OHMIC RESISTANCE OF AN INHOMOGENEOUS PLASMA

B. N. BREĬZMAN, V. V. MIRNOV, and D. D. RYUTOV

Institute of Nuclear Physics, Siberian Division, U.S.S.R. Academy of Sciences

Submitted November 21, 1969

Zh. Eksp. Teor. Fiz. 58, 1770–1783 (May, 1970)

The flow of a current through an inhomogeneous plasma is considered under conditions when the electron mean free path considerably exceeds the characteristic inhomogeneity dimensions. It is shown that the presence of a large number of captured electrons leads to a strong increase of the ohmic resistance, compared with the case of a homogeneous plasma. The effective conductivity is calculated by simultaneously taking into account electron–electron and electron–hole collisions. The spatial distribution of the electric field applied to the plasma is found.

1. INTRODUCTION

THE theory of ohmic resistance of a fully ionized homogeneous plasma has been developed by now in great detail (see the review[1,2]). In the case of a Lorentz plasma the conductivity can be readily obtained analytically, and the use of numerical methods makes it possible to take into account also the effect of electron-electron collisions.

There is a well known theoretical deduction that the conductivity of a fully ionized plasma does not depend on its concentration. This means, in particular, that if the plasma density is not homogeneous in the direction of the external electric field, then the current in the plasma does not depend on the value of the inhomogeneity.

The latter statement is valid, however, only if the scale of the inhomogeneity L is large compared with the mean free path λ. As to the inverse limiting case

$$L \ll \lambda, \tag{1}$$

Insofar as we know, it has not been considered in detail to this day. We investigate it in the present paper.

We assume that the plasma concentration varies periodically along the z axis (with period 2L), and the constant electric field applied to the plasma is parallel to this axis. As to the external magnetic field, if it does exist at all, we assume that it is homogeneous and also directed along the z axis. Such a formulation of the problem corresponds well, for example, to many experiments on Joule heating of the plasma in toroidal systems (see, in particular, [3]). The extension of our results to the case when the plasma concentration is not a periodic function of the coordinate does not entail fundamental difficulties, but the corresponding calculations become too cumbersome. We shall therefore dwell below only on a study of the periodic problem. Further, we confine ourselves to an investigation of the steady state of the plasma, assuming the current to be stationary, a valid assumption only if the concentration profile is stationary.

In fact, under the influence of the pressure gradient connected with the inhomogeneity of the plasma, the concentration profile will vary, and the time scale of this process is equal approximately to $Lv_T^{-1}(M/m)^{1/2}$, where v_T is the thermal velocity of the electrons, and m and M are respectively the masses of the electron and of the ion. But since the transient time of the current is of the same order of magnitude as the reciprocal of electron-ion collision frequency $\nu^{-1} \sim \lambda/v_T$, then we can calculate the current under the condition

$$\frac{L}{v_T}\left(\frac{M}{m}\right)^{1/2} \ll \nu^{-1}, \tag{2}$$

which we assume to be satisfied, by regarding the concentration distribution as given. By combining inequalities (1) and (2) we obtain the following conditions for the applicability of the results presented below:

$$\frac{v_T}{L}\left(\frac{m}{M}\right)^{1/2} \ll \nu \ll \frac{v_T}{L}. \tag{3}$$

These inequalities will be made more precise in Sec. 7.

The influence of the inhomogeneities of the plasma on its resistance is connected with the presence in the plasma of a polarization electric current which balances the electron-pressure gradient. The potential of this field $\varphi(z)$ can be connected with the plasma density n(z) by the quasi-neutrality condition,[1)] assuming that the electrons in the equilibrium state have a Boltzmann distribution with a temperature T:

$$\varphi(z) = \frac{T}{e}\ln\frac{n(z)}{n_{max}}. \tag{4}$$

The arbitrary constant in formula (4) is chosen from the condition that the potential vanish where the plasma concentration is maximal ($n = n_{max}$). The functions n(z) and $\psi(z) = -e\varphi(z)/2m$, and the phase trajectories of the electrons moving in the polarization electric field, are shown in Fig. 1.

If the longitudinal energy of the particle $mv_{\parallel}^2/2 - e\varphi$ is smaller than the depth of the potential well $T\ln(n_{max}/n_{min})$, then the particle executes a finite motion between two neighboring peaks of the potential, corresponding to a closed trajectory on the phase plane (captured electrons). The motion of particles with large longitudinal energy ($mv_{\parallel}^2/2 - e\varphi > T\ln(n_{max}/n_{min})$) is infinite, and the corresponding phase trajectories are not closed. Such particles will henceforth be called transiting particles. In a plasma with small inhomogeneity scale ($L \ll v_T/\nu$) the contribution to the current is made, obviously, only by transiting electrons,

[1)]In any real situation, L greatly exceeds the Debye radius.

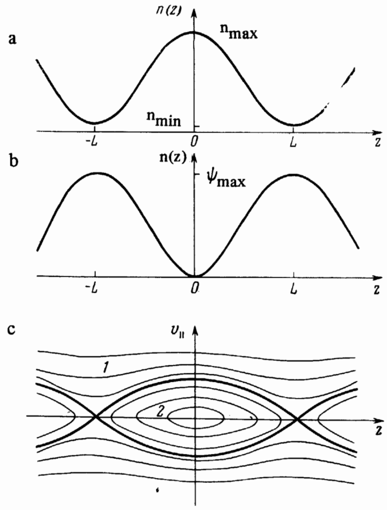

FIG. 1. a–Profile of plasma density n(z); b–distribution of the potential of the polarization electric field; c–trajectories of electrons in the phase plane $(z, v_{||})$: 1–free particle, 2–captured particles.

since the effective conductivity of the plasma should be smaller than in the case of a plasma with a large inhomogeneity scale. This circumstance was noted already by Kadomtsev.[4] We mention also the work by Barkhudarov et al.,[5] in which certain calculations were made pertaining to a weakly inhomogeneous plasma $|n_{max} - n_{min}| \ll n_{max}$.

It should be noted that the longitudinal energy is no longer an integral of motion when account is taken of the collisions, and it is impossible to distinguish clearly between transiting and captured particles, since the boundary between them in phase space becomes smeared out. However, since the thickness of the corresponding transition layer is proportional to $(\nu L/v_T)^{1/2}$ (see Sec. 3), it follows that in the case of a small inhomogeneity scale the subdivision of the particles into transiting and captured is still meaningful to some degree. To the contrary, in the case of a large inhomogeneity scale ($\nu L/v_T \gg 1$), the very concept of the transition layer becomes meaningless (since this layer becomes too thick), and all the particles take part in the current transport.

The main purpose of our investigation is to calculate the effective conductivity of the plasma

$$\sigma^* = \frac{2L}{\Delta\varphi}|j|,$$

where j is the current density and $\Delta\varphi$ is the potential difference of the external electric field over a length 2L. In Sec. 2 below, on the basis of qualitative consideration, we obtain an estimate for σ^* neglecting electron–electron collisions[2)] (the so-called Lorentz plasma). A quantitative analysis of the problem of conductivity of an inhomogeneous Lorentz plasma is based on the use of the kinetic equation, which in a spherical coordinate system (v, θ, φ) with polar axis directed along z, takes the form

$$\frac{\partial f}{\partial t} + v\cos\theta\frac{\partial f}{\partial z} + \frac{e}{m}\left[\frac{\partial\varphi}{\partial z} - E(z)\right]\left(\frac{\partial f}{\partial v}\cos\theta - \frac{\sin\theta}{v}\frac{\partial f}{\partial\theta}\right) = \frac{\nu(v,z)}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f}{\partial\theta}, \quad \nu(v,z) = \frac{2\pi\Lambda e^4 n(z)}{m^2 v^3}, \tag{5}$$

where E(z) is the additional to the polarization electric field, due to the potential difference $\Delta\varphi$. In Eq. (5) we took into account only the scattering of the electrons by ions, neglecting the change of the electron energy in the collisions (since it is proportional to the ratio m/M). With the aid of this equation we obtain in Sec. 3 the electron distribution function in the presence of a small external electric field applied to the plasma. In Sec. 4 we calculate the effective conductivity of a Lorentz plasma. The distribution of the external electric field is obtained in Sec. 5. In Sec. 6 we show that in the case when the plasma density is modulated sufficiently strongly ($n_{max} \gg n_{min}$) we can obtain an analytic expression for the effective conductivity by taking simultaneous account of electron–electron and electron–ion collisions, and we present the corresponding calculations. In Sec. 7 we refine the conditions for applicability of the results.

2. QUALITATIVE ANALYSIS

In the stationary state, the momentum acquired by the transiting electrons in an external electric field should be equal to the momentum lost as a result of collisions. This makes it possible to estimate the plasma conductivity.

At the point $z = -L$, where all the electrons are transiting, the density of their momentum can be expressed by

$$\mathscr{P} = mn_{min}u, \tag{6}$$

where u is the current velocity of the electrons at this point.

Recognizing that the characteristic velocity of the transiting electrons at the point $z = -L$ is equal to v_T, we can find the momentum acquired by such electrons within the inhomogeneity period:

$$\Delta\mathscr{P} = -\frac{e\Delta\varphi}{v_T}n_{min}. \tag{7}$$

On the other hand, the same quantity should be equal to the momentum transferred to the ions by collisions:

$$\Delta\mathscr{P} = \mathscr{P}\int_{-L}^{L}\nu^*(z)\frac{dz}{v(z)}. \tag{8}$$

Here $v(z) = (v_T^2 + \psi(L) - \psi(z))^{1/2}$ is the velocity of the transiting electron at the point z, and $\nu^*(z)$ is the effective collision frequency (which differs, as we shall presently show, from the quantity ν in Eq. (5)).

The spread of the distribution function of the transiting electrons with respect to the angle θ varies along z. Its order of magnitude is

$$\Delta\theta(z) \sim v_T / v(z).$$

Since the collision integral in the kinetic equation (5) contains two differentiations with respect to θ, we get

$$\nu^*(z) = \nu(z) / \Delta\theta^2(z), \tag{9}$$

where $\nu(z)$ can be written in the form

$$\nu(z) = \nu_{max}\frac{n(z)}{n_{max}}\left(\frac{v_T}{v(z)}\right)^3, \tag{10}$$

[2)]Such an approximation is of methodological interest and, in addition, results in good accuracy in the calculation of the conductivity of a plasma with multiply charged ions.

with $\nu_{max} \equiv \nu(v_T, 0)$.

We now obtain in accordance with the determination of the effective conductivity, with the aid of formulas (6)–(10), the following estimate:

$$\sigma_\Lambda^* \sim \sigma_\Lambda \cdot 2L\left[\int_{-L}^{L}\left(\frac{v_T}{v(z)}\right)^2\frac{n(z)\,dz}{n_{min}}\right]^{-1} = \sigma_\Lambda 2L\left[\int_{-L}^{L}\frac{\mu(z)\,dz}{1+\ln\mu(z)}\right]^{-1} \quad (11)$$

Here σ_Λ is the conductivity of a homogeneous Lorentz plasma, and $\mu(z) \equiv n(z)/n_{min}$.

Greatest interest attaches to the case of a strongly inhomogeneous plasma ($\mu_{max} \equiv n_{max}/n_{min} \gg 1$), for it is precisely in this case that the effective conductivity differs significantly from σ_Λ. The main contribution to the integral in (11) is then made by the region of values of z in which $\mu \sim \mu_{max}$, so that

$$\int_{-L}^{L}\frac{\mu(z)\,dz}{1+\ln\mu(z)} \sim 2L\frac{\mu_{max}}{\ln\mu_{max}}.$$

The corresponding estimates will be carried out more rigorously in Sec. 4.

Finally, for the conductivity of a strongly inhomogeneous Lorentz plasma we get

$$\sigma_\Lambda^* \sim \sigma_\Lambda\mu_{max}^{-1}\ln\mu_{max} \ll \sigma_\Lambda, \quad (12)$$

i.e., the inhomogeneity of the plasma in the case $(\nu L/v_T) \ll 1$ indeed leads to a noticeable decrease of the conductivity.

3. ELECTRON DISTRIBUTION FUNCTION IN A LORENTZ PLASMA

Assuming the external field applied to the plasma to be sufficiently small, we consider the linear problem. In the linear approximation, the deviation δf of the distribution function from the equilibrium value satisfies the equation

$$v\cos\theta\frac{\partial\delta f}{\partial z} - \frac{1}{2}\frac{\partial\psi}{\partial z}\left(\cos\theta\frac{\partial\delta f}{\partial v} - \frac{\sin\theta}{v}\frac{\partial\delta f}{\partial\theta}\right) = \frac{\nu}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\delta f}{\partial\theta} + \frac{eE}{m}\cos\theta\frac{\partial f_0}{\partial v}, \quad (13)$$

$$\psi(z) = -2e\varphi(z)/m,$$

where f_0 is the equilibrium distribution function, which depends only on the total energy, $f_0 \equiv f_0[v^2 + \psi(z)]$.

Instead of one equation (13) it is more convenient to deal with a system of equations for the functions p and q, which represent, respectively, the even and odd parts (in terms of $v_\parallel = v\cos\theta$) of the function δf:

$$p = {}^1\!/_2[\delta f(\cos\theta) + \delta f(-\cos\theta)], \quad q = {}^1\!/_2[\delta f(\cos\theta) - \delta f(-\cos\theta)].$$

The first of them determines the plasma density and the second the current. In the equations for p and q we change over to new variables $\epsilon = v^2 + \psi(z)$ and $\epsilon_\perp = v^2\sin^2\theta$, which represent the integrals of motion in the absence of collisions and in the absence of the electric field E(z)

$$\frac{\partial q}{\partial z} = 4\nu(\varepsilon, z)\frac{\partial}{\partial\varepsilon_\perp}\varepsilon_\perp r^{1/2}\frac{\partial p}{\partial\varepsilon_\perp}, \quad (14)$$

$$\frac{\partial p}{\partial z} = 4\nu(\varepsilon, z)\frac{\partial}{\partial\varepsilon_\perp}\varepsilon_\perp r^{1/2}\frac{\partial q}{\partial\varepsilon_\perp} + 2\frac{e}{m}E\frac{\partial f_0}{\partial\varepsilon}, \quad (15)$$

where $r \equiv r(\epsilon, \epsilon_\perp, z) = \epsilon - \epsilon_\perp - \psi(z)$. We shall use below also the symbol $s \equiv s(\epsilon, \epsilon_\perp) = \epsilon - \epsilon_\perp - \psi_{max}$.

The system (14) and (15) must be supplemented also with boundary conditions. For the transiting particles, the functions p and q depend periodically on z with a period equal to the period of the inhomogeneity, i.e.,

$$p(L) = p(-L), \quad q(L) = q(-L) \quad \text{for } s(\varepsilon, \varepsilon_\perp) > 0. \quad (16)$$

The boundary conditions for the trapped particles are imposed at the turning points $(r(\epsilon, \epsilon_\perp, z) = 0)$, where the velocity of these particles is directed across the electric field, i.e., $\cos\theta = 0$. Inasmuch as p is an even function of $\cos\theta$ and q is an odd function, we must stipulate

$$\left.\frac{\partial p}{\partial\cos\theta}\right|_{\cos\theta=0} = 0, \quad q|_{\cos\theta=0} = 0$$

or in terms of the variables ϵ and $\epsilon_\perp$,

$$\lim_{\varepsilon_\perp\to\varepsilon-\psi(z)} r^{1/2}\frac{\partial p}{\partial\varepsilon_\perp} = 0, \quad q|_{\varepsilon_\perp=\varepsilon-\psi(z)} = 0. \quad (17)$$

The electric field E is not assumed to be given; it is itself determined during solution of the problem, and must satisfy the following two conditions

$$E(L) = E(-L), \quad \int_{-L}^{L} E\,dz = -\Delta\varphi.$$

We assume for simplicity that the concentration of the ions n(z), and with it also the potential ψ, are even functions of z. It is clear that in this case q and E must also be even functions of z, and p an odd function. It therefore suffices to consider Eqs. (14) and (15) only in the interval $0 \le z \le L$, and stipulate besides the boundary conditions (16) and (17)

$$p|_{z=0} = 0, \quad \left.\frac{\partial q}{\partial z}\right|_{z=0} = 0. \quad (18)$$

We seek the solution of the system (14) and (15) with boundary conditions (16)–(18) by expanding them in powers of the small parameter $\nu L/v_T$. We first consider the region 1 (see Fig. 1) of phase space, corresponding to transiting particles. From physical considerations it is clear that when the collision frequency ν tends to zero, the current, and consequently the function q, becomes infinite, while the function p, which determines the correction to the plasma density, remains finite. It is also easy to show formally that the expansions for p and q in region 1 are the form

$$p = A + B\nu^2 + \ldots, \quad q = C/\nu + D\nu + \ldots.$$

Substitution of these expressions in (14) yields in the first approximation $\partial q/\partial z = 0$. The dependence of q on ϵ and $\epsilon_\perp$ can be obtained from Eq. (15) by integrating this equation with respect to z from z = 0 to z = L. By virtue of the boundary conditions, the left side of the obtained relation turns out to be equal to zero, and we obtain finally for q

$$q = \frac{e\Delta\varphi}{4m}\frac{\partial f_0}{\partial\varepsilon}\int_{\varepsilon-\psi_{max}}^{\varepsilon_\perp} d\varepsilon_\perp'\left[\int_0^L \nu r^{1/2}(\varepsilon, \varepsilon_\perp', z)\,dz\right]^{-1} + C_1(\varepsilon). \quad (19)$$

$$s(\varepsilon, \varepsilon_\perp) > 0.$$

The question of the integration constant C_1 in this formula will be clarified later. The function p does not depend on this constant and is determined by Eq. (15) with allowance for the first of the conditions (18):

$$p=\frac{2e}{m}\frac{\partial f_0}{\partial \varepsilon}\int_0^z E\,dz'+\frac{e\Delta\varphi}{m}\frac{\partial f_0}{\partial \varepsilon}\frac{\partial}{\partial \varepsilon_\perp}\varepsilon_\perp\frac{\int_0^z \nu r^{1/2}\,dz'}{\int_0^L \nu r^{1/2}\,dz'},\quad s(\varepsilon,\varepsilon_\perp)>0. \tag{20}$$

The functions p and q have entirely different forms in region 2 which corresponds to the captured particles. These particles cannot participate in the transport of the current, and it is therefore natural to assume that

$$q=0 \tag{21}$$

when $s(\epsilon, \epsilon_\perp) < 0$. We then get from (15)

$$p=\frac{2e}{m}\frac{\partial f_0}{\partial \varepsilon}\int_0^z E\,dz+C_2(\varepsilon,\varepsilon_\perp),\qquad s(\varepsilon,\varepsilon_\perp)<0. \tag{22}$$

Since p is an odd function, we find that $C_2(\epsilon, \epsilon_\perp) = 0$. It can be shown that in the general case, when the distribution of the concentration is not an even function of z, the constant C_2 must be determined from the condition

$$\int_{z_1(\varepsilon)}^{z_2(\varepsilon)} dz\int_0^{\varepsilon-\psi(z)} pr^{-1/2}\,d\varepsilon_\perp=0,$$

Where the limit of integration is determined by the relations

$$z_1=-L,\ z_2=L\quad \text{for}\quad \varepsilon-\psi_{max}>0,$$
$$\varepsilon-\psi(z_{1,2})=0\qquad \text{for}\quad \varepsilon-\psi_{max}<0.$$

The foregoing solution (formulas (19)–(22)) satisfies all the boundary conditions for arbitrary $C_1(\epsilon)$. This quantity can be determined by joining together the solutions (19), (20) and (21), (22) on the boundary between the regions 1 and 2. In fact as we have already mentioned, the collisions smear out this boundary. Since the collision integral in the kinetic equation (5) has a Fokker–Planck form, the thickness of the transition layer satisfies the relation $\delta \sim \nu^{1/2}$ (more accurately, $\delta \sim v_T^2(\nu L/v_T)^{1/2}$). In the transition layer itself formulas (19)–(22) are not suitable: the small parameter of the problem, which is equal to the product of the collision frequency by the transit time of the electron through the potential well, cannot be used here (in view of the fact that the transit time diverges logarithmically on approaching the transition layer). As a result the problem reduces to joining the approximate solutions on both sides of the layer. We note that a similar problem was considered in [6,7], devoted to the collision damping of a monochromatic Langmuir wave of finite amplitude.

In order to carry out the indicated joining, we first compare the orders of magnitude of p and q in the transition layer with the aid of Eq. (14): $q \sim \nu p/\delta^2 \sim p$. Further, integrating (15) with respect to $\epsilon_\perp$, we can estimate p in the layer from the known values of the derivative $\partial q/\partial\epsilon_\perp$ on its boundaries, where formulas (19) and (21) hold true:

$$p\sim\frac{\nu}{\delta}\left(\left.\frac{\partial q}{\partial \varepsilon_\perp}\right|_{\varepsilon_\perp=\varepsilon-\psi_{max}+\delta}-\left.\frac{\partial q}{\partial \varepsilon_\perp}\right|_{\varepsilon_\perp=\varepsilon-\psi_{max}-\delta}\right)\sim\nu^{-1/2}. \tag{23}$$

The fact that in the transition layer $q \sim \nu^{-1/2}$, and consequently is much smaller than in region 1, makes it possible to write down, accurate to terms of order $(\nu L/v_T)^{1/2}$, the sought joining condition in the form $q = 0$ when $s(\epsilon, \epsilon_\perp) = 0$, i.e., to put $C_1(\epsilon) = 0$. Finally, p and q are given by formulas (19) and (20) with $C_1 = 0$ in region 1 and by formulas (21) and (22) with $C_2 = 0$ in region 2.

4. CONDUCTIVITY OF THE INHOMOGENEOUS LORENTZ PLASMA

The obtained electron distribution function makes it possible to calculate the current density and to determine the effective conductivity of the plasma. The current density is connected with the function q by the relation

$$j=-e\pi\int_\psi^\infty d\varepsilon\int_0^{\varepsilon-\psi} d\varepsilon_\perp q. \tag{24}$$

The contribution made to the integral (24) by the transition layer is small compared with the current of the transiting particles (the former is proportional to $\nu^{-1/2}$, whereas the latter is proportional to ν^{-1}). Since furthermore $q = 0$ when $\epsilon_\perp > \epsilon - \psi_{max}$, the current is determined completely by the transiting electrons:

$$j=-\frac{\pi e^2\Delta\varphi}{4m}\int_{\psi_{max}}^\infty d\varepsilon\frac{\partial f_0}{\partial \varepsilon}\int_0^{\varepsilon-\psi_{max}} d\varepsilon_\perp\int_{\varepsilon-\psi_{max}}^{\varepsilon_\perp} d\varepsilon_\perp'\left[\int_0^L \nu r^{1/2}(\varepsilon,\varepsilon_\perp',z)\,dz\right]^{-1}. \tag{25}$$

If the time intervals from the instant of turning on the electric field are not too large, then the initial distribution does not have time to become "spoiled" noticeably as a result of Joule heating, and therefore we can regard f_0, with sufficient accuracy as a Maxwellian distribution function. Substituting in (25) the explicit expressions for f_0 and ψ, and taking into account the definition of the effective conductivity, we obtain

$$\sigma_\Lambda^*=\sigma_\Lambda\frac{L}{8}\int_0^\infty e^{-\xi}\,d\xi\int_0^\xi \eta\,d\eta\left[\int_0^L \mu\frac{(\xi-\eta+\ln\mu)^{1/2}}{(\xi+\ln\mu)^{3/2}}\,dz\right]^{-1}, \tag{26}$$

where σ_Λ is the conductivity of the homogeneous Lorentz plasma, and ξ and η are dimensionless variables defined by the equations $\xi = (\epsilon - \psi_{max})/v_T^2$ and $\eta = \epsilon_\perp/v_T^2$. Formula (26) solves the problem of the effective conductivity of a Lorentz plasma with a small inhomogeneity scale ($L \ll v_T/\nu$).

In the most interesting case of a strongly inhomogeneous plasma ($\mu_{max} \gg 1$), formula (26) can be greatly simplified. To this end it should be noted that the main contribution to σ_Λ^* is made by the region $\xi, \eta \sim 1$. At these values of ξ and η it is easy to obtain the following asymptotic (in the parameter μ_{max}) representation of the integral with respect to z in (26):

$$\int_0^L \mu\frac{(\xi-\eta+\ln\mu)^{1/2}}{(\xi+\ln\mu)^{3/2}}\,dz\approx\frac{\bar\mu L}{\ln\mu_{max}},\qquad \bar\mu=\frac{1}{L}\int_0^L \mu\,dz.$$

The last result is valid only for sufficiently smooth concentration distributions, such that $\bar\mu \sim \mu_{max}$.

Finally, for the conductivity of a strongly inhomogeneous Lorentz plasma with a small inhomogeneity scale, we obtain

$$\sigma_\Lambda^*={}^1/_8\sigma_\Lambda\bar\mu^{-1}\ln\mu_{max}\ll\sigma_\Lambda. \tag{27}$$

5. DISTRIBUTION OF ELECTRIC FIELD

In order to find the distribution of the electric field, it is necessary to use the quasineutrality condition

$$\delta n = \pi \int_{\psi}^{\infty} d\varepsilon \int_{0}^{\varepsilon-\psi} p r^{-1/2} d\varepsilon_{\perp} = 0. \tag{28}$$

The function p, which enters in this condition, is not known in the region of the transition layer ($|s(\epsilon, \epsilon_{\perp})| \lesssim \delta$). On the other hand, as seen from the estimates in Sec. 3, the contribution from this region is not small compared with the contribution from the remaining part of the phase space. Nonetheless, at points not too close to the minima of the concentration, namely where

$$|\psi(z) - \psi_{max}| \gg \delta \sim v_T^2 \left(\frac{\nu L}{v_T}\right)^{1/2}, \tag{29}$$

the integral in relation (28) can be calculated. To this end it is convenient to separate in explicit manner the contribution to it from the transition layer:

$$\frac{\delta n}{\pi} = \int_{\psi_{max}+\alpha}^{\infty} d\varepsilon \int_{0}^{\varepsilon-\psi_{max}-\alpha} p r^{-1/2} d\varepsilon_{\perp} + \int_{\psi_{max}+\alpha}^{\infty} d\varepsilon \int_{\varepsilon-\psi_{max}-\alpha}^{\varepsilon-\psi_{max}+\alpha} p r^{-1/2} d\varepsilon_{\perp}$$
$$+ \int_{\psi_{max}+\alpha}^{\infty} d\varepsilon \int_{\varepsilon-\psi_{max}+\alpha}^{\varepsilon-\psi} p r^{-1/2} d\varepsilon_{\perp} + \int_{\psi}^{\psi_{max}-\alpha} d\varepsilon \int_{0}^{\varepsilon-\psi} p r^{-1/2} d\varepsilon_{\perp}, \tag{30}$$

where the quantity α in the integration limits is chosen from the conditions

$$\delta \ll \alpha \ll \min\{v_T^2, |\psi(z) - \psi_{max}|\}. \tag{31}$$

Such a choice of α is always possible if condition (29) holds. The first term in (30) corresponds to transiting particles, the second to the transition layer, and the last two to captured particles. Bearing in mind the second of the conditions (31), we can write for the value of $\epsilon_{\perp}$ in the expression for $r(\epsilon, \epsilon_{\perp}, z)$ simply $\epsilon - \epsilon_{max}$ when calculating the integral over the transition layer:

$$\int_{\varepsilon-\psi_{max}-\alpha}^{\varepsilon-\psi_{max}+\alpha} p r^{-1/2} d\varepsilon_{\perp} = (\psi_{max} - \psi)^{-1/2} \int_{\varepsilon-\psi_{max}-\alpha}^{\varepsilon-\psi_{max}+\alpha} p\, d\varepsilon_{\perp}.$$

With the same accuracy we get from (15)

$$\int_{\varepsilon-\psi_{max}-\alpha}^{\varepsilon-\psi_{max}+\alpha} p\, d\varepsilon_{\perp} = \int_0^z dz' 4\nu(\varepsilon - \psi_{max})(\psi_{max} - \psi) \frac{\partial q}{\partial \varepsilon_{\perp}}\Bigg|_{\varepsilon_{\perp}=\varepsilon-\psi_{max}-\alpha}^{\varepsilon_{\perp}=\varepsilon-\psi_{max}+\alpha}$$

When the first inequality of (31) is satisfied, the values of the derivative $\partial q/\partial\epsilon_{\perp}$ at the integration limits can be regarded as known, since formulas (19) and (21) can be used for their calculation. Therefore

$$\int_{\varepsilon-\psi_{max}-\alpha}^{\varepsilon-\psi_{max}+\alpha} p\, d\varepsilon_{\perp} = -\frac{e\Delta\varphi}{m}\frac{\partial f_0}{\partial\varepsilon}(\varepsilon - \psi_{max}) \frac{\int_0^z \nu(\psi_{max} - \psi)^{1/2} dz'}{\int_0^L \nu(\psi_{max} - \psi)^{1/2} dz'}.$$

The sought contribution to the density from the particles of the layers is equal to the integral of this expression with respect to ϵ. Since the relating integrals in (30) converge when $\alpha \to 0$, we put in them $\alpha = 0$, substitute formulas (20) and (21), and obtain as a result

$$E(z) = \frac{\Delta\varphi}{8}\frac{\partial}{\partial z}\left[\left(\int_{\psi}^{\infty}\frac{\partial f_0}{\partial\varepsilon}(\varepsilon-\psi)^{1/2} d\varepsilon\right)^{-1}\right.$$
$$\left.\times \int_{\psi_{max}}^{\infty} d\varepsilon \frac{\partial f_0}{\partial\varepsilon}\int_0^{\varepsilon-\psi_{max}} \varepsilon_{\perp} s^{-3/2} d\varepsilon_{\perp} \int_0^z \nu r^{1/2} dz' \Big/ \int_0^L \nu r^{1/2} dz'\right]. \tag{32}$$

This result, which is valid for any value of μ_{max}, can be greatly simplified in the case of a strongly inhomogeneous plasma with a Maxwellian distribution function f_0, by replacing the integrals in formula (32) by their asymptotic expressions, in analogy with the procedure used in Sec. 4:

$$E = \frac{\Delta\varphi}{2\sqrt{\pi}L\bar{\mu}}\frac{\partial}{\partial z}\left\{\mu^{-1}\left(\int_0^z \mu\, dz'\right)\left[(\ln\mu)^{-1/2} + 2(\ln\mu)^{1/2}\right.\right.$$
$$\left.\left. - 2\int_0^{\infty} e^{-x}(x + \ln\mu)^{1/2} dx\right]\right\}. \tag{33}$$

The limits of applicability of formulas (32) and (33) are given by the condition (29), which can also be transformed as applied to the case of a strongly inhomogeneous plasma:

$$|z - L| \gg a \sim L\mu_{max}^{-1/2}(\nu L/v_T)^{1/4}.$$

The most interesting circumstance that must be noted here is that formula (32) yields, in the region of its applicability, an electric field of sign opposite to the applied field, with

$$\left|\int_0^{L-a} E\, dz\right| \sim |\Delta\varphi|\left(\frac{v_T}{\nu L}\right)^{1/4}.$$

Since the total potential difference $\Delta\varphi$ on the period is specified, this means that in a narrow vicinity of the concentration minimum $|z - L| \lesssim a$ there is a potential drop on the order of $\Delta\varphi(v_T/\nu L)^{1/4}$, and the electric field here can then be estimated as follows:

$$E \sim \frac{\Delta\varphi}{a}\left(\frac{\nu L}{v_T}\right)^{-1/4} \sim \frac{\Delta\varphi}{L}\left(\frac{v_T}{\nu L}\mu_{max}\right)^{1/2}.$$

The distribution of the applied field is illustrated qualitatively in Fig. 2.

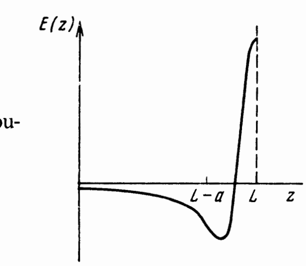

FIG. 2. Qualitative form of distribution of external electric field.

6. ALLOWANCE FOR ELECTRON-ELECTRON COLLISIONS

In the preceding sections, in determining the electric conductivity, the plasma was assumed to be of the Lorentz type. We consider below the influence exerted on the conductivity of electron–electron collisions. It is convenient here to represent the correction δf to the unperturbed (Maxwellian) distribution function f_0 in the form

$$\delta f = f_0\Phi,$$

where Φ is a new unknown function. The electron–electron collision integral is expressed in terms of Φ as follows:

$$\mathrm{St}_{ee}\delta f = -\nu(v, z)n(z)\left(\frac{mv}{2\pi T}\right)^3\frac{\partial}{\partial v_\beta}\int \exp\left\{-\frac{m}{2T}(v^2 + v'^2)\right\}$$
$$\times\left[\frac{\partial\Phi(\mathbf{v}', z)}{\partial v_\alpha'} - \frac{\partial\Phi(\mathbf{v}, z)}{\partial v_\alpha}\right]U_{\alpha\beta}\, d^3\mathbf{v}', \tag{34}$$

$$U_{\alpha\beta}=\frac{\delta_{\alpha\beta}}{u}-\frac{u_\alpha u_\beta}{u^3},\qquad u_\alpha=v_\alpha-v_\alpha'.$$

In a homogeneous plasma, the kinetic equation for finding the distribution function with allowance for electron–electron collisions turns out to be an integro-differential equation, and greatly complicates the problem. In an inhomogeneous plasma with $\mu_{max}\gg 1$, the situation is much simpler. Indeed, it is seen from (34) that the collision integral is proportional to the plasma density and consequently it can be assumed that on passing through the interval $[-L, L]$ the electron experiences collisions only over the potential well, where the plasma density is of the order of n_{max}. In this region, the concentration ratio of the transiting and captured electrons is approximately equal to $\mu_{max}^{-1}\ll 1$, so that the collisions between the transiting electrons can be neglected. This circumstance makes it possible to neglect the terms $\partial\Phi(v', z)/\partial v'_\alpha$ in the collision integral for the transiting particles. The simplified collision integral obtained in this manner can be transformed into

$$\mathrm{St}_{ee}=\frac{v(v,z)}{n(z)}v^3 f_0\left\{\frac{1}{v^2}\frac{\partial}{\partial v}v^2\mathscr{D}_{vv}\frac{\partial\Phi}{\partial v}+\frac{\mathscr{D}_{\theta\theta}}{v^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta}-\frac{m}{T}v\mathscr{D}_{vv}\frac{\partial\Phi}{\partial v}\right\},$$

where

$$\mathscr{D}_{vv}=\frac{1}{v^2}\int f_0(v')v_\beta v_\gamma U_{\beta\gamma}\,d^3v',\quad \mathscr{D}_{\theta\theta}=\frac{1}{2}\int f_0(v')\left[U_{\beta\beta}-\frac{v_\beta v_\gamma}{v^2}U_{\beta\gamma}\right]d^3v'.$$

In the region where the collisions of the transiting electrons are significant ($n\sim n_{max}$), their velocity $v\sim v_T(\ln\mu_{max})^{1/2}$ greatly exceeds the average velocity v' of the captured electrons, which is of the same order of magnitude as v_T. Therefore in calculating $\mathscr{D}_{vv}$ and $\mathscr{D}_{\theta\theta}$ we can confine ourselves to the principal terms of the expansion in the parameter $v'/v\sim(\ln\mu_{max})^{-1/2}$. Omitting the simple intermediate steps, we present here only the final result:

$$\mathscr{D}_{vv}=2n\frac{T}{mv^3},\qquad \mathscr{D}_{\theta\theta}=\frac{n}{v}.$$

In analogy with the procedure in Sec. 3, we represent Φ in the form of a sum of functions p and q which are even and odd in $\cos\theta$, respectively, change over to the variables z, ϵ, and $\epsilon_\perp$, and write down the system of equations for the determination of p and q:

$$\frac{\partial q}{\partial z}=\hat{\mathscr{L}}p, \tag{35}$$

$$\frac{\partial p}{\partial z}=\hat{\mathscr{L}}q-\frac{e}{T}E, \tag{36}$$

the linear operator $\hat{\mathscr{L}}$ is defined by the relation

$$\hat{\mathscr{L}}=8\nu\frac{\partial}{\partial\varepsilon_\perp}\varepsilon_\perp r^{1/2}\frac{\partial}{\partial\varepsilon_\perp}+\frac{8\nu T(\varepsilon-\psi)}{mr^{1/2}}\left[\frac{\partial^2}{\partial\varepsilon^2}+2\frac{\varepsilon_\perp}{\varepsilon-\psi}\frac{\partial^2}{\partial\varepsilon\,\partial\varepsilon_\perp}+\left(\frac{\varepsilon_\perp}{\varepsilon-\psi}\right)^2\frac{\partial^2}{\partial\varepsilon_\perp^2}\right]-\frac{4\nu(\varepsilon-\psi)}{r^{1/2}}\left[\frac{\partial}{\partial\varepsilon}+\frac{\varepsilon_\perp}{\varepsilon-\psi}\frac{\partial}{\partial\varepsilon_\perp}\right].$$

The principal term of the expansion of q in powers of $\nu L/v_T$, as seen from (35), does not depend on z and can be determined from the expression (36) averaged over the coordinate:

$$\left(\int_{-L}^{L}\hat{\mathscr{L}}\,dz\right)q=-\frac{e\Delta\varphi}{T}.$$

Replacing the integrals with respect to z in the formula for $\int_{-L}^{L}\hat{\mathscr{L}}\,dz$ by their asymptotic expansions at $\mu_{max}\gg 1$ (see Sec. 4), we obtain the final expression for the function q, which is valid accurate to terms of order $(\ln\mu_{max})^{-1}$:

$$-\frac{e\Delta\varphi}{8Tv_Tv_{max}L}\frac{\mu_{max}}{\bar{\mu}}\ln\mu_{max}=2\varepsilon_\perp\frac{\partial^2 q}{\partial\varepsilon_\perp^2}+\frac{2T}{m}\frac{\partial^2 q}{\partial\varepsilon^2}+\frac{4\varepsilon_\perp}{\ln\mu_{max}}\frac{\partial^2 q}{\partial\varepsilon\,\partial\varepsilon_\perp}+2\frac{\partial q}{\partial\varepsilon_\perp}-\frac{\partial q}{\partial\varepsilon}.$$

It is easy to find a solution of this equation, finite at $\epsilon_\perp\to 0$, and satisfying the joining condition $q|_{\epsilon-\epsilon_\perp=\psi_{max}}=0$:

$$q=\frac{e\Delta\varphi}{24Tv_Tv_{max}L}\frac{\mu_{max}}{\bar{\mu}}\ln\mu_{max}s(\varepsilon,\varepsilon_\perp).$$

Substituting the obtained expression in (24), we can determine the current and the effective conductivity σ^* of a strongly inhomogeneous plasma. The value of the electric conductivity calculated with allowance for the electron–electron collisions turns out to be one third as large as the Lorentz-plasma conductivity obtained in Sec. 4:

$$\sigma^*=\frac{\sigma_\Lambda}{24}\bar{\mu}^{-1}\ln\mu_{max}.$$

Bearing in mind that the conductivity σ of a homogeneous plasma, calculated with allowance for electron–electron collisions, amounts to 0.57 of σ_Λ, the last formula can be also written in the form

$$\sigma^*=0.073\,\sigma\bar{\mu}^{-1}\ln\mu_{max}.$$

It is interesting to note that in an inhomogeneous plasma a finite conductivity arises even when account is taken of only the electron–electron collisions, and amounts in this case to $\sigma^*_\Lambda/2$.

7. DISCUSSION OF RESULTS

In this section we shall trace the variation of the conductivity of an inhomogeneous plasma with increasing scale of its inhomogeneity, and also refine the limits of applicability of the results obtained above.

The exact solution of the problem of conductivity of an inhomogeneous plasma can be obtained in two limiting cases: $L\gg v_T/\nu$ (large-scale inhomogeneity) and $L\ll v_T/\nu$ (small-scale inhomogeneity). For the remaining values of L we used qualitative considerations.

Let us ascertain first the variation of the number of particles taking part in the current transport with changing L. We start with the case of small L, when the current is carried only by the transiting electrons. Their scatter with respect to $\epsilon_\perp$ does not depend on z and is equal to v_T^2. It is clear that the very concept of the transiting particles is meaningful only in the case when the variation of the transverse energy of the transiting electron, due to collisions during the time of its motion over one period of the inhomogeneity, is small compared with T. It is easy to write down the corresponding estimate by using the collision frequency ν^* introduced in Sec. 2, which represents the reciprocal time needed for $\epsilon_\perp$ to change by an amount of the order of v_T^2:

$$\int_{-L}^{L}\nu^*\frac{dz}{v(z)}\ll 1.$$

We see therefore that the criterion for the applicability of the results pertaining to a plasma with a small inhomogeneity scale is the following inequality:

$$L \ll \frac{v_T}{\nu_{max}} \ln \mu_{max}, \qquad \nu_{max} = \frac{2\pi\Lambda e^4 n_{max}}{m^2 v_T^3}.$$

With increasing L, this inequality no longer holds, and when $L \sim (v_T/v_{max}) \ln \mu_{max}$ it is no longer possible to separate the transiting particles from the transition layer: both types of particles make approximately equal contributions to the current.

As to the captured particles, when

$$L \sim \frac{v_T}{\nu_{max}} \ln \mu_{max}$$

only a small fraction of these particles still participates in the current. Indeed, if

$$\int_{-L}^{L} \frac{\nu^* \, dz}{v(z)} \sim 1,$$

then the collisions have time to change the transverse energy of the electron during one period of its motion by only an amount T, whereas the scatter of these particles with respect to $\epsilon_\perp$, owing to the large depth of the potential well, is obviously equal to $v_T^2 \ln \mu_{max}$. In order for all of them to be able to carry current, it is necessary to stipulate

$$\int_{-L}^{L} \frac{\nu^* \, dz}{v(z)} \gg \ln^2 \mu_{max}.$$

The corresponding limitation on L is

$$L \gg \frac{v_T}{\nu_{max}} (\ln \mu_{max})^2. \tag{37}$$

It is precisely in the sense of this inequality that one must understand the statement that the inhomogeneity becomes large-scale. Satisfaction of the condition (37) makes it possible to use for the conductivity calculation the formulas obtained for a homogeneous plasma.

The foregoing considerations explain qualitatively the dependence of σ^* on L shown in Fig. 3. At small L ($\nu_{max} L/v_T \ll \ln \mu_{max}$) the effective conductivity is determined by the transiting particles and is independent of L up to $L \sim v_T \ln \mu_{max}/\nu_{max}$. In the intermediate region $\ln \mu_{max} \ll \nu_{max} L/v_T \ll (\ln \mu_{max})^2$ the conductivity increases in proportion to the number of particles that carry current, from $\sigma^* = 0.073\, \sigma \bar{\mu}^{-1} \times \ln \mu_{max}$ to $\sigma^* \sim \sigma$, which is reached when $L \sim v_T (\ln \mu_{max})^2/\nu_{max}$. Further increase of L cannot lead to an increase of σ^*, for when $L \gtrsim v_T (\ln \mu_{max})^2/\nu_{max}$ all the electrons take part in the transport of the current.

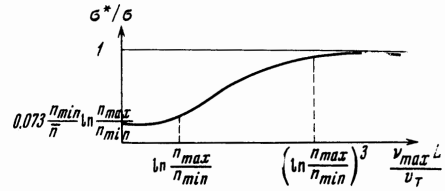

FIG. 3. Plot of σ^*/σ against the inhomogeneity scale L.

[1] B. A. Trubnikov, in: Voprosy teorii plazmy (Problems of Plasma Theory), Gosatomizdat, 1963, p. 98.

[2] S. I. Braginskiĭ, ibid. p. 183.

[3] B. A. Demidov, N. I. Elagin, D. D. Ryutov, and S. D. Fanchenko, Zh. Eksp. Teor. Fiz. 48, 454 (1965) [Sov. Phys.-JETP **21**, 302 (1965)].

[4] B. B. Kadomtsev, in: Voprosy teorii plazmy (Problems of Plasma Theory), v. 4, Atomizdat, 1964, p. 327.

[5] E. M. Barkhudarov, F. Baimbetov, N. A. Kervalishvili, V. P. Kortkhonjia, and N. L. Tsintsadze, Proc. of the III European Conf. on Controlled Fusion and Plasma Phys., Utrecht, 1969, p. 95.

[6] V. E. Zakharov and V. I. Karpman, Zh. Eksp. Teor. Fiz. 43, 490 (1962) [Sov. Phys.-JETP **16**, 351 (1963)].

[7] V. I. Karpman, Prikl. Mat. Teor. Fiz. No. 4, 3 (1964).

Translated by J. G. Adashko
214