# *THE GINZBURG-LANDAU EQUATION FOR PAIRING WITH NONZERO ANGULAR MOMENTUM*

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An analog of the Ginzburg-Landau equation is obtained for a superconductor where Cooper pairing occurs in pair states with nonzero angular momentum $l$. The equation has the form of a radial Schrödinger equation for the $l$th partial wave with a cubic nonlinearity. Its solution is a "spherical vortex" which is a completely isotropic but spatially inhomogeneous state in which Cooper pairs as a whole rotate about the center of the vortex, where the order parameter $\Delta$ vanishes. It is shown that the vortex state has a minimal free energy as compared with the solutions proposed earlier.

THE problem of superconductivity in systems where Fermi particles attract each other in pairing states with nonzero angular momentum $l$ has been considered in a number of papers (for example,[1-3]). Different schemes for an approximate decoupling of the equations of motion are used, and lead to different solutions. The "isotropic" solution (cf.[1]) corresponds to the presence of $2l + 1$ "condensates" of pairs with different angular momentum projection m. At present, the solution regarded as best is the "anisotropic" solution (cf.[2]), where the wave function of the Cooper pairs depends on the direction **n** of the relative momentum of the pair in the form of a certain linear combination of spherical harmonics, $\psi^{(l)}(\mathbf{n}) = \sum_m c_m Y_{lm}(\mathbf{n})$. For temperature T = 0 it has been shown[4] that the solution is stable[2] against a certain class of perturbations.

In all papers mentioned above the equilibrium state of the system has been assumed homogeneous in space (one or several condensates of Cooper pairs which are at rest as a whole). If we extend the class of admissible states and also consider pairs with total momentum $\mathbf{k} \neq 0$ (which is equivalent to a dependence of the order parameter $\Delta$ on the coordinates), then the pair acquires a natural distinguished direction, the degeneracy in the angular momentum projections is lifted,[1)] and the state as a whole may be made completely isotropic since all directions **k** are equally probable. It will be shown below that in this case the gain in interaction energy owing to a fuller use of the angular dependence of the attractive potential is more important than the loss on account of the spatial inhomogeneity of the state thus obtained.[2)]

The solution of the exact Gor'kov equations in the inhomogeneous case presents considerable difficulties. We restrict ourselves to a region of temperatures close to the critical value $T_c$, where we obtain the analog of the Ginzburg-Landau (GL) equation for pairing with angular momentum $l$.[3)]

The general method for deriving the GL equations from the microscopic equations for superconductors with arbitrary interaction is given in the paper of Gor'kov and Melik-Barkhudarova.[6] As shown in[6], the equation for the pairing parameter $\Delta_{\mathbf{k}}(\mathbf{n})$ [$\mathbf{k} = \mathbf{p}_1 + \mathbf{p}_2$ is the total momentum of the pair, $\mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2 = |\mathbf{p}|\mathbf{n}$ is the relative momentum, $p \sim p_0$, $k \ll p_0$] for $T \lesssim T_c$ and in the absence of external fields, has the form

$$\Delta_{\mathbf{k}}(\mathbf{n}) = mp_0\left(\Lambda + \frac{g}{(2\pi)^3}\ln\frac{T_c}{T}\right)\int d\mathbf{n}' U(\mathbf{n}',\mathbf{n})\Delta_{\mathbf{k}}(\mathbf{n}') - \frac{\lambda g p_0^3}{2(2\pi)^3 m}\int d\mathbf{n}' U(\mathbf{n}',\mathbf{n})(\mathbf{k}\mathbf{n}')^2\Delta_{\mathbf{k}}(\mathbf{n}') - \frac{\lambda g p_0 m}{(2\pi)^3\Omega^2}\sum_{\mathbf{k}_1\mathbf{k}_2}\int d\mathbf{n}' U(\mathbf{n}',\mathbf{n})\Delta_{\mathbf{k}_1}(\mathbf{n}')\Delta_{\mathbf{k}_2}(\mathbf{n}')\Delta^*_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(\mathbf{n}'). \quad (1)$$

As usual, we neglect the weak dependence of the pairing parameter $\Delta$ and the interaction potential U on the value of the momenta near the Fermi boundary. The notation in (1) agrees with that of[6]; the Fermi surface is considered spherical.

If we are interested in the case of attraction in a state with angular momentum $l$,

$$U(\mathbf{n},\mathbf{n}') = U_l P_l(\mathbf{n}\mathbf{n}'). \quad (2)$$

For simplicity we restrict ourselves to the weak coupling approximation[7] and find then for the eigenvalue, in the zero-order approximation in g, $\Lambda = (2l+1)/4\pi p_0 m U_l$; here the critical temperature agrees with that found in[1,2]. The condition for the solvability of the first approximation yields the equation

$$\ln\left(\frac{T_c}{T}\right)\int d\mathbf{n}[\Delta_{\mathbf{k}}^{(0)}(\mathbf{n})]^2 - \frac{\lambda p_0^2}{2m^2}\int d\mathbf{n}[\Delta_{\mathbf{k}}^{(0)}(\mathbf{n})]^2(\mathbf{k}\mathbf{n})^2 - \frac{\lambda}{\Omega^2}\sum_{\mathbf{k}_1\mathbf{k}_2}\int d\mathbf{n}\Delta_{\mathbf{k}}^{(0)}(\mathbf{n})\Delta_{\mathbf{k}_1}^{(0)}(\mathbf{n})\Delta_{\mathbf{k}_2}^{(0)}(\mathbf{n})\Delta^{*(0)}_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}}(\mathbf{n}) = 0. \quad (3)$$

It is seen from (1) that the choice of the interaction in the form (2) leads to a zero-order solution $\Delta_{\mathbf{k}}^{(0)}(\mathbf{n})$ in the form of an arbitrary superposition of spherical

[1)]This is easily seen by investigating the singularities of the vertex part leading to a phase transition, for a normal (non-superconducting) state with $k \neq 0$ for the considered interaction in the $l$ th harmonic.

[2)]Another example of a physical situation where the inhomogeneous state is advantageous is given in [5].

[3)]We consider only singlet spin states of the pair; then the angular momentum $l$ must be even.

harmonics $Y_{LM}(\mathbf{n})$ with fixed $L = l$ (degeneracy). We shall seek a completely isotropic state of the system in which $\Delta_{\mathbf{k}}^{(0)}(\mathbf{n})$ can only depend on the angle between the vectors $\mathbf{k}$ and $\mathbf{n}$. Therefore, the "correct linear combination" is

$$\Delta_{\mathbf{k}}^{(0)}(\mathbf{n}) = \Delta_{\mathbf{k}} P_l\left(\frac{\mathbf{k}}{k}\mathbf{n}\right). \tag{4}$$

We now substitute (4) in (3) and integrate. [To this end it is convenient to introduce in the last term of (3) an additional integration over $\mathbf{k}_3$ with $\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k} - \mathbf{k}_3)$, to use the integral representation of the $\delta$ function, and to expand the exponential in spherical waves.] After simple but tedious algebra we find

$$-\frac{a^2}{2\pi^2}\Delta_k = \frac{1}{k^2 - \varkappa^2}\int_0^\infty dR\cdot R^2\Delta(R)\,|\Delta(R)|^2 j_l(kR). \tag{5}$$

Here $\Delta(R)$ is the radial Fourier transform of the pairing parameter taken in the "mixed" $(\mathbf{R}, \mathbf{p})$ representation,

$$\Delta_{\mathbf{k}}(\mathbf{p}) = \int d\mathbf{R}\, e^{-i\mathbf{k}\mathbf{R}}\Delta(\mathbf{R}, \mathbf{p}), \tag{6}$$

$$\Delta(\mathbf{R}, \mathbf{p}) = i^l\Delta(R) P_l\left(\frac{\mathbf{pR}}{pR}\right), \quad \Delta(R) = \frac{1}{2\pi^2}\int_0^\infty dk\cdot k^2 j_l(kR)\Delta_k; \tag{7}$$

we have introduced the notation

$$a^2 = \frac{\pi}{12}\frac{p_0^2}{m^2}\frac{\beta_l}{(2l+1)f_l}, \quad \beta_l = 1 + \frac{2l(l+1)}{(2l-1)(2l+3)}, \quad \varkappa^2 = \frac{6\ln(T_c/T)}{\lambda\beta_l p_0^2/m^2},$$

$$f_l = \sum_L (2L+1)\begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^4, \tag{8}$$

where $\begin{pmatrix} l\,l\,L \\ 000 \end{pmatrix}$ is the Wigner 3j symbol.

Multiplying both parts of (5) by $k^2 j_l(kR)$, integrating over k and using the definition of $\Delta(R)$ in (7), we obtain

$$-\frac{2a^2}{\pi}\Delta(R) = \frac{K_\nu(-i\varkappa R)}{\sqrt{R}}\int_0^R d\rho\cdot\rho^{3/2} I_\nu(-i\varkappa\rho)\,|\Delta(\rho)|^2\Delta(\rho)$$

$$+ \frac{I_\nu(-i\varkappa R)}{\sqrt{R}}\int_R^\infty d\rho\cdot\rho^{3/2} K_\nu(-i\varkappa\rho)\,|\Delta(\rho)|^2\Delta(\rho), \quad \nu = l + \frac{1}{2} \tag{9}$$

In the dimensionless variables

$$z = \varkappa R, \quad \psi(z) = \left[\frac{\lambda}{\ln(T_c/T)}(2l+1)f_l\right]^{1/2}\sqrt{z}\,\Delta\left(\frac{z}{\varkappa}\right), \tag{10}$$

equation (9) takes the form

$$\frac{2i}{\pi}\psi(z) = H_\nu^{(1)}(z)\int_0^z J_\nu(t)\psi(t)\,|\psi(t)|^2 dt + J_\nu(z)\int_z^\infty H_\nu^{(1)}(t)\psi(t)\,|\psi(t)|^2 dt. \tag{11}$$

From (11) we easily obtain the equivalent differential equation

$$Z_\nu'' + \frac{1}{z}Z_\nu' + \left\{\frac{|\psi|^2}{z^2} - \frac{1}{z}\frac{\psi'}{\psi} - \frac{\psi''}{\psi}\right\}Z_\nu = 0, \tag{12}$$

where $Z_\nu(z)$ is the cylindrical function of order $\nu$. Comparing (12) with the equation for the cylinder functions,

$$Z_\nu'' + \frac{1}{z}Z_\nu' + \left(1 - \frac{\nu^2}{z^2}\right)Z_\nu = 0$$

and introducing $\Phi = \psi/\sqrt{z}$, we find

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{d\Phi}{dz}\right) + \left(1 - \frac{l(l+1)}{z^2}\right)\Phi - \Phi|\Phi|^2 = 0. \tag{13}$$

Equation (13) is the desired GL equation which determines the coordinate dependence of the "pair wave function." For $l = 0$, eq. (13) goes over into the usual GL equation[6] for the case of spherical symmetry [the identity of the coefficients is easily verified with the help of (10) and (7)]. By its form, eq. (13) represents a radial Schrödinger equation for the $l$th partial wave with an additional nonlinear term. We emphasize that the fact that such an equation exists is nontrivial, since owing to the nonlinearity the form of the GL equation is in this case determined by the angular dependence separated earlier [cf. (4)], so that it is not possible to write down a single three-dimensional GL equation.

The equilibrium solution of the usual GL equation ($l = 0$) corresponds to a spatially homogeneous state ($\Phi = 1$). For $l \neq 0$ the pair wave function must vanish at the origin owing to the centrifugal barrier, $\Phi(z) \sim z^l$. Thus the full isotropy of the state is obtained at the price of giving up spatial homogeneity. The choice of the point $R = 0$ is arbitrary in the same sense as the choice of the direction of the anisotropy in the solution of Anderson and Morel.[2] Far from the origin, the solution of (13) tends to a constant

$$\Phi(z) \approx 1 - l(l+1)/2z^2, \tag{14}$$

so that the state represents a localized "spherical vortex." The characteristic dimension of the vortex is $R_0 \sim l/\kappa \to \infty$ for $T \to T_c$. As is seen from the derivation, total pair momenta $k \sim \kappa$ are important, i.e., angular momenta of the pairs relative to the common center of the order $L \sim kR_0 \sim l$. In other words, the dimensions of the vortex are determined by the condition that at the boundary the "magnetic field" from the rotation of the pair as a whole breaks the intrinsic internal binding. Owing to the spherical symmetry the vortex state carries no current.

The thermodynamics of the system can be constructed in analogy with the usual case. The standard calculations[8] give for the difference of the free energies of the superconducting and normal states

$$\mathcal{F}_s - \mathcal{F}_n = -\Omega\frac{mp_0\lambda}{4\pi^2}\overline{|\Delta(R)|^4}, \tag{15}$$

where the bar denotes the average over the volume of the system $\Omega$. It is easy to see that the correction terms to the asymptotic form (14) give a contribution $\Omega^{-2/3}$ after averaging. Therefore all macroscopic properties are determined by the asymptotic value:

$$\Delta^{(l)} \equiv \Delta(\infty) = \Gamma_l\left(\frac{\ln(T_c/T)}{\lambda}\right)^{1/2}, \quad \Gamma_l = [(2l+1)f_l]^{-1/2}. \tag{16}$$

The results of the Gor'kov-Galitskiĭ (GG) and Anderson-Morel (AM) approaches can also be expressed in the form of (15) and (16), where in the isotropic solution $\Gamma_l^{(GG)} = 1$, and for the anisotropic solutions $\Gamma_l = [4\pi\int d\mathbf{n}|\psi^{(l)}(\mathbf{n})|^4]^{-1/2}$, where $\psi^{(l)}(\mathbf{n})$ is the normalized angular part of the pair wave function.[7] In particular, for quadrupole pairing ($l = 2$) we have[2]

$$\psi_{AM}^{(2)}(\mathbf{n}) = \frac{1}{\sqrt{2}}Y_{20}(\mathbf{n}) + \frac{1}{2}(Y_{22}(\mathbf{n}) - Y_{2,-2}(\mathbf{n})), \quad \Gamma_2^{AM} = \sqrt{\frac{7}{10}}.$$

In the vortex solution considered $\Gamma_0 = 1$, $\Gamma_{l>0} > 1$ (e.g., $\Gamma_2 = \sqrt{7/3}$). Thus, we here obtain a lower free energy than with the solutions known so far. This is natural, since the advantageous angular dependence is effective in all space, while the smallness of $\Delta$ is concentrated in the finite region of the centrifugal barrier.

It is easy to see that for temperatures above the critical temperature, the GL equation has the form

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{d\Phi}{dz}\right)-\left(1+\frac{l(l+1)}{z^2}\right)\Phi-\Phi|\Phi|^2=0. \quad (17)$$

The difference between (13) and (17) corresponds to the transition from the continuous to the discrete spectrum in the Schrödinger equation. Indeed, instead of the asymptotic form (14), the solution of (17) tends to zero at large distances. Therefore the nuclei of the superconducting phase for $T > T_c$ are spherical drops whose density decreases smoothly ($\sim z^l$) towards the center of the sphere, and sharply outwards.

The problems of the stability of the vortex state, of the behavior of such a system in an external field, and of its collective excitations require a separate investigation.

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