SECOND SOUND AND HYDRODYNAMICAL THERMAL CONDUCTIVITY IN ANTIFERROMAGNETICS

I. N. OLEĬNIK

Institute for Radiophysics and Electronics, Ukrainian Academy of Sciences

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We consider second sound and thermal conductivity in uniaxial antiferromagnetics for which the Néel temperature is lower than the Debye temperature: $\Theta_N^3 \ll \Theta_D^3$. We show that the second sound velocity decreases with increasing temperature, since in the temperature range $T \ll \epsilon_0$ (ϵ_0 is the magnon activation energy) the second sound is carried by the phonons, but when $T \gg \epsilon_0$ by the magnons. We also show that although under the conditions considered the heat transfer is mainly realized by magnons, the magnon-phonon gas viscosity may be determined by the phonons.

TRANSFER processes in magnetically ordered systems have a number of peculiar features when the normal collisions between quasi-particles are more probable than processes in which the total quasi-momentum of the system is not conserved, i.e., when a hydrodynamical situation is possible. Gurzhi[1] (see also[2]) has considered these problems theoretically for ferrodielectrics. Recently Tsarev[3] has apparently observed experimentally the occurrence of the hydrodynamic situation in the ferromagnetic compound $CrBr_3$. The level of development of experimental techniques nowadays apparently makes it possible to observe second sound and hydrodynamical thermal conductivity in magnetically ordered systems.

In our consideration of second sound and hydrodynamic thermal conductivity we shall in which follows restrict ourselves to uni-axial antiferromagnetics with two sublattices. In such antiferromagnetics there will exist at low temperatures two kinds of quasi-particles: the acoustic phonons and the magnons with dispersion laws[4]

$$\Omega = qs, \quad \varepsilon_\alpha = \sqrt{\varepsilon_0^2 + (\Theta_N ap/\hbar)^2} \pm \mu H_0.$$

Here Ω and q are the phonon energy and quasi-momentum, $s = \Theta_D a/\hbar$ the sound velocity, a the lattice constant, Θ_D the Debye temperature, ϵ_α, p the magnon energy and quasi-momentum, $\alpha = 1, 2$ the index which numbers the magnon energy branches, $\epsilon_0 \approx \sqrt{(\mu M_0 \Theta_N)}$ the magnon activation energy, Θ_N the Néel temperature, μ the Bohr magneton, M_0 the equilibrium sublattice magnetization, and H_0 a constant magnetic field parallel to the easy axis.

We shall in what follows assume that $\Theta_N^3 \ll \Theta_D^3$. There is no special interest in the opposite case $\Theta_D^3 \ll \Theta_N^3$, since we shall show below that in that case second sound and hydrodynamical thermal conductivity are connected with phonons in the whole of the admissible temperature range.

In the temperature range $\epsilon_0 \ll T \ll \Theta_N$ and for fields $\mu H_0 \ll \epsilon_0$ the normal collisions in the spin subsystem are determined by the quaternary exchange interactions between magnons, which occur with a probability[5]

$$W_{ss} = \tau_{ss}^{-1} \sim \frac{\Theta_N}{\hbar}\left(\frac{T}{\Theta_N}\right)^5.$$

In the phonon subsystem the normal collisions are connected with ternary collisions between phonons. A theoretical estimate leads to the following expression for the probability (see, e.g.,[1]):

$$W_{pp}^{\text{theor}} = \tau_{pp}^{-1} \sim \frac{\Theta_D}{\hbar}\frac{\Theta_D}{Ms^2}\left(\frac{T}{\Theta_D}\right)^5,$$

where M is the mass of an atom. However, a T^4-behavior appears to agree better with experiment[6], i.e.,

$$W_{pp}^{\exp} \sim \frac{\Theta_D}{\hbar}\frac{\Theta_D}{Ms^2}\left(\frac{T}{\Theta_D}\right)^4.$$

The phonon-magnon interaction may take place both through the emission or absorption of a phonon by a magnon, or through the decay of a phonon into two magnons, or the reverse process, viz., the combination of two magnons into one phonon. Both processes are connected with the exchange interaction and in the case $\Theta_N \sim \Theta_D$ they have probabilities of the same order of magnitude. Thus, for magnons[7]

$$W_{sp} = \tau_{sp}^{-1} \approx \frac{\Theta_N}{\hbar}\frac{\Theta_D}{Ms^2}\left(\frac{\Theta_N}{\Theta_D}\right)^3\left(\frac{T}{\Theta_N}\right)^5$$

and for phonons[7]

$$W_{ps} = \tau_{ps}^{-1} = W_{sp}(\Theta_D/\Theta_N)^3, \quad T \gg \varepsilon_0.$$

One can easily show, using the energy and momentum conservation laws, that when $\Theta_N < \Theta_D$ only the decay process of one phonon into two magnons and the reverse process are allowed, while for $\Theta_N > \Theta_D$ only the absorption (or emission) of a phonon by a magnon is allowed. In the temperature region $T \ll \epsilon_0$ the probabilities W_{ss} and W_{ps} contain a small factor $e^{-\epsilon_0/T}$, since in that case the number of magnons is exponentially small.

1. SECOND SOUND

Temperature waves or second sound, as they are called, are oscillations in the elementary excitation density, i.e., ordinary sound in a gas of quasiparticles. The conditions for the occurrence of second sound consist in the requirement that the normal collisions between quasi-particles be more probable than processes in which quasi-momentum is not conserved (Umklapp processes, impurity scattering, and so on). The second sound frequency must thus satisfy the inequalities

$$\tau_U^{-1} \ll \omega \ll \tau_N^{-1}, \tag{1.1}$$

where τ_N and τ_U are the relaxation times for normal collisions and processes which do not conserve quasi-momentum.

At low frequencies $\omega\tau_{sp} \ll 1$, which means $\omega \sim 10^5$ to $10^6\ \mathrm{sec}^{-1}$ at $T \sim 10°K$, $\Theta_N \sim \Theta_D \sim 10^{2}°K$, and because of their strong mutual drag the phonons and magnons must be considered to be a single system. The hydrodynamical equations describing the propagation of second sound in antiferromagnetics can in this case be obtained by solving, by successive approximations, a set of kinetic equations for the phonons and magnons. The zeroth approximation leads to the drift solutions for the phonon and magnon distribution functions:

$$N^0 = \left[\exp\left(\frac{\Omega - \mathbf{qu}}{T(1+\vartheta)}\right) - 1\right]^{-1}, \quad f_\alpha{}^0 = \left[\exp\left(\frac{\varepsilon_\alpha - \mathbf{pu}}{T(1+\vartheta)}\right) - 1\right]^{-1}, \tag{1.2}$$

where $\mathbf{u}(\mathbf{r}, t)$ is the drift velocity and $\vartheta(\mathbf{r}, t)$ the relative change in the temperature. The conditions that the set of kinetic equations have a solution in first approximation (which appear as a consequence of the total energy and momentum conservation in normal collisions) lead to a set of hydrodynamic equations:

$$\begin{aligned} &\dot{\vartheta}(\langle\Omega^2\rangle + \langle\varepsilon^2\rangle) + {}^1/_3 \operatorname{div} \mathbf{u}(\langle \mathbf{q}s\Omega\rangle + \langle \mathbf{p}v\varepsilon\rangle) = 0, \\ &a_{ik}\dot{u}_k + \frac{1}{3}\frac{\partial\vartheta}{\partial x_i}(\langle \mathbf{q}s\Omega\rangle + \langle \mathbf{p}v\varepsilon\rangle) = 0. \end{aligned} \tag{1.3}$$

We have used here the notation

$$\langle\psi(\mathbf{q})\rangle = -\frac{3}{h^3}\int d\mathbf{q}\,\psi(\mathbf{q})\frac{\partial N_0}{\partial\Omega}, \quad \langle\varphi(\mathbf{p})\rangle = -\sum_\alpha \frac{1}{h^3}\int d\mathbf{p}\,\varphi_\alpha(\mathbf{p})\frac{\partial f_0}{\partial\varepsilon_\alpha},$$

$$a_{ik} = \langle q_i q_k\rangle + \langle p_i p_k\rangle, \quad N_0 = \left[\exp\frac{\Omega}{T} - 1\right]^{-1}, \quad f_0 = \left[\exp\frac{\varepsilon_\alpha}{T} - 1\right]^{-1},$$

N_0 and f_0 are the equilibrium Bose distribution functions for the phonons and magnons.

Putting ϑ, $\mathbf{u} \sim \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$ we get for the second sound velocity $V = \omega/k$

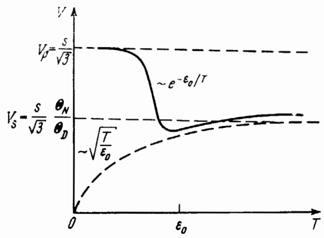
$$V^2 = \frac{(\hat{a}^{-1})_{\varkappa\varkappa}(\langle \mathbf{q}s\Omega\rangle + \langle \mathbf{p}v\varepsilon\rangle)^2}{9(\langle\Omega^2\rangle + \langle\varepsilon^2\rangle)}, \tag{1.4}$$

where $(\hat{\alpha}^{-1})_{\kappa\kappa} = (\hat{\alpha}^{-1})_{ik}\kappa_i\kappa_k$, $\kappa = \mathbf{k}/k$ is a unit vector along the direction of sound propagation. If $\Theta_N^3 \ll \Theta_D^3$ we easily get from (1.4) an asymptotic expression for the magnon-phonon second sound velocity for the high $(T \gg \epsilon_0)$ and low $(T \ll \epsilon_0)$ temperature cases:

$$V = \begin{cases} V_s\left(1 + \left(\dfrac{\Theta_N}{\Theta_D}\right)^3 - \left(\dfrac{\varepsilon_0}{T}\right)^2\right), & T \gg \varepsilon_0 \\ V_p\left(1 - \left(\dfrac{\Theta_N}{\Theta_D}\right)^3\left(\dfrac{\varepsilon_0}{T}\right)^{7/2} e^{-\varepsilon_0/T}\right), & T \ll \varepsilon_0 \end{cases}. \tag{1.5}$$

We have used here the notation $V_p = s/\sqrt{3}$, $V_s = \Theta_N a/\hbar\sqrt{3}$.

It follows from (1.5) that when $T \ll \epsilon_0$ second sound is mainly carried by phonons, since the magnetic energy branches are "frozen in" and the number of magnons is exponentially small. If $T \gg \epsilon_0$, however, second sound will be predominantly magnon second sound, since the magnon specific heat for $\Theta_N^3 \ll \Theta_D^3$ will dominate the phonon specific heat. The coefficient $\sqrt{3}$ occurring in the expression for V_s may be important to distinguish second sound from a magnon beam. The transition from phonon to magnon second sound in antiferromagnetics leads to a decrease in the propagation velocity of temperature waves with increasing temperature. It is interesting that the V(T) curve has a minimum at some intermediate temperature (see solid curve in the figure).



Second sound velocity V as function of the temperature T. The solid curve refers to magnon-phonon sound, the upper dotted curve to purely phonon sound, and the lower dotted curve to purely magnon sound (s—is the first sound velocity, Θ_D and Θ_N the Debye and Néel temperatures, and ϵ_0 the magnon activation energy).

We must note that the function V(T) in antiferromagnetics differs appreciably from the analogous function in ferromagnetics, where the situation turns out to be different: in the temperature range $T \ll \Theta_D^2/\Theta_C$ (Θ_C is the Curie temperature) $V \approx a\sqrt{(\Theta_C T)}/\hbar$ and second sound is connected with the magnons, while for $T \gg \Theta_D^2/\Theta_C$ it is connected with the phonons and $V = s/\sqrt{3}$.[8] One can show that the variable magnetic field $\mathbf{h}$ caused by the oscillations in the magnon density is small:

$$\frac{\mu h}{\vartheta T} \sim \frac{\mu M_0}{T}\left(\frac{T}{\Theta_N}\right)^3 \ll 1,$$

and taking it into account does not lead to a significant change in the second sound dispersion law (1.4). The second sound wave in antiferromagnetics, as in ferromagnetics[1], will thus be predominantly a temperature wave. One can easily obtain the estimate for $\mathbf{h}$ given here by using the magnetostatics equations

$$\operatorname{div}\left(\mathbf{h} + 4\pi\sum_\alpha \mathbf{m}_\alpha\right) = 0, \quad \operatorname{rot}\mathbf{h} = 0$$

and the expression for the density of the non-equilibrium magnetic moment

$$m_\alpha \approx \frac{\mu}{h^3}\int d\mathbf{p}\,(f_\alpha{}^0 - f_{0\alpha}).$$

The field $\mathbf{h}$ is, however, large enough to be experimentally observable and for $T \sim 10°K$, $\Theta_N \sim 10^{2}°K$, $\vartheta \sim 0.1$ we have $h \sim 0.1$ Oe.

We note that the result given in the figure for the temperature dependence of the second sound velocity is qualitatively independent of the expression for the relaxation times given above, and is connected only with two facts: the satisfaction of the inequality $\Theta_N^3 \ll \Theta_D^3$ and the presence of a finite magnon activation energy ϵ_0. If the stronger inequality $\Theta_N \ll \Theta_D$ is satisfied, the propagation of low-frequency second sound propagation becomes impossible, since the probability $W_{sp} = W_{ps}(\Theta_N/\Theta_D)^3$ is small.

In the high frequency case the magnons and phonons must be considered to be two different subsystems, in each of which second sound can propagate if the inequality $\tau_{ps}^{-1} \ll \omega \ll \tau_{pp}^{-1}$ is satisfied with a sufficient margin (from the above made estimates for the relaxation times it is clear that this situation is in principle possible). The second sound velocities in a system of magnons and phonons have the usual form of velocities in systems with conserved and non-conserved numbers of particles:[1]

$$V_s^2 = \frac{(\hat{\alpha}_s^{-1})_{\varkappa\varkappa}}{9} \frac{2\langle \mathbf{p}\mathbf{v}\varepsilon\rangle\langle \mathbf{p}\mathbf{v}\rangle\langle\varepsilon\rangle - \langle \mathbf{p}\mathbf{v}\varepsilon\rangle^2\langle 1\rangle - \langle\mathbf{p}\mathbf{v}\rangle^2\langle\varepsilon^2\rangle}{\langle\varepsilon\rangle^2 - \langle\varepsilon^2\rangle\langle 1\rangle},$$

$$V_p^2 = \frac{(\hat{\alpha}_p^{-1})_{\varkappa\varkappa}}{9} \frac{\langle \mathbf{q}s\Omega\rangle^2}{\langle\Omega^2\rangle}, \qquad (1.6)$$

where $(\hat{\alpha}_s)_{ik} = \langle p_i p_k\rangle$, $(\hat{\alpha}_p)_{ik} = \langle q_i q_k\rangle$. It follows from (1.6) that $V_p = s/\sqrt{3}$ in the entire permissible temperature range (the upper dotted curve in the figure). At the same time the velocity of purely magnon sound depends on the temperature: when $T \ll \epsilon_0$ we have, in order of magnitude, $V_s \approx \Theta_N a\sqrt{(T/\epsilon_0)}/\hbar$, if, however, $T \gg \epsilon_0$, we have $V_s = \Theta_N a/\hbar\sqrt{3}$ (lower dotted curve in figure). In both limiting cases the second sound velocity is of the same order of magnitude as the average thermal magnon velocity.

To observe the change in the temperature dependence of the second sound velocity in antiferromagnetics experimentally one could, apparently use some fluorides of transition metals (e.g., MnF_2: $\epsilon_0 \sim 12^\circ$K, $\Theta_N \sim 72^\circ$K, $\Theta_D \sim 240^\circ$K).

2. HYDRODYNAMIC THERMAL CONDUCTIVITY

In antiferromagnetic dielectrics with a sufficiently low Néel temperature, the heat transfer is predominantly through magnons when $\Theta_N \ll \Theta_D$. The hydrodynamic magnon thermal conductivity mechanism leads to a well-known temperature dependence of the thermal conductivity coefficient: $\kappa \sim T^6$.[9] However, when the much weaker inequality $\Theta_N^3 \ll \Theta_D^3$ is satisfied, allowance for the phonons and their interaction with the magnons may essentially affect the thermal conductivity, because the viscosity of the magnon-phonon gas in that case may be determined by the phonons, while the specific heat as before is determined by the magnons. (A similar situation is observed in metals.[10])

The hydrodynamic equations describing the propagation of a heat flux in antiferromagnetics must be found again, since Eqs. (1.3) of the preceding section do not take into account the viscosity of the magnon-phonon gas. To do this it is necessary to substitute the solution of a set of kinetic equations which is obtained up to and including first order terms, $N = N^{(0)}, + N^{(1)}$, $f = f^{(0)} + f^{(1)}$ into the equation

$$\operatorname{div}\{\langle\!\langle p_i \mathbf{v} f\rangle\!\rangle + \langle\!\langle q_i \mathbf{s} N\rangle\!\rangle\} = \langle\!\langle p_i \hat{J}_s{}^V(f)\rangle\!\rangle + \langle\!\langle q_i \hat{J}_p{}^V(N)\rangle\!\rangle, \qquad (2.1)$$
$$\operatorname{div}\{\langle\!\langle \varepsilon\mathbf{v} f\rangle\!\rangle + \langle\!\langle \Omega\mathbf{s} N\rangle\!\rangle\} = 0,$$

where

$$\langle\!\langle \Phi(\mathbf{p})\rangle\!\rangle = \frac{2}{h^3}\int d\mathbf{p}\,\Phi(\mathbf{p}), \quad \langle\!\langle \Psi(\mathbf{q})\rangle\!\rangle = \frac{3}{h^3}\int d\mathbf{q}\,\Psi(\mathbf{q}),$$

$\hat{J}_s^V$ and $\hat{J}_p^V$ are the magnon and phonon collision integrals for collisions which are accompanied by the loss of quasi-momentum. Equations (2.1) express the quasi-momentum conservation law for normal collisions and the energy conservation law for all collisions in the stationary case.

It is well known that under conditions when there is a hydrodynamic flow in a gas of quasi-particles the results are not very sensitive to the shape of the perpendicular cross-section of the specimen.[1] We shall therefore restrict ourselves in what follows to the simplest case of a plate with thickness d which is small compared to its length and width. We choose the x-axis in the direction of the constant temperature gradient and the z-axis are right-angles to the surface of the plate. To solve the set of kinetic equations approximately we use the Chapman-Enskog method.[11] Bearing in mind that $\vartheta = \vartheta(x)$, $\partial\vartheta/\partial x$ = constant, $\mathbf{u} = (u(z), 0, 0)$ we get in first approximation

$$-\frac{\partial f_0}{\partial\varepsilon} p_x v_z \frac{\partial u}{\partial z} = \hat{J}_{ss}^N(f^{(1)}) + \hat{J}_{sp}^N(f^{(1)};\ N^{(1)}),$$
$$-\frac{\partial N_0}{\partial\Omega} q_x s_z \frac{\partial u}{\partial z} = \hat{J}_{pp}^N(N^{(1)}) + \hat{J}_{ps}^N(N^{(1)};\ f^{(1)}). \qquad (2.2)$$

Here $\hat{J}_{ss}^N$, $\hat{J}_{sp}^N$, $\hat{J}_{pp}^N$, and $\hat{J}_{ps}^N$ denote the linearized integrals for normal collisions between the quasi-particles. We have dropped in (2.2) $\hat{J}_s^V$, $\hat{J}_p^V$ and terms such as

$$\frac{\partial f_0}{\partial\varepsilon}\varepsilon v_x \frac{\partial\vartheta}{\partial x}, \quad \frac{\partial N_0}{\partial\Omega}\Omega s_x \frac{\partial\vartheta}{\partial x},$$

which do not depend on the coordinates and which are therefore unimportant for the subsequent discussion.

The set (2.2) is a set of linear, inhomogeneous integral equations. The general solution of the corresponding homogeneous set is well known: $N^{(0)}$ and $f^{(0)}$; we need thus only find the particular solution $N^{(1)}$ and $f^{(1)}$. In what follows we restrict ourselves to the uniform case when we can split off the angular dependence and it is convenient to look for the solution in the form

$$f^{(1)} = -\tau_{ss}\frac{\partial f_0}{\partial\varepsilon} p_x v_z \frac{\partial u}{\partial z}\varphi\left(\frac{\varepsilon}{T}\right), \quad N^{(1)} = -\tau_{ps}\frac{\partial N_0}{\partial\Omega} q_x s_z \frac{\partial u}{\partial z}\psi\left(\frac{\Omega}{T}\right). \qquad (2.3)$$

Apart from numerical coefficients of order unity we get for the unknown functions $\varphi(\epsilon/T)$ and $\psi(\Omega/T)$ the following set of equations:

$$-\hat{K}_{ps}{}^{\mathrm{I}}\{\psi\} + \frac{\Theta_D}{Ms^2}\hat{K}_{ps}{}^{\mathrm{II}}\{\varphi\} = e^y f_0^2(y)\,y, \qquad (2.4)$$
$$-\hat{K}_{ss}\{\varphi\} + \left(\frac{\Theta_N}{\Theta_D}\right)^3 \hat{K}_{sp}{}^{\mathrm{II}}\{\psi\} = e^x f_0^2(x)\,x,$$

where $x = \epsilon/T$, $y = \Omega/T$, $f_0(x) = (e^x - 1)^{-1}$ while the operators $\hat{K}$ have the form

$$\hat{K}_{ps}{}^{\mathrm{I}}\{\psi\} = \int_0^\infty dx\, x^2(y-x)^2[f_0(y)+1]f_0(x)f_0(y-x)\,y\psi(y),$$

$$\hat{K}_{ps}{}^{\mathrm{II}}\{\varphi\} =$$
$$= \int_0^\infty dx\, x^2(y-x)^2[f_0(y)+1]f_0(x)f_0(y-x)\,[x\varphi(x) + (y-x)\varphi(y-x)],$$

$$\hat{K}_{ss}\{\varphi\} = \int_0^\infty dx'\,dx''\,\frac{x''}{x^2}(x^2 + x'^2 + x''^2 + (x + x' - x'')^2)^2$$
$$\times[f_0(x)+1][f_0(x')+1]f_0(x'')f_0(x+x'-x'')$$
$$\times[x\varphi(x) + x'\varphi(x') - x''\varphi(x'') - (x + x' - x'')\varphi(x + x' - x'')],$$

$$\hat{K}_{sp}{}^{\mathrm{II}}\{\psi\} = \int_0^\infty dy\, y^2(y-x)^2[f_0(x)+1][f_0(y-x)+1]f_0(y)\,y\psi(y).$$

The operators $\hat{K}_{pp}\{\psi\}$ and $\hat{K}_{sp}^{\mathrm{I}}\{\varphi\}$ are dropped as they occur in Eqs. (2.4) with small factors $(\Theta_N/\Theta_D)^4$

and $(\Theta_N/\Theta_D)^3\Theta_D/Ms^2$, respectively. To obtain (2.4) we used the explicit form of the matrix elements describing the interaction between quasi-particles:[5,7,8]

$$M_{ss}=\frac{\mu\Theta_N}{M_0V}\frac{\varepsilon^2(\mathbf{p}_1)+\varepsilon^2(\mathbf{p}_2)+\varepsilon^2(\mathbf{p}_3)+\varepsilon^2(\mathbf{p}_4)}{\sqrt{\varepsilon(\mathbf{p}_1)\varepsilon(\mathbf{p}_2)\varepsilon(\mathbf{p}_3)\varepsilon(\mathbf{p}_4)}},$$

$$M_{sp}=\sqrt{\frac{\varepsilon(\mathbf{p}_1)\varepsilon(\mathbf{p}_2)\Omega(\mathbf{q})}{\rho s^2V}},\quad M_{pp}=\sqrt{\frac{\Omega(\mathbf{q}_1)\Omega(\mathbf{q}_2)\Omega(\mathbf{q}_3)}{\rho s^2V}}.$$

Here ρ is the density of the substance, V its volume, and we have omitted factors which are functions of the directions of the momenta and the quasi-particle polarizations, which were of order unity. As the $\hat{K}$ are integral operators which do not have the parity property, with norm unity and as the factors in front of the operators $\hat{K}^{II}_{sp}$ and $\hat{K}^{II}_{ps}$ are small, the functions $\varphi(x)$ and $\psi(y)$ also have the norm unity and hence

$$f^{(1)}=-\tau_{ss}\frac{\partial f_0}{\partial\varepsilon}p_xv_z\frac{\partial u}{\partial z},\quad N^{(1)}=-\tau_{ps}\frac{\partial N_0}{\partial\Omega}q_xs_z\frac{\partial u}{\partial z}. \tag{2.5}$$

Substituting $N = N^{(0)} + N^{(1)}$, $f = f^{(0)} + f^{(1)}$, where $N^{(1)}$ and $f^{(1)}$ are determined by Eqs. (2.5), into the first of Eqs. (2.1), which expresses the quasi-momentum conservation law, we get the hydrodynamical equation

$$a\frac{\partial\vartheta}{\partial x}=\nu^{\text{eff}}\frac{\partial^2u}{\partial z^2}-\frac{u}{\tau_s^U}, \tag{2.6}$$

where

$$a=\frac{\langle pv\varepsilon\rangle+\langle qs\Omega\rangle}{\langle p^2\rangle}\approx v^2,\quad l_{ss}^N=v\tau_{ss},\quad l_{ps}^N=s\tau_{ps},$$

$$\nu^{\text{eff}}=\frac{\tau_{ss}\langle p^2v^2\rangle+\tau_{ps}\langle q^2s^2\rangle}{\langle p^2\rangle}=vl_{ss}^N+sl_{ps}^N\left(\frac{\Theta_N}{\Theta_D}\right)^5,\quad v=\frac{\Theta_Na}{\hbar}.$$

The second of Eqs. (2.1), which expresses the energy conservation law, vanishes identically because $\operatorname{div}\mathbf{u} = 0$, $\partial\vartheta/\partial x$ = constant.

The magnons can interact with one another both directly and via the phonons. Both mechanisms contribute to the kinematic viscosity (see (2.6)).

We put $(\Theta_N/\Theta_D)^3 \sim 0.1$; $\Theta_N \sim \Theta_D \sim 10^{2}$°K; $M \sim 10^{-22}$ g; $s \sim 10^5$ cm/sec. Direct estimates then show that the viscosity is determined by the second interaction mechanism, since a larger mean free path corresponds to it and thus $\nu^{\text{eff}} = sl^N_{ps}(\Theta_N/\Theta_D)^5$. If the stronger inequality $\Theta_N \ll \Theta_D$ holds, not only the specific heat, but - as follows from (2.6) - also the viscosity is determined by only the magnons, and $\nu^{\text{eff}} = vl^N_{ss}$.[9]

The thermal conductivity coefficient is defined by the relation

$$\varkappa=-Q\left(T\frac{\partial\vartheta}{\partial x}\right)^{-1},$$

where $Q = \ll v\epsilon f^{(0)}\gg + \ll s\Omega N^{(0)}\gg \approx TC_Su$ is the heat flux density, and C_S is the magnon specific heat. As a result of solving (2.6) we have

$$\varkappa=C_svl^{\text{eff}}, \tag{2.7}$$

where l^{eff} is the effective mean free path:

$$l^{\text{eff}}=l^U(1-z^{-1}\operatorname{th}z),\quad l^U=v\tau_s^U,\quad z=\frac{d}{2}(\nu^{\text{eff}}\tau_s^U)^{-1/2}.$$

We consider limiting cases. If $z \ll 1$, we have

$$l^{\text{eff}}=\frac{vd^2}{\nu^{\text{eff}}};\quad\varkappa=C_s\frac{v^2d^2}{\nu^{\text{eff}}}\sim d^2T^8\text{ for }T_1\ll T\ll T_2, \tag{2.8}$$

where T_1 and T_2 are determined from the conditions $l^{\text{eff}}(T_1) = d$, $l^{\text{eff}}(T_2) = l^U_S(T_2)$. The result obtained, (2.8), can be elucidated starting from intuitive physical considerations. Under the influence of normal collisions a quasi-particle experiences random walks which increase the effective mean free path between two collisions involving the loss of quasi-momentum. According to well-known formulae for Brownian motion, the path traversed by a particle between two collisions with the boundaries is d^2/l^N, where l^N is the largest of the free paths connected with normal collisions, in the given case l^N_{ps}. The factor $(\Theta_N/\Theta_D)^5$ takes into account the contribution of phonons as compared to magnons. When $z \gg 1$, we have $l^{\text{eff}} \approx l^U_S$ and the loss of momentum occurs because of Umklapp processes and scattering by impurities and lattice defects.

At low temperatures, when $l^N_{ps} \gg d$, the Knudsen situation occurs: $l^{\text{eff}} = d$ and $\kappa = C_Svd \sim T^3d$. (The inequality $T \gg \epsilon_0$ must then, of course, be satisfied.) This case is not contained in the equations obtained earlier and is given to complete the picture. Finally, if $T \ll \epsilon_0$ the thermal conductivity is determined by the phonons as there are exponentially few magnons.

We must note that if we use the experimental values for the probability for phonon-phonon collisions[6] in the temperature range $T \ll (\Theta_N/\Theta_D)^3\Theta_N$ we have

$$\nu^{\text{eff}}=vl_{ss}^N+sl_{pp}^N(\Theta_N/\Theta_D)^5,\quad\varkappa\sim T^7.$$

In the opposite case when $\Theta_D^3 \ll \Theta_N^3$ analysis shows that the thermal conductivity and the viscosity are determined by the phonons.

Of most interest apparently is the case of a ferromagnetic for which the probabilities describing the interactions between the quasi-particles have different temperature dependences. Different mechanisms for the interactions between the quasi-particles can therefore contribute to the viscosity in different temperature ranges.

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