SUPERCONDUCTIVITY OF ALLOYED SEMIMETALS

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Electron-electron and electron-hole pairings are simultaneously taken into account in a model of an impure semimetal with an isotropic electron spectrum, neglecting the scattering of electrons by impurities. For the case when the dielectric gap which is obtained because of electron-hole pairing is much larger than the superconducting gap, a condition is obtained for the ratio of the constants of the intraband and interband interactions at which such a pairing is possible. In the expression for the superconducting gap a large factor is obtained for the effective coupling constant because of the increase in the density of states at the edge of the allowed band in the model of an exciton insulator. In the limit of weak alloying, when the energy of degeneracy of the "excess" electrons is smaller than the superconducting gap, the latter is expressed in terms of an effective coupling constant by a power-law function instead of an exponential function, i.e., in the sense of electron-electron pairing such a system behaves like a one-dimensional system.

FOR an isotropic spectrum of electrons and holes in a semimetal an imaginary pole at T = 0, indicating instability of the system, exists both in the particle-hole channel[1] and in the particle-particle channel for scattering within the limits of a single band. In this sense the present model is analogous to a model of a one-dimensional metal, and in the problem about rearrangement of the energy spectrum it is necessary to take "parquet" diagrams into account.[2]

However one can easily verify that the linkage of electron-electron and electron-hole "ladder" diagrams in a "parquet" is accomplished in our model only in terms of a matrix element associated with a transition of a particle from one band to a particle in the other band. We shall assume this matrix element to be small because of the orthogonality of the states of different bands for identical momenta, and therefore we shall not take "parquet" diagrams into consideration. As will be evident from what follows, simultaneous electron-electron and electron-hole pairings are impossible in this approximation for equal concentrations of electrons and holes.

In the present article a situation with different concentrations of electrons and holes is considered. In this connection, for the case of only electron-hole pairing without taking the electron-electron pairing into account, a shift $\Delta\mu$ of the Fermi level occurs from the middle of the dielectric gap Δ_d into the allowed band. In what follows we shall assume that the superconducting gap Δ_c is much smaller than the dielectric gap. Therefore, for $\Delta\mu$ one can use the expression associated with $\Delta_c = 0$ to within a correction of the order of Δ_c/Δ_d.

Although the amplitude for the scattering of an electron by an electron from different bands does not have any singularities in the semimetallic phase, in the same way as the amplitude for the scattering of an electron by a hole from a single band, after taking account of electron-hole pairing for unequal concentrations of electrons and holes the distribution function of the "excess" electrons for bands 1 and 2 at T = 0 has the form (see Eq. (9) below)

$$f_{1,2}(\mathbf{p}) = \begin{cases} \frac{1}{2}\left(1 \mp \frac{\varepsilon(\mathbf{p})}{\sqrt{\varepsilon^2(\mathbf{p}) + \Delta_д^2}}\right) & \text{for } \varepsilon(\mathbf{p}) < \sqrt{\Delta\mu^2 - \Delta_д^2} \\ 0 & \text{for } \varepsilon(\mathbf{p}) > \sqrt{\Delta\mu^2 - \Delta_д^2} \end{cases}$$

instead of the Fermi distribution.

The amplitude for the scattering of an electron by an electron with such a distribution function does not have any singularities. As to taking simultaneous account of the scattering of an electron by an electron from both the same band as well as from different bands, the system possesses the usual logarithmic singularity, since the distribution function summed over both bands is the usual Fermi distribution with limiting energy $\sqrt{\Delta\mu^2 - \Delta_d^2}$.

This singularity will be taken into consideration below by the introduction of anomalous Greens' functions of two types, corresponding to superconducting pairings of the electrons from one and the same band and from different bands.

We write the Hamiltonian of the system in the form

$$H = \sum_{\substack{\alpha=1,2 \\ \sigma}} \left\{ \int \varepsilon_\alpha(\hat{\mathbf{p}})\psi_{\alpha\sigma}^+(\mathbf{r})\psi_{\alpha\sigma}(\mathbf{r})\,d\mathbf{r} + \lambda_{\alpha\alpha}\int \psi_{\alpha\sigma}^+(\mathbf{r})\psi_{\alpha-\sigma}^+(\mathbf{r})\psi_{\alpha-\sigma}(\mathbf{r})\psi_{\alpha\sigma}(\mathbf{r})\,d\mathbf{r} \right\} + \sum_{\sigma,\sigma'} \lambda_{21}\int \psi_{1\sigma}^+(\mathbf{r})\psi_{2\sigma'}(\mathbf{r})\psi_{2\sigma'}(\mathbf{r})\psi_{1\sigma}(\mathbf{r})\,d\mathbf{r},$$

$$\varepsilon_1(\mathbf{p}) = \frac{p^2 - p_{1F}^2}{2m}, \qquad \varepsilon_2(\mathbf{p}) = -\frac{p^2 - p_{2F}^2}{2m}, \tag{1}$$

i.e., for simplicity we shall assume the electron and hole masses to be equal. Let us consider the case when the concentration of electrons is larger than the concentration of holes (a donor impurity), i.e., $p_{1F} > p_{2F}$, but the difference in concentration δn is smaller than the critical value[3] above which electron-hole pairing does not exist.

The last term in Eq. (1) corresponds to an interband screened Coulomb interaction, where[1]

$$\frac{mp_F\lambda_{21}}{2\pi^2} \equiv \lambda_{21}' \approx \frac{e^2}{\varepsilon\hbar v_F \ln(p_F^2/\varkappa_D^2)} > 0$$

and the cutoff energy of this interaction is $\tilde{\omega}_2 \approx \omega_p$, where ω_p is the plasma frequency, v_F denotes the velocity of electrons and holes at the Fermi surface, which is approximately the same for electrons and holes when the difference in their concentrations is smaller than the critical value, κ_D^{-1} is the Debye screening radius, and ϵ is the dielectric constant of the semimetal.

Let us denote the cutoff energy of the intraband interactions by $\tilde{\omega}_1$. It will be shown below that λ_{11} and λ_{22} must be negative in order for superconductivity to exist, i.e., an attraction must exist between the electrons inside each band, where this attraction may be related, for example, to the electron-phonon interaction or to an electronic mechanism.[4]

Then relations will be established between λ_{11}, λ_{21}, $\tilde{\omega}_1$, and $\tilde{\omega}_2$ so that superconducting pairing can be realized together with dielectric pairing.

First let us consider the case T = 0. Let us introduce the following Green's functions:

$$\begin{aligned} G_{11}(\mathbf{r},\mathbf{r}',t,t') &= i\langle T\psi_{1\sigma}^{+}(\mathbf{r},t)\psi_{1\sigma}(\mathbf{r}',t')\rangle,\\ G_{21}(\mathbf{r},\mathbf{r}',t,t') &= i\langle T\psi_{2-\sigma}(\mathbf{r},t)\psi_{1\sigma}^{+}(\mathbf{r}',t')\rangle,\\ F_{11}^{+}(\mathbf{r},\mathbf{r}',t,t') &= i\langle T\psi_{1\sigma}^{+}(\mathbf{r},t)\psi_{1-\sigma}^{+}(\mathbf{r}',t')\rangle,\\ F_{21}^{+}(\mathbf{r},\mathbf{r}',t,t') &= i\langle T\psi_{2\sigma}^{+}(\mathbf{r},t)\psi_{1\sigma}^{+}(\mathbf{r}',t')\rangle. \end{aligned} \tag{2}$$

If one takes, as in (2), the singlet spin state in F_{11}^{+}, then a simultaneous solution for the electron-electron and electron-hole pairings is realized either when the arrangement of the spins in G_{21} and F_{21}^{+} is the opposite of that in (2) (singlet electron-hole pairs and singlet electron-electron pairs from different bands), or else, as in (2), for triplet electron-hole and triplet electron-electron pairings from different bands. The latter is possible because of the absence of the Pauli exclusion principle for electrons from different bands.

For other directions of the spins in pairs (singlet in G_{21} and triplet in F_{21}^{+} or triplet in G_{21} and singlet in F_{21}^{+}) the anomalous Green's functions G_{21}, F_{11}^{+}, and F_{21}^{+} are not coupled with each other and therefore, as one can easily verify, no singularities at all appear in the amplitude for simultaneous interband and intraband scatterings of an electron by an electron (upon taking account of just the fundamental electron-hole pairing).

Writing down the equations of motion for the operators ψ_1 and ψ_2 with the Hamiltonian (1) and changing in them to the Green's functions (2), after taking the Fourier transforms with respect to coordinates and time we obtain

$$\begin{aligned} &(\omega-\varepsilon(\mathbf{p})+\Delta\mu)G_{11}(\omega,\mathbf{p})-i\Delta_d G_{21}(\omega,\mathbf{p})+i\Delta_1 F_{11}^{+}(\omega,\mathbf{p})\\ &\quad+i\Delta_2 F_{21}^{+}(\omega,\mathbf{p})=1, \end{aligned} \tag{3}$$

$$\begin{aligned} &(\omega+\varepsilon(\mathbf{p})+\Delta\mu)G_{21}(\omega,\mathbf{p})+i\Delta_d^{*}G_{11}(\omega,\mathbf{p})-i\Delta_2 F_{11}^{+}(\omega,\mathbf{p})\\ &\quad+i\Delta_1 F_{21}^{+}(\omega,\mathbf{p})=0, \end{aligned} \tag{4}$$

$$\begin{aligned} &(\omega+\varepsilon(\mathbf{p})-\Delta\mu)F_{11}^{+}(\omega,\mathbf{p})-i\Delta_1^{*}G_{11}(\omega,\mathbf{p})+i\Delta_2^{*}G_{21}(\omega,\mathbf{p})\\ &\quad+i\Delta_d^{*}F_{21}^{+}(\omega,\mathbf{p})=0, \end{aligned} \tag{5}$$

$$\begin{aligned} &(\omega-\varepsilon(\mathbf{p})-\Delta\mu)F_{21}^{+}(\omega,\mathbf{p})-i\Delta_2^{*}G_{11}(\omega,\mathbf{p})-i\Delta_1^{*}G_{21}(\omega,\mathbf{p})\\ &\quad+i\Delta_d F_{11}^{+}(\omega,\mathbf{p})=0, \end{aligned} \tag{6}$$

where

$$\Delta_d^{*}=\lambda_{21}G_{21}(0),\quad \Delta_1^{*}=\lambda_{11}F_{11}^{+}(0),\quad \Delta_2^{*}=\lambda_{21}F_{21}^{+}(0) \tag{7}$$

and, for example,

$$G_{21}(0)=\frac{1}{(2\pi)^4}\int G_{21}(\omega,\mathbf{p})\,d\omega\,d\mathbf{p}.$$

An expression for $\Delta\mu$ in Eqs. (3)–(6) is obtained from the condition for electrical neutrality of the system:

$$\delta n=\frac{2}{(2\pi)^4}\int[G_{11}(\omega,\mathbf{p})-G_{22}(\omega,\mathbf{p})]\,d\omega\,d\mathbf{p}. \tag{8}$$

Considering

$$|\Delta_1|,\ |\Delta_2|\ll\sqrt{\Delta\mu^2-\Delta_d^2}\ll\Delta_d, \tag{9}$$

after substituting G_{11} from the system of equations (3)–(4) for $\Delta_1 = \Delta_2 = 0$ into Eq. (8),

$$\begin{aligned} G_{11}^{0}(\omega,\mathbf{p})=&\frac{1/2\,(1-\varepsilon(\mathbf{p})/\sqrt{\varepsilon^2(\mathbf{p})+\Delta_d^2})}{\omega+\Delta\mu-\sqrt{\varepsilon^2(\mathbf{p})+\Delta_d^2}+i\delta\,\mathrm{sign}\,(-\Delta\mu+\sqrt{\varepsilon^2(\mathbf{p})+\Delta_d^2})}\\ &+\frac{1/2\,(1+\varepsilon(\mathbf{p})/\sqrt{\varepsilon^2(\mathbf{p})+\Delta_d^2})}{\omega+\Delta\mu+\sqrt{\varepsilon^2(\mathbf{p})+\Delta_d^2}-i\delta} \end{aligned} \tag{10}$$

(and a similar expression for G_{22}^{0} with only a change of the sign in front of $\Delta\mu$), from Eq. (8) we obtain the following expression for $\Delta\mu$:

$$\Delta\mu=\left(\frac{\pi^4}{m^2p_F^2}\delta n^2+\Delta_d^2\right)^{1/2}. \tag{11}$$

From formula (10) and from the analogous expression for G_{22}^{0} one obtains the distribution function indicated above for the "excess" electrons. The second inequality in (9) means $\delta n \ll \delta n_{cr}$.[3]

Solving Eqs. (3)–(6) with regard to the functions G_{11}, G_{21}, F_{11}^{+}, and F_{21}^{+} we obtain

$$\begin{aligned} G_{11}(\omega,\mathbf{p})=&\{(\omega+\varepsilon(\mathbf{p})+\Delta\mu)[(\omega-\Delta\mu)^2-\varepsilon^2(\mathbf{p})-\Delta_d^2]\\ &-\Delta_1^2(\omega+\varepsilon(\mathbf{p})-\Delta\mu)-\Delta_2^2(\omega-\varepsilon(\mathbf{p})-\Delta\mu)\}D^{-1}, \end{aligned} \tag{12}$$

$$\begin{aligned} G_{21}(\omega,\mathbf{p})=&\{-i\Delta_d^{*}[(\omega-\Delta\mu)^2-\varepsilon^2(\mathbf{p})-\Delta_d^2-\Delta_1^2-\Delta_2^2]\\ &+2\varepsilon\Delta_1\Delta_2^{*}\}D^{-1}, \end{aligned} \tag{13}$$

$$\begin{aligned} F_{11}^{+}(\omega,\mathbf{p})=&\{i\Delta_1^{*}[\omega^2-(\varepsilon(\mathbf{p})+\Delta\mu)^2-\Delta_d^2]+\\ &+2(\varepsilon(\mathbf{p})+\Delta\mu)\Delta_d\Delta_2^{*}\}D^{-1}, \end{aligned} \tag{14}$$

$$F_{21}^{+}(\omega,\mathbf{p})=\{i\Delta_2^{*}[(\omega+\varepsilon(\mathbf{p}))^2-\Delta\mu^2-\Delta_d^2]+2\Delta\mu\Delta_d\Delta_1^{*}\}D^{-1}, \tag{15}$$

where

$$\begin{aligned} D=&\{[(\omega+\Delta\mu)^2-\varepsilon^2(\mathbf{p})-\Delta_d^2][(\omega-\Delta\mu)^2-\varepsilon^2(\mathbf{p})-\Delta_d^2]-\\ &-2\Delta_1^2(\omega^2-\varepsilon^2(\mathbf{p})-\Delta\mu^2-\Delta_d^2)-2\Delta_2^2(\omega^2+\varepsilon^2(\mathbf{p})-\Delta\mu^2-\Delta_d^2)\}. \end{aligned}$$

The spectrum of the system is determined by the poles of G_{11}:

$$\omega_{1,4}\approx\pm(\Delta\mu+\sqrt{\varepsilon^2(\mathbf{p})+\Delta_d^2}), \tag{16}$$

$$\begin{aligned} \omega_{2,3}\approx\pm\Big[&\Delta\mu-\sqrt{\varepsilon^2(\mathbf{p})+\Delta_d^2}+\frac{\Delta_1^2}{\Delta_d^2}(\varepsilon^2(\mathbf{p})+\Delta\mu^2+\Delta_d^2)\\ &+\frac{\Delta_2^2}{\Delta_d^2}(\Delta\mu^2+\Delta_d^2-\varepsilon^2(\mathbf{p}))\Big]^{1/2}. \end{aligned} \tag{17}$$

In what follows we shall neglect the contributions to Δ_d coming from Δ_1 and Δ_2, and therefore we may choose the phase of Δ_d to be arbitrary. We shall assume that Δ_d is real. A system of equations for the quantities Δ_1^{*} and Δ_2^{*} is obtained from Eq. (7). We also choose Δ_1 to be real. Then, as will be evident from the following, Δ_2 will be pure imaginary, and the term $i\cdot 4\Delta\mu\Delta_d\Delta_1(\Delta_2^{*}+\Delta_2)$ vanishes identically in the denominators of the functions G_{11}, G_{21}, F_{11}^{+}, and F_{21}^{+}. This has been taken into account in expressions (3)–(6).

Substituting the expressions for the functions F_{11}^{+} and F_{21}^{+} obtained from Eqs. (14) and (15) into Eq. (7), we obtain the following system of equations:

$$\Delta_1^{*}=\frac{\lambda_{11}}{(2\pi)^4}\int\frac{i\Delta_1^{*}[\omega^2-(\varepsilon(\mathbf{p})+\Delta\mu)^2-\Delta_d^2]+2\Delta_2^{*}\Delta_d(\varepsilon(\mathbf{p})+\Delta\mu)}{D}\,d\omega\,d\mathbf{p}, \tag{18}$$

$$\Delta_2^* = \frac{\lambda_{21}}{(2\pi)^4}\int \frac{i\Delta_2^*[(\omega+\varepsilon(\mathbf{p}))^2-\Delta\mu^2-\Delta_d^2]-2\Delta_1^*\Delta_d\Delta\mu}{D}\,d\omega\,d\mathbf{p}. \tag{19}$$

After integration in Eqs. (18) and (19) with respect to ω and $\mathbf{p}$ under the assumption that $|\Delta_1|$, $|\Delta_2| \ll \sqrt{\Delta\mu^2-\Delta_d^2} \ll \Delta_d \ll \tilde{\omega}_1, \tilde{\omega}_2$ we obtain

$$\Delta_1^* = -\lambda_{11}'\Delta_1^*\left(\ln\frac{\tilde{\omega}_1}{\Delta\mu+\sqrt{\Delta\mu^2-\Delta_d^2}}+\frac{\Delta_d}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_1^2+\Delta_2^2}\right) - \frac{i\Delta_d\lambda_{11}'\Delta_2^*}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_1^2+\Delta_2^2}, \tag{20}$$

$$\Delta_2^* = -\lambda_{21}'\Delta_2^*\left(\ln\frac{\tilde{\omega}_2}{\Delta\mu+\sqrt{\Delta\mu^2-\Delta_d^2}}+\frac{\Delta_d}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_1^2+\Delta_2^2}\right) + \frac{i\Delta_d\lambda_{21}'\Delta_1^*}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_1^2+\Delta_2^2} \tag{21}$$

where $\lambda_{11}' \equiv \lambda_{11}mp_F/2\pi^2$.

An account of the corrections to Δ_d and $\Delta\mu$ due to Δ_1 and Δ_2 leads to a renormalization of the factor $\Delta_d/\sqrt{\Delta\mu^2-\Delta_d^2}$ in Eqs. (20) and (21), which is small under the hypothesis (9). In these equations the terms containing logarithms, which are cut off at the lower limit by the superconducting gap $\Delta_c = (\Delta_1^2+\Delta_2^2)^{1/2}$, are cut off at the upper limit not at $\tilde{\omega}_1$ or $\tilde{\omega}_2$ but at $(\Delta_d\cdot\sqrt{\Delta\mu^2-\Delta_d^2})^{1/2}$, which is physically related to the fact that outside of this energy range there are no "excess" electrons which are able to form Cooper pairs, and cutoff is automatically realized.

Nontrivial solutions associated with the vanishing of the determinant of the system of Eqs. (20)–(21) exist for the functions Δ_1^* and Δ_2^*. Substituting the expression for G_{21} obtained from Eqs. (3) and (4) under the assumption $\Delta_1 = \Delta_2 = 0$ into Eq. (7) for Δ_d, we obtain

$$1 = \lambda_{21}'\ln\frac{\tilde{\omega}_2}{\Delta\mu+\sqrt{\Delta\mu^2-\Delta_d^2}}. \tag{22}$$

With (22) taken into account, the condition for the solvability of the system (20)–(21) takes the form

$$\frac{\Delta_d}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_1^2+\Delta_2^2} = -2\frac{1+\lambda_{11}'\ln(\tilde{\omega}_1/\tilde{\omega}_2)+\lambda_{11}'/\lambda_{21}'}{\lambda_{21}'+3\lambda_{11}'+\lambda_{11}'\lambda_{21}'\ln(\tilde{\omega}_1/\tilde{\omega}_2)} \equiv \frac{1}{\lambda^*}. \tag{23}$$

Let us clarify what sign the interaction λ_{11} should have in order to satisfy condition (23); here we assume that $\lambda_{21} > 0$ since otherwise Eq. (22) would not have any solutions for the dielectric gap Δ_d, which everywhere above was assumed to be much larger than the superconducting gaps $|\Delta_1|$ and $|\Delta_2|$. Starting from the assumptions (9) used by us, the left hand side of Eq. (23) is always positive; hence the right hand side is also positive, i.e., the effective interaction λ^* must be positive.

For $\lambda_{11} > 0$ the positiveness of λ^* is guaranteed only under the condition

$$\ln\frac{\Delta_d}{\tilde{\omega}_1} > \frac{1}{\lambda_{11}'}, \tag{24}$$

which contradicts the initial assumptions under which the system of equations (20)–(21) is obtained. For $\lambda_{11} < 0$ the condition (23) is satisfied for

$$\left(\ln\frac{\tilde{\omega}_1}{\tilde{\omega}_2}+\frac{1}{\lambda_{21}'}\right)^{-1} > |\lambda_{11}'| > \left(\ln\frac{\tilde{\omega}_1}{\tilde{\omega}_2}+\frac{3}{\lambda_{21}'}\right)^{-1}, \tag{25}$$

at the same time

$$\Delta_1^2+\Delta_2^2 = 4\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}\exp\left(-\frac{2\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_d\lambda^*}\right). \tag{26}$$

Formally λ^* varies from zero to infinity in the interval (25), but in fact there is a restriction on the side of large values of λ^* because of the use of condition (9):

$$\left(\frac{3}{\lambda_{21}'}+\ln\frac{\tilde{\omega}_1}{\tilde{\omega}_2}\right)^{-1} < |\lambda_{11}'| < \left\{\lambda_{21}'+\frac{2\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_d}\ln\frac{\Delta_d}{\sqrt{\Delta\mu^2-\Delta_d^2}}\right\} \times\left\{3+\lambda_{21}'\ln\frac{\tilde{\omega}_1}{\tilde{\omega}_2}+\left(\ln\frac{\tilde{\omega}_1}{\tilde{\omega}_2}+\frac{1}{\lambda_{21}'}\right)\frac{2\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_d}\ln\frac{\Delta_d}{\sqrt{\Delta\mu^2-\Delta_d^2}}\right\}^{-1}.$$

The appearance of the factor $\Delta_d/\sqrt{\Delta\mu^2-\Delta_d^2}$ in expression (26) for λ^* is associated with the increase in the model of an exciton insulator[1] of the density of states as the edge of the allowed band is approached, and with its becoming infinite at the band edge for an isotropic energy spectrum of the electrons and holes in a semimetal. An analogous effect is well known for many-valley semiconductors, where for the superconducting gap the effective density of states in the exponential contains the number of valleys as a factor.[5]

With a decrease in the concentration of "excess" electrons, still a large fraction of them turn out to be in the region of an increased density of states, and $\Delta_c = (\Delta_1^2+\Delta_2^2)^{1/2}$ increases.

For $\Delta_1^2+\Delta_2^2 > \Delta\mu^2-\Delta_d^2$ (weak alloying) formula (26) becomes invalid. In this connection one obtains a system of equations analogous to (20)–(21) for Δ_1^* and Δ_2^* with the following replacement of terms:

$$\frac{\Delta_d}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_1^2+\Delta_2^2} \to \left[\frac{\Delta_d}{(\Delta_1^2+\Delta_2^2)^{1/2}}\right]^{1/2},$$

i.e., terms appear just like in the problem of the eigenvalues of a shallow one-dimensional well.[6] In this case

$$(\Delta_1^2+\Delta_2^2)^{1/2} \approx \Delta_d\lambda^{*2}. \tag{27}$$

In fact, for such a small concentration of "excess" electrons apparently a coupling of these electrons with impurity ions[7] will be more advantageous, which is also a source of "excess" electrons. According to Eq. (21) the relation between Δ_2^* and Δ_1^* has the form

$$\Delta_2^* = i\Delta_1^*\frac{\lambda_{21}'}{2\lambda^*+\lambda_{21}'}, \tag{28}$$

i.e., if Δ_1 is chosen to be real then Δ_2 must be imaginary which we have used in Eqs. (12)–(17). From the simultaneous solution of Eqs. (26) and (28) we have

$$\Delta_1^2 = \frac{\Delta_d(2\lambda^*+\lambda_{21}')^2\sqrt{\Delta\mu^2-\Delta_d^2}\exp(-2\sqrt{\Delta\mu^2-\Delta_d^2}/\Delta_d\lambda^*)}{2(2\lambda^{*2}+\lambda_{21}'^2+2\lambda^*\lambda_{21}')}, \tag{29}$$

$$\Delta_2^2 = \frac{\Delta_d\lambda_{21}'^2\sqrt{\Delta\mu^2-\Delta_d^2}\exp(-2\sqrt{\Delta\mu^2-\Delta_d^2}/\Delta_d\lambda^*)}{2(2\lambda^{*2}+\lambda_{21}'^2+2\lambda^*\lambda_{21}')}. \tag{30}$$

As is clear from Eqs. (29) and (30), formally as $\lambda^* \to \infty$, i.e., as

$$\frac{1}{\lambda_{11}'} \to \ln\frac{\tilde{\omega}_1}{\tilde{\omega}_2}+\frac{1}{\lambda_{21}'},$$

$\Delta_2 \to 0$ but Δ_1 remains a finite quantity. As $\lambda^* \to 0$, i.e.,

$$\frac{1}{\lambda_{11}'} \to \ln\frac{\tilde{\omega}_1}{\tilde{\omega}^2}+\frac{3}{\lambda_{21}'},$$

both $|\Delta_1|$ and $|\Delta_2|$ approach zero. We note that a nontrivial solution for Δ_2 exists associated with a repulsive interaction ($\lambda_{21} > 0$).

If $\Delta\mu$ is set equal to zero in Eqs. (18) and (19), i.e.,

if one takes identical concentrations of electrons and holes, then no terms appear which diverge for $\Delta_1 = \Delta_2 = 0$, and therefore in the case of weak interactions considered by us the system of equations (18)–(19) does not have a solution under these conditions.

Let us determine the temperature of the superconducting transition. For this purpose it is convenient to use the temperature technique for the Green's functions.[8] In this connection, as is well known, in the expressions obtained at T = 0 it is necessary to replace an integration over ω by a summation over $\omega_n = (2n + 1)\pi T$, where $n = 0, \pm 1, \pm 2, \ldots$. The system of equations for $\Delta_1^*(T)$ and $\Delta_2^*(T)$ will differ from Eqs. (18) and (19) only by the indicated replacement of an integration by a summation.

In order to determine the superconducting transition temperature, in this system of equations it is necessary to regard Δ_1 and Δ_2 as infinitesimal, and terms which are quadratic in these quantities are neglected. In this connection it is clear that Δ_1^* and Δ_2^* can only vanish simultaneously.

Under the condition (9) the superconducting transition temperature T_c will be much smaller than the temperature T_d at which the dielectric gap Δ_d vanishes. Therefore, for $T = T_c$ we shall use expressions (11) and (22) with T = 0 for Δ_d and $\Delta\mu$. In this limit we obtain

$$\Delta_1^*(T_c-0)=-\lambda_{11}'\Delta_1^*(T_c-0)\left(\ln\frac{\tilde{\omega}_1}{\Delta\mu+\sqrt{\Delta\mu^2-\Delta_d^2}}+\frac{\Delta_d}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\gamma^2\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\pi^2T_c^2}\right)-$$
$$-i\frac{\Delta_2^*(T_c-0)\Delta_d\lambda_{11}'}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\gamma^2\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\pi^2T_c^2}, \qquad (31)$$

$$\Delta_2^*(T_c-0)=-\lambda_{21}'\Delta_2^*(T_c-0)\left(\ln\frac{\tilde{\omega}_2}{\Delta\mu+2\sqrt{\Delta\mu^2-\Delta_d^2}}+\frac{\Delta_d}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\gamma^2\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\pi^2T_c^2}\right)+$$
$$+\frac{i\Delta_1^*(T_c-0)\Delta_d\lambda_{21}'}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\gamma^2\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\pi^2T_c^2}, \qquad (32)$$

where $\ln\gamma = C = 0.577$.

The temperature T_c of the superconducting transition is determined from the condition that the determinant of the system of equations (31), (32) vanishes. As a result

$$T_c=\frac{2\gamma}{\pi}(\Delta\mu^2-\Delta_d^2)^{1/2}\Delta_d^{1/2}\exp\left(-\frac{\sqrt{\Delta\mu^2-\Delta_d^2}}{\Delta_d\lambda^*}\right), \qquad (33)$$

i.e., the relation between T_c and $(\Delta_1^2 + \Delta_2^2)^{1/2}$ at T = 0 is the same as in the BCS model.[8]

If the condition $|\Delta_1|, |\Delta_2| \ll \sqrt{\Delta\mu^2 - \Delta_d^2}$ for T = 0 is not satisfied, then in order to determine the superconducting transition temperature it is necessary to solve the equations for $\Delta_1^*(T)$ and $\Delta_2^*(T)$ simultaneously with the equation for $\Delta\mu(T)$.

If the critical temperature expressed by formula (33) is determined by a broadening of the Fermi level, then the dependence of $\Delta\mu$ on T takes into account the additional shift of the Fermi level as a whole.

It is easy to verify that for $|\Delta_1^*(T = 0)|, |\Delta_2(T = 0)| \gg \sqrt{\Delta\mu^2 - \Delta_d^2}$ the system of equations for $\Delta\mu(T_c)$ and $\Delta_1^*(T_c - 0)$, $\Delta_2^*(T_c - 0)$ obtained from (31) and (32) by the replacement

$$\frac{\Delta_d}{2\sqrt{\Delta\mu^2-\Delta_d^2}}\ln\frac{4\gamma^2\Delta_d\sqrt{\Delta\mu^2-\Delta_d^2}}{\pi^2T_c^2}\to\left(\frac{\Delta_d}{T_c}\right)^{1/2}$$

is not compatible, i.e., a phase transition of second order is impossible. We note that it is precisely in this limiting case that the superconducting gap $(\Delta_1^2 + \Delta_2^2)^{1/2}$ enters into the system of equations for Δ_1^* and Δ_2^* not under a logarithmic sign but inside a square-root sign. This case has not been investigated in detail in the present article since apparently in this case the "excess" electrons are more favorably localized on impurities.

Since the argument of the exponential function in expression (33) may be very small because of the fact that $\Delta_d/\sqrt{\Delta\mu^2 - \Delta_d^2} \gg 1$, then one can hope for a critical temperature of the superconducting transition in such a metal which is higher than that of an ordinary metal in the phonon model of BCS.[8]

All of the above considerations were carried out for the case when the concentration of electrons in the semimetallic phase is larger than the number of holes (a donor impurity). Analogous results are obtained for the opposite case (an acceptor impurity) by only replacing $\Delta\mu$ by $-\Delta\mu$ everywhere.

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