CONTRIBUTION TO THE THEORY OF QUASILINEAR RELAXATION OF AN ION-ACOUSTIC WAVE SPECTRUM IN A PLASMA

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The problem of relaxation of an anisotropic spectrum of ion-acoustic noise in a plasma is solved in the quasilinear approximation. It is shown that at a sufficient noise intensity, quasilinear relaxation causes a three-dimensional spectrum to change into a one-dimensional spectrum. A self-similar solution of the quasilinear relaxation equations that describe the process is obtained for the case of a strongly anisotropic spectrum.

THE purpose of the present paper is to explain certain regularities in the process of quasilinear evolution of the anisotropic spectrum of ion-acoustic noise in a plasma, i.e., the evolution of the spectrum as the results of induced Cerenkov emission and absorption of waves by the plasma particles. It is assumed that the noise level is sufficiently low to make the nonlinear interactions of the waves negligible. As shown by Sizonenko and Stepanov[1] and by Bernstein and Engelman,[2] the quasilinear relaxation of the spectrum of the potential oscillations leads inevitably either to their complete attenuation, or to the establishment of a stationary one-dimensional spectrum. It is of interest to investigate the dynamics of such a "collapse" of the three-dimensional spectrum into a one-dimensional one. This process plays an important role, for example, in the development of current instability in a plasma.[3]

TRANSFORMATION OF A THREE-DIMENSIONAL WAVE PACKET INTO A ONE-DIMENSIONAL ONE

Assume that there is an intense anisotropic spectrum of ion-acoustic waves in the plasma. We introduce spherical coordinates in velocity space (v, ϑ, φ) and in wave-vector space (k, ϑ', φ'). The problem simplifies greatly for a packet having an azimuthal symmetry in φ' and a small width in modulus of the wave vector. Apart from small corrections, the number of the waves $N(\mathbf{k})$, can be represented in the form

$$N(\mathbf{k})=\frac{\partial\varepsilon}{\partial\omega_k}\frac{k^2|\varphi|_{\mathbf{k}}^2}{8\pi}=N(\vartheta',t)\,\delta(k-k_0) \tag{1}$$

(the symbols are standard).

Rudakov and Korablev[4] have shown that a spectrum of precisely this type builds up when current instability develops in the plasma.

When $T_e \gg T_i$, the interaction of the ion-acoustic waves with the ions can be regarded as negligible for the evolution of the spectrum, since the density of the resonant ions is an exponentially small quantity, and the ions become rapidly heated by interacting with the oscillations. As a result, their contribution to the total increment tends to zero (see [5], and also [3]).

By virtue of the assumptions made above, the problem of the quasilinear evolution of the spectrum becomes mathematically very similar to the problem of instability of a current in a plasma,[4] the only difference being that in the present paper we consider a case in which there is no external electric field. The system of quasilinear equations is of the form[4]

$$\frac{\partial f}{\partial t}=\frac{1}{v^3\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\left(A\frac{\partial f}{\partial\vartheta}+B\frac{\partial f}{\partial v}\right)+\frac{1}{v^2}\frac{\partial}{\partial v}\frac{1}{v}\left(D\frac{\partial f}{\partial v}+B\frac{\partial f}{\partial\vartheta}\right), \tag{2}$$

$$\partial N(\mathbf{k})\,/\,\partial t=2\gamma(\mathbf{k})N(\mathbf{k}), \tag{3}$$

where

$$\gamma=\frac{\pi}{2}\frac{\omega^3}{k^2}\frac{M}{mn}\left(\omega\int\frac{1}{v}\frac{\partial f}{\partial v}\delta(\omega-\mathbf{kv})\,d\mathbf{v}+\int\frac{k_\vartheta}{v}\frac{\partial f}{\partial\vartheta}\delta(\omega-\mathbf{kv})\,d\mathbf{v}\right), \tag{4}$$

$$\left.\begin{matrix}A\\B\\D\end{matrix}\right\}=\pi\left(\frac{e}{m}\right)^2\int d\mathbf{k}v\,|\varphi|_{\mathbf{k}}^2\delta(\omega-\mathbf{kv})\left\{\begin{matrix}k_\vartheta^2\\ \omega k_\vartheta,\\ \omega^2\end{matrix}\right. \tag{5}$$

where $f(\mathbf{v})$ is the time-averaged ("background") electron distribution function.

In view of the smallness of the parameter ω/kv_{Te}, where v_{Te} is the thermal velocity of the electrons, the principal term in the right side of (2) will be the one containing the coefficient A; the next in magnitude ($\sim A\omega/kv_{Te}$) will be the term containing $B\,\partial f/\partial v$. The remaining terms in the right side of (2) are of still higher order of smallness in ω/kv_{Te}, and we shall disregard them. The diffusion coefficients A and B in velocity space are proportional to the energy density of the waves, and for large oscillation amplitudes it is possible to satisfy the condition

$$\frac{\partial f}{\partial t}\ll\frac{1}{v^3\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta B\frac{\partial f}{\partial v}. \tag{6}$$

We assume the inequality (6) to be satisfied, and we shall subsequently establish the limits of applicability of this assumption.

Thus, Eq. (2) can be solved by successive approximations in A^{-1}. The zeroth approximation yields $\partial f_0/\partial\vartheta = 0$, i.e., the interaction with the noise leads to symmetrization of the particle distribution function. In the next approximation in ω/kv_{Te} we obtain

$$f_1(v,\vartheta)=-\int_0^\vartheta\frac{B}{A}\frac{\partial f_0}{\partial v}d\vartheta. \tag{7}$$

It is clear even from this that sufficiently intense noise cannot attenuate completely as a result of quasilinear relaxation. Indeed, it follows from (7) that the density of the maximum momentum that can be transferred to the electrons does not exceed $nmv_{Te}(\omega/kv_{Te})$, and that if the momentum of the waves is larger, then the quasilinear relaxation can lead to realization of only one of two possible stationary states obtained in [1,2], namely to a transformation of the three-dimensional wave packet into a one-dimensional one.

It is necessary next to substitute in (3) and (4) the solution (7) and to perform calculations perfectly analogous to those performed by Rudakov and Korablev.[4] As a result we arrive at the following equation, which describes the evolution of the spectrum of the ion-acoustic waves:

$$\frac{\partial}{\partial t}\ln N(z,t)=2\gamma_0\int_z^1\frac{y\,dy}{\sqrt{1-y^2}\sqrt{y^2-z^2}}\times\int_0^y\frac{z'(z-z')N(z',t)}{\sqrt{y^2-z'^2}}dz'\Big/\int_0^y\frac{z'^2N(z',t)\,dz'}{\sqrt{y^2-z'^2}}. \tag{8}$$

Here $z = \cos\vartheta'$, $y = \sin\vartheta$, and

$$\gamma_0=\pi^2\frac{\omega^4}{k^3}\frac{M}{m}f_0\left(\frac{\omega}{k}\right). \tag{9}$$

It is seen from (8) that when $\vartheta' = 0$ the increment $\gamma(k)$ is positive, i.e., the interaction of the electrons with the oscillations in such an anisotropic spectrum leads to an energy and momentum flux in the region of small angles. As a result, the anisotropy of the packet becomes more intense, i.e., the three-dimensional spectrum actually becomes one-dimensional.

SELF-SIMILAR SOLUTION OF THE SPECTRUM EVOLUTION EQUATION

Let the anisotropy of the spectrum be so strong that it can be regarded as concentrated in a narrow cone with aperture $\vartheta_0 \ll 1$. Then Eq. (8) can be rewritten in simpler form

$$\frac{d}{d\tau}\ln N(\zeta,\tau)=\int_0^\zeta\frac{d\psi}{\sqrt{\psi(\zeta-\psi)}}\times\int_\psi^{1-\cos\vartheta_0}\frac{(\chi-\zeta)N(\chi,\tau)\,d\chi}{\sqrt{\chi-\psi}}\Big/\int_\psi^{1-\cos\vartheta_0}\frac{N(\chi,\tau)\,d\chi}{\sqrt{\chi-\psi}},$$

$$\zeta=1-\cos\vartheta',\quad \psi=1-y,\quad \chi=1-z',\quad \tau=2\gamma_0 t. \tag{10}$$

Equation (10) admits of a self-similar separation of the variables. We choose the basic scales to be the angle ϑ_0 and a certain characteristic time τ_0, which must be determined from the solution itself. We seek the solution of Eq. (10) in the form

$$1-\cos\vartheta_0=(\tau_0/\tau)^m,\quad N(\vartheta',\tau)=(\tau/\tau_0)^n n(\xi),\quad \xi=\frac{1-\cos\vartheta'}{1-\cos\vartheta_0}=\zeta\left(\frac{\tau}{\tau_0}\right)^m. \tag{11}$$

The energy density of the oscillations is written in terms of the new variables as follows:

$$W=\omega\int N(\mathbf{k})\,d\mathbf{k}=2\pi k_0^2\omega(\tau/\tau_0)^{n-m}\int_0^1 n(\xi)\,d\xi, \tag{12}$$

and since the oscillation energy in the final state should be finite, we get m = n.

Let us rewrite Eq. (10) in terms of new variables, introducing $\eta = \psi(\tau/\tau_0)^m$ and $\xi' = \chi(\tau/\tau_0)^m$:

$$\frac{m}{\tau}\left[1+\xi\frac{dn/d\xi}{n(\xi)}\right]=\left(\frac{\tau_0}{\tau}\right)^m\int_0^\xi\frac{d\eta}{\sqrt{\eta(\xi-\eta)}}\times\int_\eta^1\frac{(\xi'-\xi)n(\xi')d\xi'}{\sqrt{\xi'-\eta}}\Big/\int_\eta^1\frac{n(\xi')d\xi'}{\sqrt{\xi'-\eta}}. \tag{10'}$$

The condition for separation of the variables is m = 1. Thus,

$$\vartheta_0=(2\tau_0/\tau)^{1/2},\quad N(\vartheta',t)=\frac{\tau}{\tau_0}n(\xi), \tag{13}$$

and the function $n(\xi)$ is a solution of the equation

$$1+\xi\frac{dn/d\xi}{n(\xi)}=\tau_0\int_0^\xi\frac{d\eta}{\sqrt{\eta(\xi-\eta)}}\int_\eta^1\frac{(\xi'-\xi)n(\xi')d\xi}{\sqrt{\xi'-\eta}}\Big/\int_\eta^1\frac{n(\xi')d\xi'}{\sqrt{\xi'-\eta}}. \tag{14}$$

Using expression (12), we normalize the solution of Eq. (14) in the following manner:

$$n(\xi)=\frac{W}{2\pi k_0^2\omega}F(\xi),\quad \int_0^1 F(\xi)\,d\xi=1. \tag{15}$$

In this case $F(1) \ll 1$, since the entire packet of the waves is concentrated in a cone with limiting angle ϑ_0.

Let us refine this inequality. Assume that in the plasma there is a certain background of oscillations with an energy density $W_f \ll W$. We introduce the numerical parameter $\Lambda = \ln(W/W_f)$, which we assume to be a large quantity. The boundary condition can be written in the form

$$F(1)\approx e^{-\Lambda}. \tag{16}$$

Bearing in mind that $F(\xi)$ decreases rapidly as $\xi \to 1$, we seek the solution of (14) in the form of the product

$$F(\xi)=F_1(\xi)e^{-\alpha\xi},\quad \alpha\gg 1. \tag{15'}$$

As the first approximation we take $F_1(\xi)$ = const. Substituting (15) in (14), we obtain

$$1+\xi\left(\frac{\pi\tau_0}{2}-\alpha\right)=\frac{\tau_0}{\alpha}\int_0^\xi\frac{d\eta}{\sqrt{\eta(\xi-\eta)}}\times\left(\frac{1}{2}\int_\eta^1\frac{e^{-\alpha\xi'}d\xi'}{\sqrt{\xi'-\eta}}-e^{-\alpha\xi'}\sqrt{\xi'-\eta}\Big|_\eta^1\right)\Big/\int_\eta\frac{e^{-\alpha\xi'}d\xi'}{\sqrt{\xi'-\eta}}. \tag{17}$$

The ratio of the second term to the first in the numerator of the integrand is a small quantity of order α^{-1}, provided $\alpha(1-\xi) \gg 1$. If on the other hand $\xi \sim 1$, so that at the upper limit of integration with respect to η the condition of smallness of the second term is not satisfied, then, breaking up the integral with respect to η into two, we can verify that the integral for which the foregoing inequality is satisfied is a large quantity of order α relative to the second integral. As a result (17) reduces, accurate to terms $\sim\alpha^{-1}$, to the following equation:

$$\left(1-\frac{\pi\tau_0}{2\alpha}\right)+\xi\left(\frac{\pi\tau_0}{2}-\alpha\right)=0,$$

which is identically satisfied when $\alpha = \pi\tau_0/2$.

At the same accuracy, we determine the constant $F_1 \approx \pi\tau_0/2$ from the normalization condition (15). Finally, from the boundary condition (16) we obtain $\pi\tau_0/2 = \Lambda$. Finally, accurate to terms of order Λ^{-1}, the self-similar solution (13) takes the form

$$\vartheta_0=2(\Lambda/\pi\gamma_0 t)^{1/2},\quad N(\vartheta',t)=\frac{W}{4k_0^2\omega}\gamma_0 t\exp\left(-\frac{\pi\gamma_0 t}{4}\vartheta'^2\right). \tag{18}$$

The solution (18) was obtained under the assumption that in the problem there is only one characteristic time

τ_0, which is valid only if the derivative $\partial f/\partial t$ is small (condition (6)). In the first approximation in ω/kv_{Te} there follows from (6) the form of the distribution function (7), and vice-versa. However, in order for such a distribution to be established, the momentum and the energy of the waves in the initial state must exceed the momentum and energy that must be transferred to the particles in order to produce a distribution of the form (7):

$$\left|\int \frac{W_{\mathbf{k}}}{\omega}\mathbf{k}\,d\mathbf{k}\right| \gg nmv_{Te}\left(\frac{\omega}{kv_{Te}}\right).$$

For a strongly anisotropic spectrum this yields the following limitation on the energy density of the noise:

$$W/nT \gg m/M. \qquad (19)$$

A more accurate condition for the applicability of our formulas can be obtained formally from the inequality (6). We substitute the solution (7) in the left side of (6):

$$\frac{\partial f_1}{\partial t} \sim -\frac{\partial f_0}{\partial v}\frac{\partial}{\partial t}\frac{B}{A} \ll \frac{\partial f_0}{\partial v}\frac{1}{v^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}(B\sin\vartheta). \qquad (6')$$

We use the concrete form of the coefficient B from the formula (5), in which we can substitute the noise density from (1) and (18). The ratio B/A for a narrow spectrum is simply ω/k. As the characteristic scale of the derivative $\partial/\partial t$ we choose the linear increment γ_0 from (8), which can only intensify the limitations on the noise energy density. This enables us to replace the inequality (6) by

$$W/nT \gg \vartheta_0 m/M, \qquad (20)$$

which determines the limits of applicability of the self-similar solution (18).

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