coefficient for electrons which hit the surface is such that it corresponds to a non-vanishing geometrically separated orbit.

[6]The 1/6th power is obtained when we take into account the fact that the dispersion law is not quadratic (see [1]).

case I when the specularity is unimportant will be Z(H) $\propto H^{-1/3}$, but in case II, 1) we shall have $Z(H) \propto H^{1/5}$, i.e., there is a difference of $8/15$ in the indexes of the powers which, of course, can be observed as the magnitude of the field can here be changed by several orders of magnitude. Indeed, the strength of the field is determined by the condition $|2\pi\gamma| = |r(1 + i\omega t_0)/l| \ll 1$. If then $|\partial q/\partial p_y| eHv_0/c \gg |i\omega + 1/t_0|$, the asymptotic behavior will be $H^{1/5}$, while in the opposite limit it will be $H^{-1/3}$. If we use for the experiment samples for which $l$ is 1 mm and the magnetic fields reach such magnitudes that $r \simeq 10^{-4}$ cm or $H = 10^4$ to $10^5$ Oe this gives already three orders of magnitude.[7)] If therefore there is in such a range of fields a change from one kind of dependence to another we can at once find the magnitude of $\partial q/\partial p_y$.

2) If, however, $\rho_{\text{eff}} \gg 1$, the impedance will in the main approximation in $(\delta_{\text{eff}}/r)^{1/2}$ be the same as in case I for $q \equiv 0$, i.e., this case corresponds to purely diffuse reflection. The specularity coefficient is thus important only in case II, 1).

Moreover, the specularity coefficient leads in the case of non-convex Fermi surfaces to a phenomenon which is completely identical to cyclotron resonance occurring only for electrons which collide with the surface. The difference consists in that the "steepness" of such a resonance is less due to an effective diminution of the mean free path because of non-specular collisions with the surface.

The author is grateful to M. Ya. Azbel' for suggesting this topic and for useful hints.

## APPENDIX

We solve Eq. (3.10) exactly using the method given in [7]:

$$-\xi^2 F_\sigma(\xi) + 1 = e^{i\sigma} \int_0^\infty d\xi' F_\sigma(\xi') \frac{1}{\sqrt{\xi\xi'}} \left\{ \frac{1}{|\xi - \xi'|^{1/2}} - \frac{1}{(\xi + \xi')^{1/2}} \right\}. \quad (A.1)$$

We are not interested here in the solution of $F_\sigma(\xi)$ itself, but in the quantity which determines the impedance

$$I_\sigma = \int_0^\infty F_\sigma(\xi)\, d\xi. \quad (A.2)$$

We change variables, $\xi = e^t$, $F_\sigma(e^t) = g(t)$. Equation (A.1) becomes

$$e^{5/2 t} g(t) + e^{i\sigma} \int_{-\infty}^{+\infty} dt' g(t') \left\{ \frac{1}{|1 - e^{t-t'}|^{1/2}} - \frac{1}{(1 + e^{t-t'})^{1/2}} \right\} = e^{t/2}. \quad (A.3)$$

In a two-sided Laplace transformation the representation of the function g(t) will be

$$M(z) = \int_{-\infty}^{+\infty} e^{-zt} g(t)\, dt, \quad g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{zt} M(z)\, dz. \quad (A.4)$$

We can write (A.2) in the form

$$I_\sigma = M(-1) = \int_{-\infty}^{+\infty} g(t) e^t\, dt. \quad (A.5)$$

One shows easily that if we choose M(z) such that a) M(z) would be analytical in the band $-3 < \operatorname{Re} z < b$ ($b > -1/2$) everywhere, except in the point $z = -2$, where there is a simple pole with residue 1, b) M(z) satisfies the difference equation

$$M(z - 5/2) + e^{i\sigma} K(z) M(z) = 0, \quad (A.6)$$

$$K(z) = \frac{2\sqrt{2}}{\sqrt{\pi}} \Gamma(-z) \Gamma\left(z + \frac{1}{2}\right) \sin\frac{\pi z}{2} \cos\frac{\pi}{2}\left(z - \frac{1}{2}\right) \quad (A.7)$$

(where $\Gamma(z)$ is the gamma-function (see [8], pp. 49 and 54–55) then g(t) defined by (A.4) satisfies the initial equation (A.3).

We put

$$M(z) = \frac{2/5\pi}{\sin 2/5\pi(z + 2)} L(z) e^u, \quad u = -\frac{z + 2}{5/2} i\sigma, \quad (A.8)$$

and then $L(z - 5/2) - K(z)L(z) = 0$, and

$$L(z) \text{ is analytical for } -3 < \operatorname{Re} z < b \text{ and } L(-2) = 1. \quad (A.9)$$

By direct substitution we can check that

$$L(z) = \frac{A(z)}{A(-2)} e^{D(z) - D(-2)} \quad (A.10)$$

satisfies all conditions (A.9), provided

$$A(z) = \left[ \frac{1}{K(z)} \frac{z + 2}{z + 3/2} \right]^{1/2}; \quad (A.11)$$

$$D(z) = \frac{1}{2} \ln \left\{ \frac{\Gamma(-2/5(z + 2)) \sin 2/5\pi(z + 2)}{\Gamma(-2/5(z + 3/2)) \sin 2/5\pi(z + 3/2)} \times \frac{\Gamma(-2/5(z - 3/2))}{\Gamma(-2/5(z - 2)) \Gamma(-2/5 z)} \left(-\frac{4\pi}{5}\right)^z \left[ \frac{\operatorname{tg} 1/2\pi(z + 1/2)}{\operatorname{tg} 1/2\pi z} \times \frac{\sin 2/5\pi(z + 2) \sin 2/5\pi(z + 3/2)}{\sin 2/5\pi(z + 1) \sin 2/5\pi(z + 1/2) \sin 2/5\pi(z + 5/2)} \right]^{1/2} \right\}^{8)}. \quad (A.12)$$

If we add to D(z) a function $\alpha(z)$ periodic with period $5/2$ and analytical in the band $-3 < \operatorname{Re} z < b$, all properties (A.9) will also be satisfied. We show that this function $\alpha(z)$ can be put identically equal to zero. First of all, $\alpha(z)$ increases more slowly than exponentially as $z \to \infty$. In the opposite case $M(z)|_{z \to \infty} \to \exp\{e^z\}$ and one could not have any Laplace transform. From the periodicity and analyticity of $\alpha(z)$ in the above-mentioned band it follows that it is analytical in the whole plane. It can therefore in the whole plane be written in the form of a finite polynomial (see, e.g., [9], p. 61). A finite polynomial can be periodic only if it is simply identically equal to a constant. This constant can be put equal to zero as it does not contribute to M(z). We have thus shown that there is a unique solution to (A.9) and hence to (A.1). Using (A.5), (A.8), (A.10) to (A.12) we find

$$I_\sigma = i e^{-2/5 i\sigma} \frac{4\pi^2}{\sin 2/5\pi} \left[ \frac{5\pi}{63} \sin\frac{\pi}{5} \left( 2\cos\frac{\pi}{5} \right)^{1/2} \frac{\Gamma(4/5)\Gamma(8/5)}{\Gamma^3(2/5)} \right]^{1/2}. \quad (A.13)$$

7) Of course, there is still an upper bound to the magnetic field in order that the conditions for the anomalous skin-effect are satisfied, $\delta/r \ll 1$ ($H < 10^6$ Oe).

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Translated by D. ter Haar
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