NONLINEAR INTERACTION OF A LOW-DENSITY RELATIVISTIC ELECTRON BEAM WITH A PLASMA

R. I. KOVTUN and A. A. RUKHADZE

P. N. Lebedev Physics Institute, USSR Academy of Sciences

Submitted November 5, 1969

Zh. Eksp. Teor. Fiz. 58, 1709–1714 (May, 1970)

The propagation of a stationary electrostatic wave of finite amplitude in an infinite plasma through which a low-density relativistic electron beam moves is investigated. The dispersion equation which relates the frequency with the wave vector and the amplitude of the wave field is derived. In the limit of infinitesimally small amplitudes this equation becomes the dispersion equation of the linear theory and contains the two-stream instability, so that the wave amplitude increases with time. At large amplitudes, for which the electrons in the beam can be trapped by the wave, the two-stream instability is stabilized and the only solutions of the dispersion equation correspond to stationary nonlinear oscillations.

1. INTRODUCTION AND SUMMARY

It is well known that a strong electrostatic instability can develop in the interaction of an electron beam with a plasma when the directed velocity of the beam is greater than the thermal velocity of the particles.[1,2] Many authors have investigated the consequences that arise in a plasma as a consequence of this instability. In particular, a large number of papers are devoted to the quasilinear relaxation of the beam in the plasma.[3-5] This approach is applicable if the thermal spread of the velocities of the beam electrons is large so that the instability can be described in terms of a kinetic theory. On the other hand, in most experiments an essentially monoenergetic electron beam interacts with the plasma; in this case the two-stream instability is hydrodynamic and the quasilinear theory does not apply, at least in the early stages of the development of the instability.

It is of interest to formulate a nonlinear theory for the interaction of a beam with a plasma and to investigate the consequences of the development of the hydrodynamic instability taking account of the fact that in a low-density beam (under certain specified conditions) it is possible for a single-mode instability to develop, i.e., only that mode grows whose growth rate is a maximum. For example, this is the case when the beam is weakly modulated at a frequency that corresponds to the plasma frequency of the plasma[6] (or, what is essentially the same thing, when an electrostatic wave at the plasma frequency is launched in the beam-plasma system). Using the asymptotic method of Krylov and Bogolyubov[7] for a single oscillatory mode it is possible to obtain a generalized dispersion equation that takes account of the finite wave amplitude. Analysis of this equation shows that the oscillations are unstable at low amplitudes so that the amplitude increases with time. On the other hand, if the amplitude of the wave exceeds some critical value, which is determined by the parameters of the beam and the plasma, the generalized dispersion equation admits only real solutions, which correspond to a stationary nonlinear wave. Starting from this picture it may be assumed that in the interaction of a low-density beam with a plasma an electrostatic wave with wavelength corresponding to the maximum growth rate for the two-stream instability will grow until the wave amplitude reaches the critical value.

The critical field for the wave can be estimated easily from simple physical considerations. It is known from the linear theory that in the absence of an external magnetic field (or with an infinite magnetic field, in which case the oscillations become one-dimensional) the wave with the maximum growth rate is the one for which $ku = \omega_{L0}$ where k is the wave vector, u is the acoustic velocity and $\omega_{L0} = [4\pi e^2 n_0/m]^{1/2}$ is the plasma frequency; the maximum growth rate is given by

$$\gamma_{max} = |\omega - ku| = \frac{\sqrt{3}}{2}\left(\frac{n_1}{2n_0}\right)^{1/3}\omega_{L0}\gamma_0^{-1},$$

where n_1 and n_0 are respectively the electron density in the beam and the plasma in the laboratory coordinate system (the system fixed in the plasma), ω is the frequency of the wave, and $\gamma_0 = (1 - u^2/c^2)^{-1/2}$. It is evident that the beam will excite a wave in the plasma i.e., its energy will be transferred to the plasma, if the beam energy in the system fixed in the wave is greater than the peak potential in the wave. This condition can be written in the form

$$\frac{m}{2}\left(u - \frac{\omega}{k}\right)^2 \gamma_0^4 > e\Phi_0\gamma_0,$$

where Φ_0 is the peak potential in the wave in the laboratory coordinate system.[1)] In the opposite limit the electrons in the beam are trapped by the wave and there is no relative motion of the beam with respect to the wave.

[1)]If there is to be no broadening of the wave spectrum owing to the excitation of thermal oscillations by the beam the following condition must be satisfied:

$$\frac{1}{L} > \frac{\sqrt{3}}{2}\frac{\omega_{L0}}{u}\left(\frac{n_1}{2n_0}\right)^{1/3}\frac{1}{\gamma_0}\left(\ln\frac{\Phi_0}{\Phi_T}\right)^{-1/2} > \frac{1}{L}\left(\ln\frac{\Phi_0}{\Phi_T}\right)^{-1/2},$$

where L is the linear dimension of the system and Φ_T is the amplitude of the thermal noise. It follows, in particular, that $\Phi_0 \gg \Phi_T$. Strictly speaking, it is only when this condition is satisfied that the one-dimensional approximation can be used.

Consequently, there is no energy exchange between the beam and the wave, both remaining in a stationary state. Equating the kinetic energy of the beam and the potential of the wave we find the following steady-state amplitude:

$$e\Phi_0 \approx \frac{m}{2}\gamma_{max}^2\gamma_0^3 = \frac{3}{8}mu^2\gamma_0\left(\frac{n_1}{2n_0}\right)^{2/3}. \quad (1.1)$$

The relative energy density of the electrostatic field in the plasma is given by the relation

$$\frac{E_0^2}{8\pi n_1 mc^2\gamma_0} \approx \frac{9}{256}\left(\frac{n_1}{2n_0}\right)^{1/3}\beta^2\gamma_0, \quad (1.2)$$

where $\beta = u/c$. It is evident that this ratio will also characterize the relative energy spread of the electrons in the beam as the instability develops. It follows from Eq. (1.2) that when

$$(n_1/n_0)^{1/3}\gamma_0\beta^2 \ll 1 \quad (1.3)$$

the relative energy density of the electrostatic oscillations excited by the electron beam in the plasma will be small, as will the energy spread of the beam. This condition also supports the assumption made above concerning the nonrelativistic nature of the motion of the beam in the coordinate system fixed in the wave.

The estimates given above will be justified and treated analytically in what follows.

2. BASIC EQUATIONS

As indicated above, we are interested in one-dimension electrostatic oscillations in a system consisting of a relativistic, monoenergetic, low-density electron beam and a plasma. The hydrodynamic equations that describe this system are well-known:[2]

$$\begin{gathered}\frac{\partial^2\Phi}{\partial z^2} = 4\pi\sum_\alpha e(n_\alpha - N_0),\\ \frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial z}(n_\alpha v_\alpha) = 0,\\ \left(\frac{\partial}{\partial t} + v_\alpha\frac{\partial}{\partial z}\right)\frac{v_\alpha}{\sqrt{1 - v_\alpha^2/c^2}} = -\frac{e_\alpha}{m_\alpha}\nabla\Phi.\end{gathered} \quad (2.1)$$

Here α refers to the electrons in the beam and the plasma while N_0 is the density of the neutralizing ion background.

For steady-state oscillations, which are of interest here, all quantities are functions of the argument $\omega t - kz$. In view of this circumstance it is convenient to transform to a reference system fixed in the wave, in which the equations in (2.1) become stationary. It is then a simple matter to find two integrals of the motion:

$$\begin{gathered} n_{\alpha r}v_{\alpha r} = \nu_{\alpha r}u_{\alpha r},\\ \frac{mc^2}{\sqrt{1 - v_{\alpha r}^2/c^2}} + e\Phi_r = \frac{mc^2}{\sqrt{1 - u_{\alpha r}^2/c^2}},\end{gathered} \quad (2.2)$$

where $u_{\alpha r}$ is the velocity of the electrons and $\nu_{\alpha r}$ is the density of the electrons at points for which $\Phi_r = 0$ (the subscript r is used to denote quantities in the moving coordinate system). Introducing the notation

$$W_{\alpha r} = \left(1 - \frac{u_{\alpha r}^2}{c^2}\right)^{-3/2}\frac{mu_{\alpha r}^2}{2}, \quad (2.3)$$

from Eq. (2.2) we find

$$n_{\alpha r} = \nu_{\alpha r}\sqrt{\frac{W_{\alpha r}}{W_{\alpha r} - e\Phi_r}}. \quad (2.4)$$

In the derivation of this expression it is assumed that $e\Phi_r \ll mc^2$, as is the case for a low-density beam that satisfies (1.3). Substituting Eq. (2.4) in Poisson's equation in the moving coordinate system we have

$$\frac{\partial^2\Phi_r}{\partial z_r^2} = 4\pi e\sum_\alpha\left(\nu_{\alpha r}\sqrt{\frac{W_{\alpha r}}{W_{\alpha r} - e\Phi_r}} - N_{0r}\right). \quad (2.5)$$

We now take account of the fact that the velocity of the plasma electrons in the moving coordinate system coincides with the velocity of the wave at points for which $\Phi_r = 0$, that is to say, $\omega_{pr} = \omega/k$ while the electron velocity in the beam is given by

$$u_{br} = \left(u - \frac{\omega}{k}\right)\Big/\left(1 - \frac{\omega^2}{c^2k^2}\right).$$

We assume further that the wave being considered here corresponds to a wave characteristic of a beam-plasma system with a maximum growth rate

$$\gamma_{max} = |\omega - ku| = \frac{\sqrt{3}}{2}\gamma_0^{-1}\left(\frac{n_1}{2n_0}\right)^{1/3},$$

while $\omega \approx ku$. As a consequence of (1.3) the beam velocity u_{br} is nonrelativistic in this case and $W_{pr} \gg W_{br}$. On the other hand, it is evident from Eq. (2.5) that nonlinear effects must be considered for wave propagation in the plasma even when $e\Phi_r \sim W_{br}$ if $W_{pr} \gg e\Phi_r$, although the plasma term can be linearized. As a result, from Eq. (2.5) we have

$$\frac{\partial^2\Phi_r}{\partial z_r^2} + \frac{4\pi e^2\nu_{pr}}{m\omega^2}k^2\gamma_0^{-3/2}\Phi_r = 4\pi e\left[\nu_{br}\sqrt{\frac{W_{br}}{W_{br} - e\Phi_r}} - \nu_{0r}\right], \quad (2.6)$$

ν_{0r} is the density of the background which neutralizes the charge of the beam in the moving coordinate system. Thus, the problem of propagation of a wave of finite amplitudes in a beam-plasma system reduces to the analysis of Eq. (2.6) with a small nonlinearity $\nu_{br} \ll \nu_{pr}$.

3. NONLINEAR DISPERSION EQUATION

In analyzing Eq. (2.6) we must consider two cases: a) $W_{br} > e\Phi_{r0}$ and b) $W_{br} \le e\Phi_{r0}$, where Φ_{r0} is the peak potential in the wave Φ_r. In the first case the electrons in the beam can overcome the potential hills in the wave and the beam is displaced with respect to the wave, transferring part of its kinetic energy to the wave. In the second case the beam electrons are trapped by the wave and, on the average, the beam does not move with respect to the wave. It will be evident that in this case the beam cannot transfer its energy to the wave, so that the system reaches a stationary state.

$W_{br} > e\Phi_{r0}$. Using the asymptotic method of Krylov-Bogolyubov[7] we seek the solution of Eq. (2.6) with its small nonlinearity in the form $\Phi_r = \Phi_{0r}\cos\Psi = \Phi_{0r}\cos(k_r z_r + \vartheta)$, where Φ_{0r} and ϑ are slowly varying functions if (1.3) is satisfied. In the zeroth approximation in the smallness parameter these quantities can be regarded as constant.[2)] Substituting the solution in this form in Eq. (2.6), multiplying by cos Ψ and averaging over Ψ, we have

2)We note that (1.3) also implies that the higher harmonics are small.

$$1-\frac{\omega_{L0}^2}{\omega^2}-\frac{16e\nu_{br}}{k^2\Phi_{0r}}\gamma_0\frac{\sqrt{1-0{,}5\eta^2}}{\eta^2}[(2-\eta^2)K(\eta)-2E(\eta)]=0, \quad (3.1)$$

where $\nu_{pr} = n_0\gamma_0$, $k = k_r\gamma_0$, $\Phi_{0r} = \Phi_0\gamma_0$ and $K(\eta)$ and $E(\eta)$ are complete eliptical integrals with the modulus

$$\eta=\left\{\frac{2e\Phi_0}{{}^1/_2m\gamma_0^3(\omega/k-u)^2+e\Phi_0}\right\}^{1/2}. \quad (3.2)$$

In this form Eq. (3.1) is still not convenient since it contains the electron density of the beam ν_{br} at the points for which $\Phi_r = 0$. It is more convenient to convert to an average spatial beam density $\bar{n}_{br} = n_{1r}$, averaging Eq. (2.4) with respect to Ψ for this purpose (the modulation of the electron density in the plasma by the wave is negligibly small). As a result we find

$$\nu_{br}=\frac{\pi n_{1r}}{2K(\eta)}(1-0{,}5\eta^2)^{-1/2}. \quad (3.3)$$

If we also assume that $n_1 = \gamma_0 n_{1r}$, then Eq. (3.1) can be transformed to the final form

$$1-\frac{\omega_{L0}^2}{\omega^2}-\frac{\omega_{L1}^2}{\gamma_0^3(\omega-ku)^2+2em^{-1}k^2\Phi_0}\frac{8C(\eta)}{K(\eta)}=0, \quad (3.4)$$

where $C(\eta) = \eta^{-4}[(2-\eta^2)K(\eta)-2E(\eta)]$. When η varies from 0 to 1 the ratio $8C(\eta)/K(\eta)$ varies from 1 to 8.

Equation (3.4) represents the desired dispersion equation for the nonlinear wave in the beam-plasma system and relates the frequency ω, the wave vector k, and the wave amplitude Φ_0 when $W_{br} > e\Phi_r$. In the linear limit, in which $\Phi_0 \to 0$, so that $\eta \to 0$, this equation becomes the well-known dispersion relation of the linear theory.[2]

$W_{br} \leq e\Phi_{r0}$. In this case the electrons in the beam are trapped by the wave and execute bounded motion between the potential hills. Solving Eq. (2.6) we keep in mind the fact that the right side becomes infinite at the electron turning points i.e., the points for which $W_{br} = e\Phi_{or}\cos(k_r z_r + \vartheta)$. Hence, in carrying out the average over $\Psi = k_r z_r + \vartheta$ the turning points must be eliminated from the region of integration by a small interval (of order $\sqrt{n_1/n_0}$). The contribution of the poles at these turning points can be introduced by means of the Green's function and it is found that this procedure leads to a correction of order $(n_1/n_0)^{2/3}$, which is beyond the accuracy of the present approximation. The remaining procedure for obtaining the dispersion equation is completely analogous to that given above and we find

$$1-\frac{\omega_{L0}^2}{\omega^2}+2\frac{\omega_{L1}^2m}{e\Phi_0k^2}\left[1-2\frac{E(\eta)}{K(\eta)}\right]=0, \quad (3.5)$$

in which the modulus of the integrals is

$$\eta=\left\{\frac{{}^1/_2\gamma_0^3m(\omega/k-u)^2+e\Phi_0}{2e\Phi_0}\right\}^{1/2}. \quad (3.6)$$

It is evident that Eq. (3.5), in contrast with Eq. (3.4), only has real solutions ω, which correspond to stable stationary oscillations of the system. This result is completely reasonable because when $W_{br} < e\Phi_r$ the beam electrons are trapped by the waves so that there is no relative motion between the beam and the wave and no energy transfer.

4. ENERGY OF THE STEADY-STATE WAVE

Thus, we find that an electrostatic wave in a beam-plasma system can only reach a stationary state at sufficiently large amplitudes such that the electrons in the beam are completely trapped by the wave. It is of interest to determine the minimum value of the peak potential of the wave Φ_{0M} as a function of the parameters of the plasma and beam, thus determining the critical value at which the wave can become stationary. For this purpose we take the limit $\eta \to 1$ in Eqs. (3.4) and (3.5). Using these equations we find

$$e\Phi_{0M}=2mu^2\omega_{L1}^2/(\omega_{L0}^2-\omega^2). \quad (4.1)$$

It is then evident that the steady-state can be reached only at frequencies below the plasma frequency. Using the condition $\eta = 1$ and eliminating the frequency ω from Eq. (4.1) we obtain the following expression for the stationary amplitude of an electrostatic wave excited by the two-stream instability:

$$e\Phi_{0M}=2^{1/3}mu^2\left(\frac{n_1}{2n_0}\right)^{2/3}\gamma_0. \quad (4.2)$$

The relative energy density of an electrostatic field in the plasma is given by the relation

$$\frac{E_0^2}{8\pi n_1mc^2\gamma_0}=\frac{1}{2\cdot2^{1/3}}\left(\frac{n_1}{2n_0}\right)^{1/3}\beta^2\gamma_0. \quad (4.3)$$

The relations in (4.2) and (4.3) differ from the approximate relations given in (1.1) and (1.2) only by numerical factors. It is interesting to note that to an accuracy within numerical factors of order unity these results also coincide with the results of the quasilinear theory[5], which are valid when the beam excites a broad spectrum of oscillations in the plasma.

[1] A. I. Akhiezer, and Ya. B. Faĭnberg, Dokl. Akad. Nauk SSSR 69, 555 (1949); ibid. 75, 1851 (1951). Ya. B. Faĭnberg, Atomnaya Energiya (Atomic Energy) 11, 313 (1961).

[2] E. E. Lovetskiĭ and A. A. Rukhadze, Zh. Eksp. Teor. Fiz. 48, 514 (1965) [Sov. Phys.-JETP 21, 526 (1965)].

[3] A. A. Vedenov, E. P. Velikhov and R. Z. Sagdeev, Usp. Fiz. Nauk 73, 701 (1961) [Sov. Phys.-Uspekhi 4, 332 (1961)]; A. A. Vedenov, E. P. Velikhov and R. Z. Sagdeev, Nuclear Fusion 1, 82 (1961); A. A. Vedenov, Voprosy Teorii plazmy (Reviews of Plasma Physics), Atomizdat, 1963 [Consultants Bureau, New York, 1967].

[4] W. E. Drummond and D. Pines, Nuclear Fusion, Supp. 3, p. 1043, 1962.

[5] V. D. Shapiro and Ya. B. Faĭnberg, Vzaimodestvie puchkov zaryazhennykh chastits c plazmoĭ (Interaction of Charged-Particle Beams with Plasma), Acad. Sci. Uk.S.S.R. Press, Kiev, 1965, p. 92; V. D. Shapiro, V. I. Shevchenko and Ya. B. Faĭnberg, Zh. Eksp. Teor. Fiz. 57, 966 (1969) [Sov. Phys.-JETP 30, 528 (1970)].

[6] Ya. B. Faĭnberg, Czech. J. Physics 57, B18, 658 (1968).

[7] N. N. Bogolyubov and Yu. A. Mitropolskiĭ, Asimptoticheskie metody v teorii nelineĭnykh kolebaniĭ (Asymptotic Methods in the Theory of Nonlinear Oscillations) Fizmatgiz, 1958 [Gordon and Breach, New York, 1962].

Translated by H. Lashinsky
206