SLOW COLLISIONS IN A SYSTEM OF THREE BODIES INTERACTING BY COULOMB'S LAW. III. SCATTERING LENGTHS FOR COLLISION PROCESSES INVOLVING MESONIC HYDROGEN ATOMS

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The method of phase functions is used to calculate the scattering lengths for symmetric charge exchange on mesonic atoms of hydrogen and to determine the limits of validity of this concept. The values of the scattering lengths for all the three processes in the even (a_g) and odd (a_u) channels are listed in a table.

1 The following reactions are of particular interest in mesoatomic and nuclear physics[1]

$$p\mu^- + p \to p + p\mu^-, \tag{1a}$$

$$d\mu^- + d \to d + d\mu^-. \tag{1b}$$

$$t\mu^- + t \to t + t\mu^-, \tag{1c}$$

of which the first two were investigated experimentally in[2] and theoretically in[3,4]. The electron shells of the hydrogen atoms, whose nuclei act as targets for the scattering of mesonic atoms, do not affect the above three reactions because the size of the mesonic atoms is much smaller than the characteristic atomic distances. The scattering processes defined by Eq. (1) are, therefore, almost pure examples of the quantum-mechanical problem of three bodies interacting by Coulomb's law. A special case of this problem is the process

$$pe^- + p \to p + pe^- \tag{2}$$

which was considered in a previous paper[5], where the method was discussed in detail and the relevent references were reproduced.

2. As in the case of the reaction defined by Eq. (2), the resonance nature of the collision processes defined by Eq. (1) enables us to restrict our attention to the two-level approximation in the calculation of the charge-exchange and total cross sections $\sigma_{ex}(k)$ and $\sigma(k)$ without taking into account the spins of the meson and nuclei:

$$\sigma_{ex}(k) = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_g - \delta_u),$$

$$\sigma(k) = \frac{2\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(\sin^2\delta_g + \sin^2\delta_u); \tag{3}$$

where

$$k^2 = 2ME, \quad 2M = \frac{M_p}{M_\mu} + \frac{1}{2} = \frac{M_p}{m}, \quad \frac{1}{m} = \frac{1}{M_\mu} + \frac{1}{2M_p}$$

(and similarly for d and t) and E is the collision energy. The phases δ_g^l and δ_u^l are calculated from the equations

$$\frac{d^2}{dR^2}\chi_{g,u}^l(R) + \left[k^2 - \frac{l(l+1)}{R^2} - V_{g,u}(R)\right]\chi_{g,u}^l(R) = 0,$$

$$V_{g,u}(R) = 2M[W_{g,u}(R) - W_{g,u}(\infty)] + [K_{g,u}(R) - K_{g,u}(\infty)], \tag{4}$$

where all the quantities are given in the system of units in which $\hbar = e = m = 1$.

The symmetric $W_g(R)$ and antisymmetric $W_u(R)$ terms for systems described by Eq. (1) and consisting of two nuclei and a negative muon, and also the corresponding matrix elements $K_g(R)$ and $K_u(R)$ taking into account adiabatic corrections for the motion of the nuclei are calculated in[6].

In the case of the processes defined by Eq. (2) the inclusion of adiabatic corrections has practically no effect on the results of calculations, because in this case $2M \approx 10^3$. In the case of scattering of mesonic atoms, however, these corrections must be taken into account. The relatively small variation in the quantity 2M between the different hydrogen isotopes (2M = 10–30) leads to considerable differences in the course of the process defined by (1a)–(1b).

Figure 1 shows graphs of $\sigma_{ex}(k)$ for the above reactions. It is clear that for collision energies $E < 0.01$ eV all the cross sections tend to the constant limit

$$\sigma_{ex} = \pi(a_g - a_u)^2, \tag{5}$$

where a_g and a_u are the scattering lengths in the even and odd channels (and, correspondingly, b_g, b_u and c_g, c_u for the processes defined by Eqs. (1b) and (1c)).

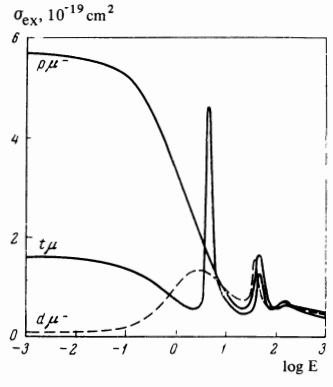

FIG. 1. Graphs of $\sigma_{ex}(k)$ for the processes defined by Eqs. (1a)-(1c) (E is in Ev). For $E \leqslant 1$ eV we have pure s-scattering. For 1 eV $\leqslant E \leqslant 100$ eV we have resonances, and $E = 10^3$ eV is the region where Born scattering has not yet been reached.

The cross section for the reaction given by Eq. (1b) is strongly suppressed in comparison with the reactions (1a) and (1c), which is explained by the same sign of scattering lengths b_g and b_u for this process. In all other respects, the dependence of $\sigma_{ex}(k)$ on the collision energy E is the same for all three reactions in Eq. (1), i.e., there is a slow fall or increase for $E < 1$ eV, sharp peaks for 1 eV $< E <$ 100 eV, and a continuous fall for $E > 100$ eV.

These features become understandable through inspection of Fig. (2) which shows the phase differences $\Delta_l = \delta_g^l - \delta_u^l$ as functions of ln E for the process (1b). The near-horizontal regions for $E < 1$ eV correspond to pure s-scattering. Resonances in the cross sections appear at the collision energies at which $\sigma_{ex}(k)$ receives contributions from partial cross sections $\sigma_l(k)$ with $l \neq 0$, for which the difference Δ_l at given E is $\Delta_l = \pi(2n+1)/2$. For $E > 100$ eV the number of such phases is large, the picture is averaged, and the cross section $\sigma_{ex}(k)$ again becomes a smooth function of k.

3. The expressions for the cross sections $\sigma_{ex}(k)$ and $\sigma(k)$ given by Eq. (3) are realistic only for collision energies $E > 1$ eV. In the energy region E = 0.01–1 eV, which is experimentally interesting, the picture is complicated by the interaction between the spins of the meson and of the nuclei. However, even without spins such investigation is of methodologic interest because it should provide an illustration of the general results in the theory of potential scattering. Moreover, it enables us to calculate the scattering lengths a_g and a_u which can subsequently be used for more realistic calculations.

Figure 3 shows the scattering phases $\delta_g^0(k)$ in the even channel for $l = 0$. These are smooth functions which for $k \to 0$ tend to the limits $\delta_g^0(0) = \pi n$ in accordance with the Levinson theorem, where n is the number of bound states in the potential $V_g(R)$, n = 1 for the system $p\mu^-p$, and n = 2 for the systems $d\mu^-d$ and $t\mu^-t$. In the odd channel the phases $\delta_u^l(0)$ are zero and decrease monotonically with increasing k.

Using the equation $\Delta_l(0) = \delta_g^l(0)$ and Levinson's theorem, we can readily establish from Fig. 2 that in each of the $d\mu^-d$ states with orbital angular momenta $l = 0, 1$ there are two vibrational levels. This resolves the old question[1] about the existence of the second vibrational level in the $d\mu^-d$ system with $l = 1$.

4. Calculations of the phases $\delta_{g,u}^l$ and partial cross sections $\sigma_l(k)$ show that the region of pure s-scatter-

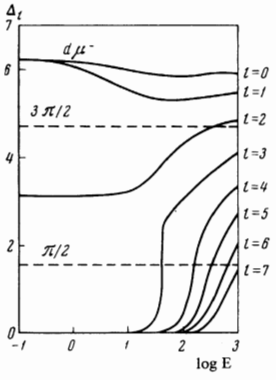

FIG. 2. Graphs of the phase difference $\Delta_L = \delta_g^l - \delta_u^l$ for process (1b), showing that in the potentials $V_g^l(R)$ with $l = 0, 1$ there are two vibrational states of the $d\mu^-d$ system, for $l = 2$ there is one state, and the resonance in the cross sections $\sigma_{ex}(k)$ at $E \approx 50$ eV is due to the phase Δ_3 which, at this energy, is equal to $\pi/2$.

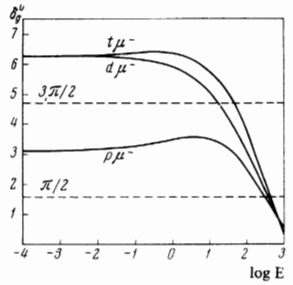

FIG. 3. Graphs of the phases $\delta_g^0(k)$ for $l = 0$ in the case of the processes (1a)-(1c).

ing for the processes defined by Eq. (1) extends up to collision energies $E \sim 1$ eV. The scattering lengths a_g and a_u, b_g and b_u, and c_g and c_u for the reactions given by Eq. (1), which are given in the table, were calculated from the formula[7]

$$k \operatorname{ctg} \delta = -\frac{1}{a} + \frac{\pi\alpha}{3a^2}k + \frac{2\alpha}{3a}k^2 \ln\frac{\alpha k^2}{16}, \quad \alpha = \frac{9M}{2}, \tag{6}$$

which is valid for potentials with the asymptotic form $V(R) = -\alpha/R^4$ for $R \to \infty$. All the values are given in mesoatomic units ($\hbar = e = M_\mu = 1$). To check the calculations the scattering lengths were then determined directly from the equations for the scattering lengths:

$$\frac{d}{dR}a(R) = V(R)[R - a(R)]^2, \quad a(0) = 0, \quad a = a(\infty). \tag{7}$$

These equations follow from the method of phase functions[7] and lead to the same results as Eq. (6).

Equation (6) is valid only for small k: for $E > 0.1$ eV the number of terms in it is no longer sufficient to ensure a constant scattering length (a = const). The range of validity of the concept of the scattering length is, in practice, narrower still because Eq. (5) follows from Eq. (3) only on the assumption that $\delta(k) \approx ak$, which is equivalent to neglecting the last two terms in Eq. (6). In this case, the range of energies in which the condition $a \approx$ const is satisfied to within $\epsilon = 0.01$ is restricted by the condition $E \leq 0.01$ eV. In this region, all the results obtained in the scattering length approximation are invalid (see Fig. 4).

It follows from the table that the previous calculations of the scattering lengths a_g and b_g for the even channel are in relatively good agreement with our own,

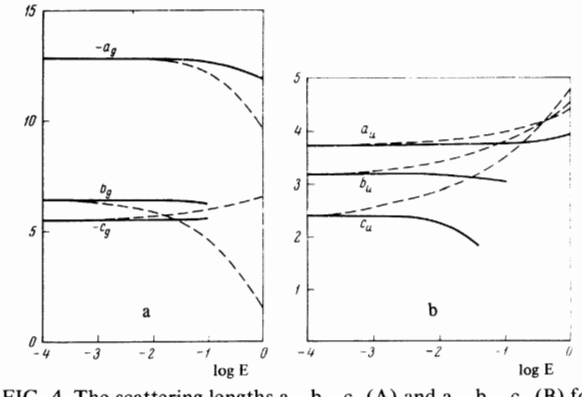

FIG. 4. The scattering lengths a_g, b_g, c_g(A) and a_u, b_u, c_u(B) for the potentials $V_g(R)$ and $V_u(R)$ in the case of the processes defined by Eqs. (1a)-(1c). Solid curve–calculated from Eq. (6). Broken curve–values of δ/k usually adopted as the scattering lengths. It is clear that this definition is satisfactory only for $E \leq 10^{-2}$ eV.

Scattering Lengths for the Reactions Given by Eq. (1)

	$p\mu^- + p$		$d\mu^- + d$		$t\mu^- + t$	
	a_g	a_u	b_g	b_u	c_g	c_u
Gershtein & Zel'dovich [4]	−17.3	5.25	6.67	5.76	—	—
Cohen, Judd, Riddel [4]	−11.0	5.0	—	—	—	—
Present work	−13.3	3.7	5.5	3.1	−6.5	2.4

whereas the quantities a_u and b_u differ by factors of 1.5. These differences are explained by the approximate nature of the previous calculations. In particular, they did not take into account the shallow minimum of the term $W_u(R)$ at $R = 12.55$, whose effect on the scattering processes turned out to be considerable.[5] Figure 5 illustrates this influence: at $R_0 = 10.5$, where the potential $V_u(R)$ becomes zero, the scattering length $a_u(R)$ reaches a maximum[1)] and $c_u(R_0) > b_u(R_0) > a_u(R_0)$. In previous calculations these values were adopted as final. However, further integration of Eq. (7), taking into account the long-range asymptotic form $V_u(R) = \alpha/R^4$ and the minimum at $R = 12.55$, is found to modify these values substantially and leads to the reverse inequality $a_u > b_u > \sigma_u$.

The accuracy of the method of phase functions is restricted only by the reliability of $W(R)$ and $K(R)$ which are known to within $\epsilon = 10^{-6}$ or better. The table shows the scattering lengths only to within two significant figures because the accuracy of the method of perturbed stationary states in the two-level approximation is restricted to

$$(2M)^{-2} \approx (M_\mu / M_p)^2 \approx 10^{-2}\ [4].$$

6. Real experiments which are usually interpreted in terms of the scattering length are carried out at collision energies $E \sim 1$ eV. This energy exceeds not only the upper limit for the validity of the concept of a scattering length, but also the size of the hyperfine splitting for mesonic atoms of hydrogen, which appears when the meson spin interacts with the spin of the proton, deuteron or triton (for the $p\mu^-$, $d\mu^-$, and $t\mu^-$ levels we have, respectively, $\Delta E = 0.183$, 0.049, and 0.241 eV). The presence of the hyperfine splitting ΔE modifies the course of the scattering processes in Eq. (1). In particular, there is an additional inelastic channel for transitions between hyperfine structure components during the collisions.

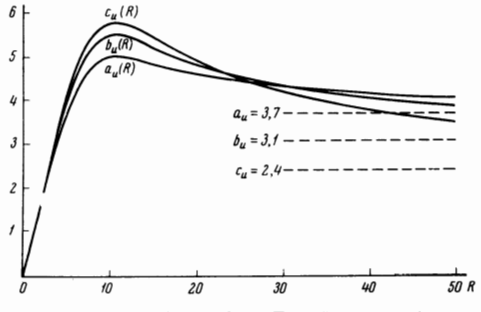

FIG. 5. Graphs of a(R) found from Eq. (7). True value $a = a(\infty)$ is reached only for $R \approx 200$. In previous calculations it was assumed that $a = a(10)$, which is clearly unsatisfactory.

[1)] This effect is readily established directly from Eq. (7).

Gershteĭn[3] has derived formulas for these processes in terms of the scattering lengths a_g and a_u. It is clear that formulas of this kind can be used only in the energy region where the concept of the scattering length is still valid ($E \lesssim 10^{-2}$ eV). Moreover, it was assumed in their derivation that the scattering matrix for the processes defined by Eq. (1) depends only on two parameters, for example, δ_g and δ_u (or a_g and a_u, which is equivalent) which are calculated from the two independent equations in Eq. (4) for the even and odd channels. In reality, when the spin interaction is taken into account, the equations in Eq. (4) can no longer be regarded as independent and a third parameter (the mixing parameter ϵ) appears in the scattering matrix. This parameter can be found by solving the coupled system of equations for the wave functions χ_g and χ_g.[3] By solving this set of equations we find the accurate limits of validity for the given formulas given by Eq. (3) simultaneously with the transition cross sections.

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