PLASMA DIFFUSION IN TOROIDAL SYSTEMS

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Turbulent plasma diffusion due to the gravitational dissipative instability in toroidal systems is considered. Expressions are obtained for the diffusion coefficient and the thermal conductivity for both low and high values of the electron collision frequency. It is found that the values of the diffusion coefficient and thermal conductivity are significantly greater than the corresponding classical values. The question of plasma diffusion due to weak violations of magnetic surfaces is also discussed.

1. INTRODUCTION

PLASMA diffusion in a toroidal system is an extremely complicated process. In addition to the natural transverse diffusion associated with the displacement of electrons in each collision by a distance of the order of the Larmor radius, in toroidal systems a significant contribution to the particle flux across the magnetic surface comes from the characteristic convection process.[1-3] This convection arises as a consequence of the small separation of charges due to the toroidal drift. Although a strong charge separation does not occur in the presence of a rotational transform, since it is compensated by current flow along the lines of force, a small transverse electrical field still remains as a consequence of the finite conductivity of the plasma. This field leads to motion of the plasma in the outward direction. At the outermost trajectory the plasma moves beyond the magnetic surfaces. On the other hand, the inner trajectories lie within the magnetic surfaces, so that to a first approximation in the curvature the flux averaged over the magnetic surfaces vanishes and it is only in higher order that there is a nonvanishing transverse flux.[1-3] In a rarefied plasma this process is complicated by the fact that the basic contribution to the convective flux comes from trapped particles.[4-5]

There is some question as to whether this theoretical laminar-convection picture is actually realized in experimental devices or whether it is really replaced by turbulent convection, as is the case in a conventional fluid with low viscosity. Since the inhomogeneity of the magnetic field that leads to the laminar convection is also the origin of a physically similar effect, the so-called gravitational dissipative instability[6,7] (the toroidal analog of the well-known flute instability), it appears that the turbulent-convection regime is more probable. The present paper is devoted to an investigation of this question.

2. GRAVITATIONAL INSTABILITY

In a dense plasma the gravitational instability leads to a local convective flux of the same order of magnitude as the laminar convection. But since the flux only appears in regions in which the field diminishes toward the outside, the corresponding average flux through the magnetic surface is not quadratic, but is linear in the curvature; hence the total flux is R/a times larger than the laminar flux (R is the major radius of the torus and a is the minor radius). In a rarefied plasma the effect can be still stronger.

For reasons of simplicity we shall consider a circular torus typical of a tokomak device. The magnetic field in this system obeys the following relation along a line of force:

$$B_z = B(1 - \varepsilon \cos k_0 l), \tag{1}$$

where l is the coordinate along the line of force and ϵ is the depth of modulation of the field, while $2\pi/k_0$ is the period of the field, where $\epsilon = r/R$, $k_0 = 1/qR$ where r is the distance from the magnetic axis, R is the radius of curvature, $q = rB_z/RB_\theta$, B_θ is the azimuthal magnetic field and l is computed for the outermost trajectory. The equation for the potential for small oscillations of the system can be obtained from the neutrality condition by equating the ion and electron densities $n_i' = n_e'$. Below we will treat oscillations with longitudinal phase velocities that are greater than the ion thermal velocity. In this case the quantity n_i' is given by the following expression, which derives from the theory of drift waves:[8]

$$n_i' = \left(\frac{\omega_*}{\omega} + \frac{\omega_*\omega_{m_i}(l)}{\omega^2} - \frac{c_s^2}{\omega^2}\frac{\partial^2}{\partial l^2}\right)\frac{en_0}{T_e}\varphi, \tag{2}$$

where $\omega_* = (-cT_e k_y/eBn_0)\, dn_0/dr$ is the drift frequency, $k_y = m/r$ is the azimuthal wave number, $\omega_{mi} = k_y cT_i \cos k_0 l/eBR$ is the frequency associated with the ion magnetic drift and $c_s = \sqrt{T_e/m_i}$.

Because of their high longitudinal mobility the electrons maintain a Boltzmann distribution along the lines of force. Hence the electron density perturbation can be written in the form

$$n_e' = \frac{en_0}{T_e}\varphi + \frac{en_0}{T_e}\hat{K}\varphi, \tag{3}$$

where the second term takes account of the small departure from the Boltzmann distribution for the perturbed density; $\hat{K}$ is an integral operator whose exact form will be given below for various values of the collision frequency ν_e.

By equating n_i' and n_e' we obtain the equation for φ:

$$\left(\frac{\partial^2}{\partial s^2} + A\cos s - E\right)\varphi = \hat{K}\varphi,$$
$$s = k_0 l, \quad A = \varepsilon\left(\frac{\omega_*}{k_0 c_s}\right)^2, \quad E = \frac{\omega - \omega_*}{\omega_*}\left(\frac{\omega_*}{k_0 c_s}\right)^2. \tag{4}$$

It can easily be shown that the nature of the solution of this equation depends on the dimensionless quantity $A = \epsilon(\omega_*/k_0 c_s)^2$. If A is small, that is to say, if in one oscillation period the ions can move a distance greater than the period of the modulated field, the equation describes drift waves and ion-acoustic waves which are coupled to each other because of the dependence of the magnetic drift frequency on s. In taking account of dissipation, for example the electron friction on the ions, it is possible for these waves to be excited. This is the so-called case of strong coupling, according to the terminology of[9]. In this case the oscillations are characterized by long wavelengths along the lines of force; these oscillations can be affected by shear and average minimum B.

The case of weak coupling $A > 1$ is more interesting. In this case the cosine in Eq. (4) can be expanded and the expansion can be limited to two terms. As a result we obtain an equation like the Schrödinger equation for a harmonic oscillator (neglecting the small terms associated with $\hat{K}\varphi$); the solutions of this equation are localized in the region of minima of the magnetic field. As a consequence these are completely insensitive to shear and average minimum B. The growth rates for these perturbations are appreciably greater than for small A and are determined completely by the non-Boltzmann part of the perturbed electron density

$$\omega = \omega_*(1-\varepsilon); \quad \gamma = \omega_* \langle \varphi_0 \hat{K}\varphi_0 \rangle / \langle \varphi_0 \varphi_0 \rangle, \tag{5}$$

where the angle brackets denote an average over s while φ_0 denotes the solution of Eq. (4) without the right side (when $A > 1$ this is the Hermite function).

We now consider the actual form of the operator $\hat{K}$. In a dense plasma, in which the mean free path is much shorter than the length of the system and in which the hydrodynamic description applies, $\hat{K}$ is of the form

$$\hat{K}\varphi = -\int \frac{\omega - \omega_*}{\omega + iDk^2 - \omega_{me}} e^{ikl} \varphi_k \, dk, \tag{6}$$

where $D = T_e/m_e\nu_e$ is the longitudinal diffusion coefficient, k is the longitudinal wave number, $\omega_{me} = \omega_{mi}T_e/T_i$. In order to estimate the growth rate (5) we note that the characteristic values of k that correspond to the solution φ_0 for $A > 1$ are of order $k \sim k_0 A^{1/4}$ so that when $|K|$ is small, that is to say, when $Dk^2 > 1$, we have

$$\gamma \sim \varepsilon\omega_*^2 / Dk_0^2 A^{1/2} \text{ for } \nu_e > \nu_2 = k_0 v_e. \tag{7}$$

The maximum growth rate with respect to A obtains when $A \sim 1$ and the corresponding transverse wavelengths are determined by the relation $k_\perp \rho_i \sim k_0 a/\sqrt{\epsilon} \sim \sqrt{\epsilon}/q$.

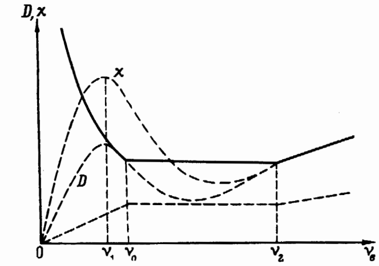
As the electron collision frequency is reduced the growth rate (7) diminishes and when $\nu_e = \nu_2 = k_0 v_e$ the transition to the collisionless case occurs. At smaller values of ν_e the collision frequency does not appear in the result and need be taken into account only in determining the proper direction for traversing the pole. In this region ($\nu_e < k_0 v_e$) the electron deviation from the Boltzmann distribution is due to the resonance interaction with the wave. Under these conditions, in the case $dT/dr = 0$, to which we limit ourselves,

$$\hat{K}\varphi \sim i\sqrt{\pi}\,\varepsilon\varphi \frac{\omega_*}{k_0 v_e}, \tag{8}$$

while the corresponding growth rate is of order

$$\gamma \sim \varepsilon\omega_*^2 / k_0 v_e. \tag{9}$$

The expression in (9) applies when ν_e is reduced to



the value $\nu_e = \nu_0 \equiv \epsilon\omega_b$ where ω_b is the bounce frequency for trapped particles between the magnetic mirrors in the field (1). In order-of-magnitude terms $\omega_b \sim \sqrt{\epsilon} k_0 v_e$. When $\nu_e < \nu_0$ the basic contribution to the non-Boltzmann part of the perturbation in the electron density comes from trapped electrons.[10] As shown in[10], in this case $\hat{K}$ is given by

$$\hat{K}\varphi = -\frac{\sqrt{\varepsilon}(\omega - \omega_*)}{\omega - \omega_{me} + i\nu_{\text{eff}}} \hat{K}_0\varphi, \tag{10}$$

where the factor $\sqrt{\epsilon}$ denotes the fraction of trapped particles, $\nu_{\text{eff}} = \nu_e/\epsilon$ is the effective collision frequency for the trapped particles, and K_0 is an integral operator of order unity which is localized in the region of the minimum of the magnetic field i.e., in the region of localization of the solutions of Eq. (4). This factor can be replaced by unity with reasonable accuracy. Using Eq. (10) and the frequency expression (5) we can find the growth rate for oscillations characterized by $\omega < \nu_{\text{eff}}$:

$$\gamma \cong \varepsilon^{5/2}\omega_*^2 / \nu_e \text{ for } \nu_e < \nu_0. \tag{11}$$

As the collision frequency is reduced still further the instability gradually becomes the trapped-ion instability.[11]

3. DIFFUSION AND THERMAL CONDUCTIVITY

Thus, over a wide range of values of the electron collision frequency there is an instability associated with the fall off of the magnetic field outward from the plasma into the region of the outermost trajectory. Since the growth rate increases with $k_\perp$, the small scale perturbations grow more rapidly than the large scale perturbations, so that it is reasonable to expect turbulent convection. The corresponding diffusion can be estimated as $D \sim \gamma/k_\perp^2$. The dependence of this coefficient on ν_e is shown in the figure by the solid line. In the region $\nu_e > \nu_0$ this coefficient is ϵ^{-1} times larger than the classical coefficient[4], which is shown in the figure by the dashed line. This difference is associated with the fact that the turbulent flux is of the order of the laminar flux, but differs from zero only in the region in which the field falls off, so that the total flux does not contain any small factor ϵ as in the case of laminar convection. When $\nu_e < \nu_0$ the coefficient D increases as ν_e diminishes; this effect is associated with the transport of trapped electrons in the drift waves.

We now consider the effect of a temperature gradient on the convection. In the collision-dominated regime $\nu_e > \nu_2$ this effect is small so that the diffusion in the presence of a temperature gradient is essentially the same as before; furthermore, there is a thermal flux with a thermal conductivity χ of the order of D. The low collision frequency regime is more sensitive to the presence of a temperature gradient. When $\nu_0 < \nu_e < \nu_2$

the temperature gradient coincides in sign with the density gradient and plays a stabilizing role so that the instability is stabilized when $\eta = d \ln T/d \ln n > 1/2$; furthermore, one can expect a transition to classical diffusion due to the trapped particles. Stabilization also occurs in the region $\nu_e > \nu_0$, where it appears in the nonlinear regime and is associated with the electrostatic trapping of the electrons by drift waves.[12] As has been shown in[12] this effect leads to small-scale convection with a thermal conductivity $\chi \approx \epsilon^{3/2} a^2 \nu_e (1 + \nu_e^2/\nu_1^2)^{-1}$ where $\nu_1 = cT/eBa^2$. The coefficient χ is ϵ^{-1} times larger than the corresponding diffusion coefficient that arises by virtue of the small phase shift, due to magnetic drift, between the density fluctuations n' and the fluctuation of the potential φ'. Thus, in the presence of a temperature gradient characterized by $\eta \sim 1$ the functional dependence of χ and D on ν_e must have the form shown qualitatively in the figure by the dashed line.

On the other hand, the presence of a longitudinal current leads to destablization, especially in a dense plasma.[8] In particular, if the directed velocity of the electrons $u = j/en$ exceeds the acoustic velocity $c_s = \sqrt{T_e/m_i}$ the stabilizing role of the temperature gradient in the region $\nu_0 < \nu_e < \nu_2$ moves out to a second stage and one can expect a noticeable increase in D and χ as compared with the values shown in the figure. In the region $\nu_e > \nu_2$ there is also an enhanced thermal conductivity that arises by virtue of the current-convective instability.

In the hydrodynamic region $\nu_e > \nu_0$ the instability being considered is characterized by very short wavelengths $\lambda_\perp \sim q\rho_i/\sqrt{\epsilon}$; for this reason it is sensitive to the ion viscosity. It can be shown that if the ion mean free path is long, specifically, if the following condition is satisfied:

$$\frac{\varepsilon}{q} > \left(\frac{m_e}{m_i}\right)^{1/4}\left(1+\frac{q^2}{\varepsilon}\left(\frac{\lambda_i}{qR}\right)^2\right)^{-1/2},$$

then the viscosity stabilizes the instability.

Thus, we have shown that a drift-type instability in a toroidal plasma always causes turbulent convection and in a rarefied plasma ($\nu_e \lesssim \nu_2$) this convection leads to diffusion and thermal conductivity that are not very much greater than the classical laminar values. Specifically when $\nu_1 \sim \nu_2$ as is the case, for example, in tokomak devices, when $u \sim c_s$ roughly speaking the turbulent diffusion coefficient is R/a times greater than the classical value while the electron thermal conductivity associated with turbulence is $(R/a)^2$ times larger. In a dense plasma $\nu_e \gtrsim \nu_2$, we again find that D and χ are increased; however, the actual loss of energy from the plasma can be determined by the ion thermal conductivity, which is much greater than the electron thermal conductivity. These conclusions are found to be in qualitative agreement with the results obtained with tokomak devices.[13-14]

4. DIFFUSION DUE TO WEAK VIOLATIONS OF MAGNETIC SURFACES

In making a comparison of various plasma loss mechanisms (classical diffusion, instability, etc.) it is necessary to take account of the possibility of imperfections in the actual magnetic surfaces, that is to say, violations of the magnetic surfaces due to external perturbations. In an earlier work one of the authors[15] has considered the question of plasma confinement in toroidal systems with rather strong perturbations of the surfaces, in which the surfaces do not close inside the plasma. It was shown that in this case the plasma is subject to a centrifugal instability which leads to plasma losses of the order of the Bohm value.

Here we analyze the case of a weaker violation of the surface, in which the widths of the islands and ripples and the distortions of the magnetic surfaces[16] are not large compared with the ion Larmor radius. In this case, the magnetic (more precisely, drift) surfaces are conserved approximately for the ions. However, because of their small Larmor radius and high mobility along the lines of force we find that the electrons can be displaced rapidly along the magnetic trap. We assume that the perturbation of the magnetic field is stochastic[17], that is to say, each line of force in circulating eventually occupies the entire available volume of the trap. We assume furthermore that the perturbation of the surface is so large that the electrons can circulate rather freely across the average magnetic field. Specifically, let δ^2 be the mean square displacement of a line of force in a distance L (of order of the length of the torus) along which the correlation of the perturbation is lost, so that the electron diffusion coefficient $D_\perp = \delta^2 v_e/L$ where $v_e = \sqrt{T/m_e}$ is the electron thermal velocity. We will also assume that $D_\perp > \rho_e v_e$ i.e., $\delta > \sqrt{\rho_e L}$, where $\rho_e = v_e m_e c/eB$ is the mean electron Larmor radius. Consequently, the electron diffusion is more rapid than the drift in the electric field. In this case, which in some sense is a limiting case, the electrons reach equilibrium rapidly and must obey a Boltzmann distribution

$$T\nabla n = en\nabla\varphi, \tag{12}$$

where T is the electron temperature, n is the electron density and φ is the potential of the electric field. Thus, the problem reduces to an investigation of ion confinement in a trap in the presence of an electric field given by (12). In order to simplify the analysis we assume that the ion temperature is much lower than the electron temperature so that the hydrodynamic equations can be used in the description of the ion motion:

$$m_i n\frac{d\mathbf{v}}{dt} = -T\nabla n + \frac{en}{c}[\mathbf{vB}] + m_i n\nu\nabla_\parallel^2 \mathbf{v}, \tag{13*}$$

where ν is the longitudinal viscosity. Together with the equation of continuity

$$\partial n/\partial t + \operatorname{div} n\mathbf{v} = 0 \tag{14}$$

Eqs. (12) and (13) determine completely the motion of the plasma. We consider an axisymmetric toroidal trap. Assuming that the curvature is small $\epsilon = r/R \ll 1$ (r is the minor radius and R is the major radius of the torus) we seek a solution of Eqs. (1)–(3) in terms of an expansion in the parameter ϵ: $n = n_0 + n' + \ldots$, $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}' + \ldots$. In the zeroth approximation in ϵ we have a straight plasma pinch in which all quantities depend

*$[\mathbf{vB}] \equiv \mathbf{v} \times \mathbf{B}$.

only on the radius. Assuming that the inertia term and the viscosity term in (13) are small we find the rotational velocity $v_0 = (cT/eBn_0)(dn_0/dr)$.

In order to examine the next approximation we substitute the expression for the transverse velocity in the equation of continuity:

$$v_\perp = \frac{cT}{eB^2 n}[\mathbf{B}\nabla n]$$

and equate terms of order ϵ to find

$$\frac{\partial n'}{\partial t} - \varepsilon \frac{\sin\theta}{r}\frac{cT}{eB}\frac{dn_0}{dr} + \frac{B_\theta}{B}\frac{1}{r}\frac{\partial}{\partial\theta} n_0 v_\parallel' = 0, \qquad (15)$$

where B_θ is the azimuthal component of the magnetic field, which is assumed to be small. The second term in this expression arises from the toroidal correction to the longitudinal field $B_z = B(1 - \epsilon\cos\theta)$.

The equation for the longitudinal velocity component $v_\parallel'$ can be obtained by multiplying Eq. (13) by **B** and linearizing with respect to ϵ:

$$n_0 m_i \frac{v_0}{r}\frac{\partial}{\partial\theta} v_\parallel' + \frac{B_\theta}{B}\frac{T}{r}\frac{\partial n'}{\partial\theta} = m_i n_0 \nu \frac{B_\theta^2}{B^2}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} v_\parallel'. \qquad (16)$$

Here, ν is the longitudinal viscosity. In Eq. (16) we have neglected the small term $n_0 m_i \partial v_\parallel'/\partial t$.

The average plasma flux across the plasma is determined by the relation

$$q = \langle (n_0 + n')v_r'(1 + \varepsilon\cos\theta)\rangle = \varepsilon n_0 \langle v_r' \cos\theta\rangle, \qquad (17)$$
$$v_r' = -\frac{cT}{eBn_0 r}\frac{\partial n'}{\partial\theta}$$

where the angle brackets denote averages with respect to θ.

In Eq. (17) we have taken account of the fact that the quantity $\langle v_r' n'\rangle$ vanishes. Thus, in the determination of the flux it is sufficient to find n' from Eqs. (15) and (16). The simplest procedure is to write the quantities n' and $v_\parallel'$ in the form $n' = n_1\cos\theta + n_2\sin\theta$ and $v_\parallel' = v_1\cos\theta + v_2\sin\theta$. We then find that n_1 and n_2 are described by a system consisting of two linear first-order differential equations in the time. This system has a stable stationary solution

$$n_1 = \varepsilon\frac{B^2}{B_\theta^2}\frac{v_0^2 m_i}{T} n_0; \qquad n_2 = \varepsilon\frac{\nu v_0 m_i}{rT} n_0. \qquad (18)$$

In this case the particle flux is found to be $q = D dn_0/dr$ where $D = 16\epsilon^2 S D_B$, $D_B = 1/16\ cT/eB$ is the Bohm diffusion coefficient and S is the parameter defined by $S = \nu m_i c/r^2 eB$.

It is evident from Eq. (18) that the longitudinal ion inertia leads to the displacement of the pinch along the major radius ($n_1 > 0$) while the viscosity causes a displacement along the axis of symmetry of the torus ($n_2 \neq 0$).

It should be recalled that Eq. (18) applies only for small displacements, in which case $n_1, n_2 \ll n_0$. In the hydrodynamic approximation (mean free path smaller than R and $\rho < r$) the quantity n_2 is small. However, the quantity n_1 can become fairly large for very small values of B_θ/B. This feature is due to the fact that the ion current does not close very well along the lines of force because of the longitudinal inertia. In particular, if the longitudinal inertia is very large then when $\nu = 0$ Eq. (2) describes the transverse motion of a single component ideal gas of charged particles.

It is clear that in an inhomogeneous magnetic field these particles will escape to the walls with the drift velocity. For the initial time interval, in which the displacement of the plasma with respect to the magnetic surfaces is still small, the plasma flux is proportional to the time so that

$$\frac{\partial q}{\partial t} = -\frac{1}{2}\frac{1}{R^2}\frac{c^2T^2}{e^2B^2}\frac{dn_0}{dr}$$

It then follows that the plasma is lost from the trap in a time $t \sim \epsilon^{-1} r^2 eB/cT$ which, in an unprecise observation, can be taken to be the Bohm diffusion time.

The analysis given here shows that in all cases in which the observed plasma loss is of the order of the Bohm value it is necessary to make a careful investigation of the quality of the magnetic surfaces before investigating more complicated loss mechanisms.

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