FLUCTUATIONS IN RING LASERS

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Submitted October 9, 1969

Zh. Eksp. Teor. Fiz. 58, 1651–1661 (May, 1970)

Fluctuations of the amplitudes and phases of opposite waves in a ring laser are considered by taking into account the coupling between them by means of back scattering. The calculations are performed for single-mode operation and various levels of output power lying between near-threshold values and such high field strengths that saturation becomes important. Concrete expressions are obtained for the spectral densities, moments and amplitude correlation coefficient for the opposite waves. Expressions are also derived for the line widths of each of the waves and for the line width of the difference frequency. The influence of coupling between waves on the shape of the amplitude and phase shift fluctuation spectra is elucidated. It is found that in the presence of coupling the wave width is not completely determined by the frequency-fluctuation spectral density at zero frequency.

IT the present time there are no papers containing a consistent calculation of the fluctuations of the amplitudes and phases of opposing waves in a ring laser in weak and strong fields, and also with allowance for the coupling between the waves via the back scattering. Several results were obtained in[1] and[2], but only for the case of a weak field. In[1] the fluctuations were calculated by a quantum approach, in which the fluctuation sources were not introduced explicitly in the equation, and were not calculated. We introduce into the equation for the field the fluctuation sources whose intensities have been calculated in[3]. We make use here of the results of[3].

The purpose of the present investigation was to calculate the fluctuations of the amplitude and of the phase of each of the opposing waves, and also the fluctuations of the phase difference between the opposing waves. The latter can be of great practical importance from the point of view of clarifying the limiting capabilities of laser gyroscopes.

1. FUNDAMENTAL EQUATIONS

We specify the field in the form of a sum of two opposing waves and write down the abbreviated equations for the amplitude and phases of these waves with allowance for the coupling between them:

$$\frac{dE_{1,2}}{dt}=\frac{\omega_0}{2}\left\{4\pi\varkappa''_{1,2}-\frac{1}{Q}\right\}E_{1,2}\mp\frac{\omega_0}{2Q}|m_{1,2}|E_{2,1}\sin(\Phi+\vartheta_{1,2})+\omega_0\xi_{a\,1,2}(t), \tag{1.1}$$

$$\frac{d\varphi_{1,2}}{dt}=\pm\frac{\Omega}{2}-\frac{\omega_0}{2}4\pi\varkappa'_{1,2}-\frac{\omega_0}{2Q}\frac{E_{2,1}}{E_{1,2}}|m_{1,2}|\cos(\Phi+\vartheta_{1,2})+\frac{\omega_0}{E_{1,2}}\xi_{\mathrm{ph}\,1,2}(t). \tag{1.2}$$

Here $\kappa'_{1,2}$ and $\kappa''_{1,2}$ are the real and imaginary parts of the complex polarizability, Ω is the resonator frequency difference for the opposing waves, due either to rotation of the laser or to the introduction of some independent element, $|m_{1,2}|$ are the moduli of the coupling coefficients via the back scattering, $\vartheta_{1,2}$ are the phases of the coupling coefficients, $\xi_{a\,1,2}(t)$ and $\xi_{\mathrm{ph}\,1,2}(t)$ are the sources of the amplitude and phase fluctuations, and Q is the quality factor of the resonator. The quantities $\kappa'_{1,2}$ and $\kappa''_{1,2}$ and the fluctuation-source intensities were calculated in[3].

2. FLUCTUATION OF AMPLITUDES OF OPPOSING WAVES IN THE ABSENCE OF A COUPLING BETWEEN THEM

Under certain conditions, which will be specified in greater detail below, the equations for the amplitudes of the opposing waves (1.1) can be linearized relative to the mean values, and the fluctuations can be calculated by using the correlation approximation. Putting $E_{1,2} = \bar{E}_0 + \delta E_{1,2}$ and expanding in a series in δE_1 or δE_2, we obtain the equations for the fluctuations:

$$\frac{d\delta E_{1,2}}{dt}+A\delta E_{1,2}+B\delta E_{2,1}=\omega_0\xi_{a1,2}(t). \tag{2.1}$$

$$A=-4\pi\omega_0 a\bar{E}_0^2\frac{\partial\varkappa_i''}{\partial(aE_i^2)},\qquad B=-4\pi\omega_0 a\bar{E}_0^2\frac{\partial\varkappa_i''}{\partial(aE_j^2)}.$$

Here (i, j = 1, 2; $i \neq j$).

From (2.1) we determine in the usual manner the spectrum of the amplitude fluctuations for the opposing waves:

$$(\delta E_i\delta E_{i,j})_\omega=\frac{\omega_0^2}{2}\left[\frac{(\xi_{ai}^2)_0+(\xi_{ai}\xi_{aj})_0}{\omega^2+(A+B)^2}\pm\frac{(\xi_{ai}^2)_0-(\xi_{ai}\xi_{aj})_0}{\omega^2+(A-B)^2}\right], \tag{2.2}$$

$$(\delta(E_i^2)\delta(E_{i,j}^2))_\omega=4\bar{E}_0^2(\delta E_i\delta E_{ij})_\omega. \tag{2.2'}$$

Thus, the spectral density of the amplitude fluctuations (and of the intensity fluctuations) constitutes a sum of two Lorentz lines: a broader line with width A + B, and a narrower one with width A – B. Since $(\xi_{ai}\xi_{aj})_0 < 0$, obviously the narrower line always has a higher intensity. It follows from this, incidentally, that the correlation function of the opposing waves $(\delta E_i\delta E_j)_\omega$ is always negative.

The total intensity of both Lorentz lines determines the amplitude dispersion

$$\langle\delta E_i^2\rangle=\frac{\omega_0^2}{2}\frac{(\xi_{ai}^2)_0A-(\xi_{ai}\xi_{aj})_0B}{A^2-B^2}. \tag{2.3}$$

It is seen from this equation that on approaching the limit of the instability region of the two-wave regime (the stability limit corresponds to A = B), just as when the generation threshold is approached, the relative dispersion of the amplitude of each of the opposing waves increases without limit. This means that in these cases the correlation approximation becomes inapplicable. The condition for the applicability of the method of linearization and of the correlation approximation follows directly from (2.3):

$$\frac{\omega_0^2}{2\bar{E}_0^2}\frac{(\xi_{ai}^2)_0 A - (\xi_{ai}\xi_{aj})_0 B}{A^2-B^2} \ll 1. \tag{2.4}$$

Let us consider the asymptotic expressions for the spectral densities of the fluctuations of the amplitude and of the intensity of the opposing waves. In a weak field

$$(\delta E_i \delta E_{i,j})_\omega = \frac{\omega_0^2 N_0}{2}\left[\frac{1}{\omega^2+(\alpha+\beta)^2\omega_0^2 d^2 a^2 \bar{E}_0^4} \pm \frac{1}{\omega^2+(\alpha-\beta)^2\omega_0^2 d^2 a^2 \bar{E}_0^4}\right]. \tag{2.5}$$

Here N_0 is the intensity of the noize in the zeroth approximation in the field.

It follows from (2.5) that even in a weak field the form of the spectrum of the amplitude fluctuations in the ring laser, generally speaking, differs from a Lorentz form. It is easy to verify, however, that at not too small deviations from the center of the Doppler line, when $\mu \gtrsim \gamma_{ab}$, the line shape is indeed close to a Lorentz shape, as confirmed by the experimental data of[4]. In a strong field we have

$$(\delta E_i\delta E_{i,j})_\omega = \pm\frac{\omega_0^2 N_\infty^{(a)}}{\omega^2+\omega_0^2 d^2\mu^4/a^2\bar{E}_0^4\gamma_{ab}^2(\mu^2+\gamma_{ab}^2)}, \tag{2.6}$$

$$\frac{\langle\delta E_i^2\rangle}{\bar{E}_0^2} = \frac{\omega_0 a\gamma_{ab}\sqrt{\mu^2+\gamma_{ab}^2}}{2d\mu^2}N_\infty^{(a)}. \tag{2.6'}$$

It follows from (2.6′) that with increasing field the relative dispersion of the amplitude and of the intensity of the opposing waves tends to a constant value.

From (2.2) and (2.2′) we can readily obtain an expression for the spectral density of the correlation coefficient of the intensities of the opposing waves:

$$\rho_\omega = \frac{(\delta E_i^2\delta E_j^2)_\omega}{(\delta(E_i^2)^2)_\omega} = \frac{(A^2+B^2+\omega^2)(\xi_{ai}\xi_{aj})_0 - 2AB(\xi_{ai}^2)_0}{(A^2+B^2+\omega^2)(\xi_{ai}^2)_0-2AB(\xi_{ai}\xi_{aj})_0}, \tag{2.7}$$

In a weak field

$$\rho_\omega = -\frac{2\alpha\beta}{\alpha^2+\beta^2}\frac{(\Delta\omega_\rho)^2}{\omega^2+(\Delta\omega_\rho)^2}, \tag{2.8}$$

where $\Delta\omega_\rho = \omega_0 da\bar{E}_0^2\sqrt{\alpha^2+\beta^2}$.

Expression (2.8) for the correlation coefficient in a weak field was investigated by us in[5]. Here we compare the result with the experimental data of[4], where a plot is given of the frequency dependence of the correlation coefficient. Unfortunately, the parameters of the laser with which this plot was obtained (detuning, resonator bandwidth, power) are not given in[4]. Calculations in accordance with formula (2.8) for two values of the correlation coefficient ρ_0 and $\rho_{\Delta\omega_\rho}$ yielded the following values of these parameters: $\mu \approx 1.1\,\gamma_{ab}$, $\omega_0 da\bar{E}_0^2 \approx 9.2\times10^4$ rad/sec, $\Delta\omega_\rho \approx 5.02\times10^4$ rad/sec. If we now use these parameters and plot the correlation coefficient of the intensities against the frequency in accordance with formula (2.8), then we obtain good agreement with experiment (Fig. 1). The circles in this figure show the experimental points. The discrepancy in the region of high frequencies can be attributed to the

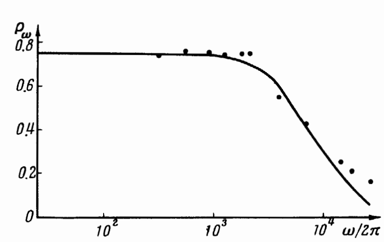

FIG. 1. Plot of the correlation coefficient.

fact that it is no longer possible in this case to neglect the quantity $(\xi_{ai}\xi_{aj})_0$, as was done in the derivation of formula (2.8). At high frequencies the correlation coefficient of the noise sources for the opposing waves $(\xi_{ai}\xi_{aj})_0/(\xi_{ai}^2)_0$ begins to play an important role, since the intensity correlation coefficient tends to this value as $\omega\to\infty$, and not to zero as would follow from (2.8).

In a strong field we obtain from (2.7)

$$\rho_\omega \approx -1. \tag{2.9}$$

We consider further the fluctuations of the amplitudes of the opposing waves in the case when the correlation approximation becomes inapplicable. We confine ourselves here to a weak field, when the equations for the amplitudes of the opposing waves can be written in the form

$$\frac{dE_{1,2}}{dt} = \frac{\omega_0 d}{2}[\eta - \alpha a E_{1,2}^2 - \beta a E_{2,1}^2]E_{1,2} + \omega_0\xi_{a1,2}(t). \tag{2.10}$$

Here $\eta = 1 - 1/Qd$ is the excess of the pump level over the threshold value.

The corresponding Fokker-Planck equation for the joint probability density of the intensities is of the form

$$\frac{\partial w}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial(E_i^2)}\left\{[\omega_0 d(\eta-\alpha a E_i^2 - \beta a E_j^2)E_i^2 + 2\omega_0^2 N_0]w + 2\omega_0^2 N_0\frac{\partial(wE_i^2)}{\partial(E_i^2)}\right\}. \tag{2.11}$$

It is easy to verify that this equation is satisfied by the following stationary solution:

$$w(E_1^2,E_2^2) = C\exp\left\{-\frac{\alpha a^2}{4\bar{N}^2}\left[\left(E_1^2-\frac{\eta}{a(\alpha+\beta)}\right)^2 + \left(E_2^2-\frac{\eta}{a(\alpha+\beta)}\right)^2 + 2\frac{\beta}{\alpha}\left(E_1^2-\frac{\eta}{a(\alpha+\beta)}\right)\left(E_2^2-\frac{\eta}{a(\alpha+\beta)}\right)\right]\right\}. \tag{2.12}$$

Here $\bar{N} = \sqrt{N_0\omega_0 a/d}$,

$$C = \left\{\sqrt{\frac{\pi}{\alpha}}\frac{\bar{N}}{a}\int_0^\infty\left[1-\Phi\left(\frac{a\beta}{\bar{N}\sqrt{2\alpha}}x - \frac{\eta}{\bar{N}\sqrt{2\alpha}}\right)\right] \times\exp\left\{-\frac{(\alpha^2-\beta^2)a^2}{4\alpha\bar{N}^2}\left(x-\frac{\eta}{(\alpha+\beta)a}\right)^2\right\}dx\right\}^{-1}.$$

From this solution we get the moments

$$\langle E_i^n\rangle = C\sqrt{\frac{\pi}{\alpha}}\frac{\bar{N}}{a}\int_0^\infty x^{n/2}\left[1-\Phi\left(\frac{a\beta x-\eta}{\bar{N}\sqrt{2\alpha}}\right)\right] \times\exp\left\{-\frac{(\alpha^2-\beta^2)a^2}{4\alpha\bar{N}^2}\left(x-\frac{\eta}{(\alpha+\beta)a}\right)^2\right\}dx, \tag{2.13}$$

$$\langle E_i^n E_j^m\rangle = C\frac{m}{2}\Gamma\left(\frac{m}{2}\right)\left(\frac{\alpha a^2}{2\bar{N}^2}\right)^{-(m+2)/4}\int_0^\infty x^{n/2}$$

$$\times \exp\left\{-\frac{(\alpha^2-\beta^2)a^2}{4a\bar{N}^2}\left(x-\frac{\eta}{(\alpha+\beta)a}\right)^2-\frac{a^2\beta^2}{8a\bar{N}^2}\left(x-\frac{\eta}{\beta a}\right)^2\right\}$$
$$\times D_{-(m+2)/2}\left(\frac{\beta a}{\bar{N}\sqrt{2a}}\left(x-\frac{\eta}{\beta a}\right)\right)dx. \qquad (2.14)$$

The integrals in these expressions cannot be evaluated in the general case. We therefore calculate them in two limiting particular cases, when the correlation approximation is not valid.

1. $\eta \ll \bar{N}$. Then in the zeroth approximation in $\eta/\bar{N}$ we have

$$\langle E_i^2\rangle = \sqrt{\frac{\pi(\alpha-\beta)}{\alpha(\alpha+\beta)}}\frac{\bar{N}}{a}\left(\operatorname{arctg}\frac{\sqrt{\alpha^2-\beta^2}}{\beta}\right)^{-1}, \qquad (2.15)$$

$$\langle E_i^4\rangle = \frac{2\alpha\bar{N}^2}{a^2(\alpha^2-\beta^2)}\left\{1-\frac{\beta\sqrt{\alpha^2-\beta^2}}{\alpha^2}\left(\operatorname{arctg}\frac{\sqrt{\alpha^2-\beta^2}}{\beta}\right)^{-1}\right\}, \qquad (2.16)$$

$$\langle E_i^2E_j^2\rangle = \frac{2\beta\bar{N}^2}{a^2(\alpha^2-\beta^2)}\left\{\frac{\sqrt{\alpha^2-\beta^2}}{\beta}\left(\operatorname{arctg}\frac{\sqrt{\alpha^2-\beta^2}}{\beta}\right)^{-1}-1\right\}. \qquad (2.17)$$

When the limit of the stability region of the two-wave regime is approached ($\beta \approx \alpha \approx 0.5$), all the moments remain bounded and equal to

$$\langle E_i^2\rangle = \sqrt{\frac{\pi}{2}}\frac{\bar{N}}{a}, \quad \langle E_i^4\rangle = \frac{8}{3}\frac{\bar{N}^2}{a^2}, \quad \langle E_i^2E_j^2\rangle = \frac{4}{3}\frac{\bar{N}^2}{a^2}. \qquad (2.18)$$

From (2.15)–(2.17) we can readily obtain expressions for the relative dispersion of the intensities of the opposing waves near the generation threshold, and also of the correlation coefficients between them. In particular, when $\mu \approx \gamma_{ab}$ ($\beta = \alpha/2 \approx 0.25$) we obtain

$$\sigma^2 = \frac{\langle E_i^4\rangle - \langle E_i^2\rangle^2}{\langle E_i^4\rangle} \approx 0.39, \quad \rho = \frac{\langle E_i^2E_j^2\rangle - \langle E_i^2\rangle^2}{\langle E_i^4\rangle - \langle E_i^2\rangle^2} \approx -0.19.$$

When $\mu \approx 0$ ($\beta = \alpha \approx 0.5$)

$$\sigma^2 \approx 0{,}41, \quad \rho \approx -0{,}22.$$

(It follows from the correlation approximation that $\rho = -\beta/\alpha$).

2. $\alpha - \beta \ll \alpha$. In the zeroth approximation in $\alpha - \beta$ we obtain

$$\langle E_i^2\rangle = \frac{\eta}{a} + 4\frac{\bar{N}^3}{a^3}\sqrt{\frac{\pi}{2}}C\left[1+\Phi\left(\frac{\eta}{\bar{N}}\right)\right], \qquad (2.19)$$

$$\langle E_i^4\rangle = 4\frac{\bar{N}^2}{a^2}\left(1+\frac{1}{3}\frac{\eta^2}{\bar{N}^2}-\frac{4}{3}\frac{\bar{N}^2}{a^2}Ce^{-\eta^2/2\bar{N}^2}\right), \qquad (2.20)$$

$$\langle E_i^2E_j^2\rangle = \langle E_i^4\rangle/2.$$

Here

$$C = \frac{a^2}{4\bar{N}^2}\left\{e^{-\eta^2/2\bar{N}^2}+\sqrt{\frac{\pi}{2}}\frac{\eta}{\bar{N}}\left[1+\Phi\left(\frac{\eta}{\bar{N}}\right)\right]\right\}^{-1}. \qquad (2.21)$$

When $\eta = 0$ these expressions go over into (2.18).

It follows from (2.19)–(2.21) that far from the generation threshold, when $\eta \gg \bar{N}$,

$$\langle E_i^2\rangle = \frac{\eta}{a}\left(1+\frac{\bar{N}^2}{\eta^2}\right), \quad \langle E_i^4\rangle = \frac{4}{3}\frac{\eta^2}{a^2}\left(1+3\frac{\bar{N}^2}{\eta^2}\right)$$
$$\langle E_i^2E_j^2\rangle = \frac{\langle E_i^4\rangle}{2}. \qquad (2.22)$$

Hence

$$\sigma^2 = 0.25, \quad \rho \approx -1.$$

These results agree with the results of [1].

3. FLUCTUATIONS OF THE PHASE AND OF THE FREQUENCY

We put in (1.2) $\varphi_{1,2} = \bar{\varphi}_{1,2} + \delta\varphi_{1,2}$ and $\Phi = \bar{\Phi} + \delta\Phi$, and write the equations for the deviations

$$\delta\dot{\varphi}_{1,2} = \frac{\omega_0}{\bar{E}_0}(C\delta E_{1,2} + D\delta E_{2,1} + \xi_{\text{ph}\,1,2}), \qquad (3.1)$$

$$\delta\dot{\Phi} = \frac{\omega_0}{\bar{E}_0}[(C-D)(\delta E_1-\delta E_2)+(\xi_{\text{ph}1}-\xi_{\text{ph}2})]. \qquad (3.2)$$

Here

$$C = -4\pi a\bar{E}_0^2\frac{\partial\varkappa_i'}{\partial(aE_i^2)}, \qquad D = -4\pi a\bar{E}_0^2\frac{\partial\varkappa_i'}{\partial(aE_j^2)}.$$

It follows from (3.1) that the spectral densities of the frequency fluctuations of both opposing waves are the same:

$$(\delta\dot{\varphi}_i^2)_\omega = \frac{\omega_0^2}{\bar{E}_0^2}[(C^2+D^2)(\delta E_i^2)_\omega + 2CD(\delta E_i\delta E_j)_\omega + (\xi_{\text{ph}i}^2)_\omega]. \quad (3.3)$$

The spectral density of the frequency-difference fluctuations is

$$(\delta\dot{\Phi}^2)_\omega = 2\frac{\omega_0^2}{\bar{E}_0^2}[(C-D)^2((\delta E_i^2)_\omega - (\delta E_i\delta E_j)_\omega) + (\xi_{\text{ph}i}^2)_\omega - (\xi_{\text{ph}i}\xi_{\text{ph}j})_\omega]. \qquad (3.3')$$

Thus, the spectrum of the fluctuations of the frequencies of the opposing waves differs significantly in form from the spectrum of fluctuations of the noise sources. This difference is due to the influence of the amplitude fluctuations on the frequency fluctuations.

Let us calculate now the phase advance and the phase difference, due to the presence of fluctuations, within a time τ. From (3.1) and (3.2) it follows that if the time τ is much larger than the correlation times of the amplitude and the sources $\xi_{\text{ph}\,i}(t)$, then we obtain a diffusion law for the phase advances and for the phase difference, i.e., $\langle\delta\varphi_{i\tau}^2\rangle = 2D_\varphi\tau$ and $\langle\delta\Phi_\tau^2\rangle = 2D_\Phi\tau$. The diffusion coefficients D_φ and D_Φ are determined respectively by the spectral density of the frequency fluctuations and of the frequency difference at zero frequency:

$$D_\varphi = {}^1\!/_2(\delta\dot{\varphi}_i^2)_0, \qquad D_\Phi = {}^1\!/_2(\delta\dot{\Phi}^2)_0.$$

The coefficients D_φ and D_Φ, as is well known, determine the width of the spectral line.

When $a\bar{E}_0^2 \ll 1$ we have[1)]

$$D_\varphi = \frac{\omega_0 d\bar{N}^2}{2a\bar{E}_0^2}\left[1+\frac{b^2(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)^2}\right], \quad D_\Phi = \frac{\omega_0 d\bar{N}^2}{a\bar{E}_0^2}\left\{1+\frac{b^2}{(\alpha-\beta)^2}\right\}.$$

When $a\bar{E}_0^2 \gg 1$

$$D_\varphi = \frac{\omega_0^2 a}{4}\frac{\mu^2+\gamma_{ab}^2}{\mu^2}N_\infty^{(a)}, \qquad D_\Phi = 4D_\varphi.$$

Thus, in a weak field the phase diffusion coefficient, and consequently the line width, decrease in inverse proportion to the power. With increasing field, the rate of decrease slows down and finally in a strong field the line width tends to a constant.

All the obtained expressions are valid in the case when the correlation approximation is applicable. It can be shown that the condition for the applicability of the correlation approximation corresponds also to the con-

1)When $b^2 \gg (\alpha-\beta)^2$ this formula coincides with the corresponding formula of Belenov [2] for natural fluctuations.

dition under which it is possible to use a diffusion law for the change of the mean-square phase advance and the phase difference. In the opposite case the correlation time of the amplitude fluctuations turn out to be comparable or larger than the time $\tau \sim 1/D$. It is necessary here to take into account the deviations from the diffusion law.

In the other extreme case, when the correlation approximation is not applicable and the correlation time of the amplitude fluctuations turns out to be much larger than $\tau \sim 1/D$, it is possible to disregard completely the amplitude fluctuations in the calculation of the mean-squared phase advance. The line width is then determined only by the intensity of the phase-fluctuation source.

4. INFLUENCE OF COUPLING BETWEEN OPPOSING WAVES

Let us assume that the frequency difference Ω and the coupling are such that the frequencies of the opposing waves are identical (the waves are mutually synchronized). In this case, in the absence of fluctuations, the phase difference Φ between the opposing waves is a constant quantity. In the presence of fluctuations, the phase difference Φ changes, but under weak-coupling conditions (see[6]) this changes much more slowly than the change of the amplitudes. It is therefore possible to regard Φ in the equations for the amplitudes (2.1) as a determined quantity.

We represent the amplitudes of the opposing waves $E_{1,2}$ in the form $E_{1,2} = \overline{E}_0 + \Delta E_{1,2} + \delta E_{1,2}$. Here $\overline{E}_0$ are the amplitudes of the opposing waves in the absence of a coupling or fluctuations, $\Delta E_{1,2}$ are the corrections due to the presence of the coupling, and $\delta E_{1,2}$ are the fluctuations of the amplitudes. It follows from (2.1) that

$$\Delta E_{1,2} = \mp \frac{\omega_0}{2Q} \overline{E}_0 \frac{A|m_{1,2}| \sin(\Phi + \vartheta_{1,2}) + B|m_{2,1}| \sin(\Phi + \vartheta_{2,1})}{A^2 - B^2}$$

and that the spectrum of the amplitude fluctuations in the presence of coupling coincides in first approximation with the spectrum of the amplitude fluctuations of the opposing waves in the absence of coupling.

Let us consider now the phase fluctuations. The equations for the small phase fluctuations take the form

$$\frac{d\delta\varphi_{1,2}}{dt} = \frac{\omega_0}{\overline{E}_0}(C\delta E_{1,2} + D\delta E_{2,1} + \xi_{\text{ph}1,2}) - \frac{\omega_0}{2Q} M_{1,2}\delta\Phi, \tag{4.1}$$

$$\frac{d\delta\Phi}{dt} = \frac{\omega_0}{\overline{E}_0}[(C-D)(\delta E_1 - \delta E_2) + (\xi_{\text{ph}1} - \xi_{\text{ph}2})] - \frac{\omega_0}{2Q} M\delta\Phi. \tag{4.2}$$

Here

$$M_{1,2} = -|m_{1,2}| \sin(\overline{\Phi} + \vartheta_{1,2}) \pm \frac{\omega_0}{A^2 - B^2}[(CA - DB)|m_{1,2}| \times \cos(\overline{\Phi} + \vartheta_{1,2}) + (CB - DA)|m_{2,1}| \cos(\overline{\Phi} + \vartheta_{2,1})],$$

$$M = M_1 - M_2 = -|m_1| \sin(\overline{\Phi} + \vartheta_1) + |m_2| \sin(\overline{\Phi} + \vartheta_2) + \omega_0 \frac{C-D}{A-B}[|m_1| \cos(\overline{\Phi} + \vartheta_1) + |m_2| \cos(\overline{\Phi} + \vartheta_2)],$$

$\overline{\Phi}$ is the stationary value of the phase difference in the absence of fluctuations. We note that M is proportional to $d\Omega/d\Phi$ in the absence of fluctuations, and consequently vanishes on the boundary of the synchronization band.

We write down the stationary solution of Eq. (4.2):

$$\delta\Phi(t) = \frac{\omega_0}{\overline{E}_0} \int_0^\infty \{(C-D)[\delta E_1(t-t') - \delta E_2(t-t')] + \xi_{\text{ph}1}(t-t') - \xi_{\text{ph}2}(t-t')\} \cdot \exp\left\{-M\frac{\omega_0}{2Q}t'\right\} dt'. \tag{4.3}$$

Substituting now (4.3) in (4.1) and (4.2), we obtain the spectral density of the fluctuations of the frequency of each of the waves and the frequency difference of the opposing waves:

$$(\delta\dot{\varphi}_i^2)_\omega = (\delta\dot{\varphi}_i^2)_\omega^{(0)} + \frac{M_1 M_2 \omega_0^2}{\omega_0^2 M^2 + 4Q^2\omega^2} (\delta\dot{\Phi}^2)_\omega^{(0)}, \tag{4.4}$$

$$(\delta\dot{\Phi}^2)_\omega = \frac{4Q^2\omega^2}{\omega_0^2 M^2 + 4Q^2\omega^2} (\delta\dot{\Phi}^2)_\omega^{(0)}. \tag{4.5}$$

Here $(\delta\dot{\varphi}_i^2)_\omega^{(0)}$ and $(\delta\dot{\Phi}^2)_\omega^{(1)}$ are respectively the spectral densities of the frequency fluctuations and of the frequency difference of the opposing waves in the absence of coupling, determined by formulas (3.2) and (3.3).

Thus, the spectral densities of the frequency fluctuations and of the frequency difference of the opposing waves depend strongly on the magnitude and phase of the coupling coefficients. The spectral density of the fluctuations of the frequency difference at $\omega = 0$, due to the coupling between the waves, vanishes. The influence of the coupling on the spectral density of the frequency fluctuations of each of the waves reduces to the addition or the subtraction, depending on the sign of the product M_1M_2, of an additional Lorentz line of width $\omega_0 M/2Q$.

Let us write out the value of the product M_1M_2:

$$M_1 M_2 = |m_1||m_2| \sin(\overline{\Phi} + \vartheta_1)\sin(\overline{\Phi} + \vartheta_2) + \frac{\omega_0}{A^2 - B^2} \times [(CA - DB)|m_1||m_2| \sin(\vartheta_1 - \vartheta_2) + {}^1/_2(CB - DA)(|m_1|^2 \times \sin 2(\overline{\Phi} + \vartheta_1) - |m_2|^2 \sin 2(\overline{\Phi} + \vartheta_2))] - \frac{\omega_0^2}{(A^2 - B^2)^2}\{[(CA - DB)^2 + (CB - DA)^2]|m_1||m_2| \cos(\overline{\Phi} + \vartheta_1)\cos(\overline{\Phi} + \vartheta_2) + (CA - DB) \times (CB - DA)[|m_1|^2 \cos^2(\overline{\Phi} + \vartheta_1) + |m_2|^2 \cos^2(\overline{\Phi} + \vartheta_2)]\}$$

In particular, at equal coupling coefficients, i.e., when $|m_1| = |m_2| = |m|$ and $\vartheta_1 = \vartheta_2 = \vartheta$, we obtain

$$M_1 M_2 = |m|^2 \sin^2(\overline{\Phi} + \vartheta) - \frac{M^2}{4} \qquad M = 2\omega_0 \frac{C-D}{A-B}|m|\cos(\overline{\Phi} + \vartheta).$$

In another particular case, when the moduli of the coupling coefficients are equal, and the phases differ by π, we have

$$M_1 M_2 = \frac{\omega_0^2(C+D)^2}{(A+B)^2}|m|^2 \cos^2(\overline{\Phi} + \vartheta_1) - \frac{M^2}{4},$$
$$M = -2|m|\sin(\overline{\Phi} + \vartheta_1).$$

From this we can conclude that near the center of the synchronization band, when $\sin(\overline{\Phi} + \vartheta) \approx 0$ in the former case and $\cos(\overline{\Phi} + \vartheta_1) \approx 0$ in the latter case, the spectral density of the frequency fluctuations at zero turns out to be much smaller than at $\omega \gtrsim \omega_0 M/2Q$. The value of the spectral density of the fluctuations of the frequency of zero is approximately

$$(\delta\dot{\varphi}_i^2)_0 \approx (\delta\dot{\varphi}_i^2)_0^{(0)} - {}^1/_4(\delta\dot{\Phi}^2)_0^{(0)}.$$

Near the boundary of the synchronization band, to the contrary, M is close to zero, and $M_1M_2 > 0$, and conse-

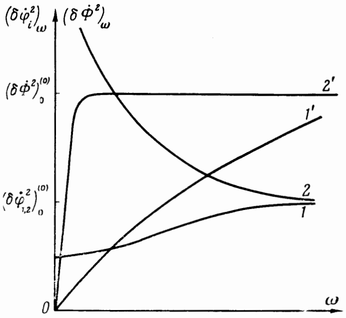

FIG. 2. Plots of the spectral densities of the frequencies of the opposing waves (curves 1, 2) and of the frequency difference between them (curves 1′, 2′) near the center of the synchronization band (1, 1′) and near its boundary (2, 2′).

quently the spectral density of the frequency fluctuations at zero turns out to be very large. Approximate plots of the dependence of the spectral densities of the frequencies of the opposing waves and of the frequency difference between them near the center of the synchronization band and near its boundary are shown in Fig. 2.

Let us determine now the mean-square phase advance of each of the opposing waves and the phase difference between them within the time τ. It follows from (4.3) that the mean-square phase advance within a time τ much larger than the correlation times of the amplitudes and of the noise sources, is equal to

$$\langle\delta\Phi_\tau^2\rangle = \frac{2Q}{\omega_0 M}\left(1-\exp\left\{-\frac{\omega_0}{2Q}M\tau\right\}\right)(\delta\dot\Phi^2)_0^{(0)}. \qquad (4.6)$$

Thus, at $\tau \ll 2Q/\omega_0 M$ the mean-square phase shift varies in accordance with the diffusion law, and the diffusion coefficient of the phase difference does not depend on the coupling and is equal to $D_\Phi^{(0)}$. On the other hand, if $\tau \gg 2Q/\omega_0 M$, then the mean-squared phase shift tends to a constant quantity.

Let us determine now the width of the fluctuation band of the phase difference. It is easy to see that the principal role in the calculation of the width of the phase-fluctuation band is played by the value of the diffusion coefficient at the instants of time $\tau \approx 1/D$. We can therefore conclude that if $\omega_0 M/2Q \ll D_\Phi^{(0)}$ (this inequality always holds near the boundary of the synchronization band), then the width of the phase-difference fluctuation band is the same as in the absence of couplings. In the opposite case, the phase-difference fluctuation band is a very narrow line (a delta function in the limit). Let us consider further the mean-square phase advance of each of the opposing waves. From (4.1) and (4.3) it follows that

$$\langle\delta\varphi_{i\tau}^2\rangle = \langle\delta\varphi_{i\tau}^2\rangle^{(0)} + \frac{M_1M_2}{M^2}(\delta\dot\Phi^2)_0^{(0)}\tau - 2\frac{M_1M_2Q}{M^3\omega_0}\left(1-\exp\left\{-\frac{M\omega_0}{2Q}\tau\right\}\right)(\delta\dot\Phi^2)_0^{(0)}.$$

Thus, at $\tau \ll 2Q/\omega_0 M$ the phase advance of each of the opposing waves varies in accordance with a diffusion law with the same diffusion coefficient as in the absence of coupling, i.e., $D_\varphi = D_\varphi^{(0)}$. When $\tau \gg 2Q/\omega_0 M$, the mean-squared phase advance is equal to

$$\langle\delta\varphi_{i\tau}^2\rangle = (\delta\dot\varphi_i^2)_0 - 4\frac{M_1M_2Q}{M^3\omega_0}D_\Phi^{(0)}.$$

It follows therefore that $\omega_0 M/2Q \ll D_\varphi^{(0)}$, then the line width of each of the waves is equal to the line width in the absence of couplings, i.e., $\Delta\omega_{ph} = D_\varphi^{(0)}$. This relation is always valid near the limit of the synchronization band. On the other hand, if

$$\frac{\omega_0 M}{2Q} \gg D_\varphi^{(0)} + \frac{M_1M_2}{M^2}D_\Phi^{(0)},$$

then the line width of each of the opposing waves is determined by the spectral density of the frequency fluctuations of these waves at zero:

$$\Delta\omega_{ph} = (\delta\dot\varphi_i^2)_0/2.$$

We can conclude from the foregoing that in the presence of coupling between the waves the spectral density of the frequency fluctuations of each of the opposing waves and of the frequency difference between the waves are not constant quantities. In addition, the line widths of the generated waves are not always determined by the value of the spectral density of the frequency fluctuations of these waves at zero frequency. Near the limit of the synchronization band, the line width of each of the waves and the width of the fluctuation band of the phase difference are equal to the spectral density of the frequency fluctuations and the frequency difference, respectively, far from zero. Near the center of the synchronization band, the line width at sufficiently large coupling is equal to the spectral density of the frequency fluctuations at zero, which is very difficult to measure because of the presence of technical fluctuations. In the intermediate case, when the coupling is insufficiently large or else the frequency deviation is closer to the limit of the synchronization band, the law governing the mean-squared phase advance differs from a diffusion law, and we obtain for the line width a more complicated expression. The foregoing conclusions are very important, since all the methods known to us of measuring the line width are based on measurements of the spectral density of the frequency fluctuations.

We do not consider here the case when the frequency deviation Ω lies outside the synchronization region. An examination of this case is of great interest from the point of view of practical applications, but entails great mathematical difficulties. It is clear from simple considerations that if the frequency difference Ω is much larger than the width of the synchronization band, then the coupling between the waves via the back scattering will have a slight effect. The results obtained without allowance for the coupling will then be valid.

The results of the present paper were discussed with Yu. L. Klimontovich. I take the opportunity to express my gratitude to him.

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Translated by J. G. Adashko
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