ROTATING RING RESONATOR IN A GRAVITATIONAL FIELD

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The effect of a gravitational field on the eigenfrequencies of a rotating ring resonator is investigated on the basis of general relativity theory and electromagnetic theory for continuous media. The covariant equations of propagation of electromagnetic waves in a medium at rest in a rotating reference system in the presence of a gravitational field are written down in vector form. The equations are used to study the resonant properties of an optical ring resonator. Formulas for the frequency shift of the traveling waves and for the frequency difference of the opposite waves in a resonator with a nonreciprocal element are derived for plane electromagnetic waves propagating in a rotating ring resonator. It is shown that at low resonator rotation velocities the frequency difference of the opposite waves, due to a static gravitational field, is greater than the effect due to resonator rotation by several orders of magnitude. The effect of a stationary gravitational field created by a rotating mass on the eigenfrequencies of a ring resonator is also considered.

RING lasers are capable of detecting very small angular displacements, on the order of 0.001 deg/hr. At such a high measurement accuracy of angular rotation, it is necessary to take into account the influence of the gravitational field on the frequency shift in the ring resonator, and also the influence of rotation of the masses (for example, the rotation of the earth around its own axis) on the change of the frequency of an electromagnetic wave propagating in the rotating ring laser along the resonator circuit. As will be shown below, the change of the natural frequency of the ring resonator, due to the gravitational field, exceeds by several orders of magnitude the frequency shift occurring at the aforementioned angular velocities. On the other hand, a ring laser makes it possible to measure gravitational effects with high accuracy, and can serve as an instrument for verifying the conclusions of general relativity theory.

In this paper we investigate, on the basis of general relativity theory, the influence of a gravitational field on the natural frequencies of a rotating ring resonator.

1. EQUATIONS OF ELECTROMAGNETIC FIELD FOR ISOTROPIC MEDIA THAT ARE AT REST RELATIVE TO A REFERENCE FRAME THAT ROTATES IN A GRAVITATIONAL FIELD

The propagation of electromagnetic waves in a dielectric medium that is at rest in an arbitrary reference frame, in the presence of masses producing a gravitational field, is described by the covariant Maxwell equations[1,2]

$$H^{\alpha\beta}{}_{;\,\beta}=0,\quad F_{\alpha\beta;\,\gamma}+F_{\beta\gamma;\,\alpha}+F_{\gamma\alpha;\,\beta}=0, \tag{1}$$

together with the covariant material equations[3]:

$$\sqrt{-g}H^{\alpha\beta}=\sqrt{-g}g^{\alpha\gamma}g^{\beta\nu}S^{\sigma\tau}{}_{\gamma\nu}F_{\sigma\tau}, \tag{2}$$

or

$$\frac{1}{\sqrt{-g}}g_{\alpha\gamma}g_{\beta\nu}\sqrt{-g}H^{\gamma\nu}=S^{\sigma\tau}{}_{\alpha\beta}F_{\sigma\tau}.$$

In formulas (1) and (2), $H^{\alpha\beta}$ and $F_{\alpha\beta}$ are respectively the contravariant and covariant tensors of the electromagnetic field; $g^{\alpha\beta}$ is the metric tensor characterizing the geometry of the four-dimensional space; g is the determinant of the metric tensor. The indices separated by a semicolon denote covariant derivatives with respect to the corresponding coordinates. Throughout the article, Greek indices α, β, γ, ... run through the values 0, 1, 2, 3 and the Latin indices i, j, k, ... run through the values 1, 2, and 3; summation is carried out over repeated indices. In the case of an isotropic medium, the fourth-rank tensor $S^{\alpha\beta}_{\gamma\nu}$ reduces to a contraction of the second-rank tensor S^{α}_{β}:

$$S^{\alpha\beta}{}_{\gamma\nu}=S^{\alpha}{}_{\gamma}S^{\beta}{}_{\nu},$$

having the components

$$S^{0}{}_{0}=\varepsilon\sqrt{\mu},\quad S^{1}{}_{1}=S^{2}{}_{2}=S^{3}{}_{3}=\frac{1}{\sqrt{\mu}}\text{ and } S^{\alpha}{}_{\beta}=0,\quad\text{if}\quad\alpha\neq\beta, \tag{3}$$

Here ϵ and μ are the values of the dielectric constant and the magnetic permeability of the medium, as measured by local observers coupled to the medium.

In a space with static gravitational field produced by a spherical body of mass m, the metric tensor has the components

$$g^{00}=1-\frac{2\varphi}{c^2},\quad g^{11}=g^{22}=g^{33}=-1-\frac{2\varphi}{c^2};\quad g^{\alpha\beta}=0,\quad\text{if}\quad\alpha\neq\beta, \tag{4}$$

where $\varphi = -km/R$ is the potential of the gravitational field, k is the gravitational constant, and R is the distance from the center of the mass to the point of observation.

Let us proceed to a reference frame that rotates in the gravitational field. The covariant metric tensor in a rotating reference frame, the z axis of which is directed along the angular-velocity vector $\mathbf{\Omega}$ of the rotation, has the following components:

$$g^{\alpha\beta}=\begin{pmatrix} 1-\frac{2\varphi}{c^2} & \frac{\Omega y}{c}\left(1-\frac{2\varphi}{c^2}\right) & -\frac{\Omega x}{c}\left(1-\frac{2\varphi}{c^2}\right) & 0 \\ \frac{\Omega y}{c}\left(1-\frac{2\varphi}{c^2}\right) & -1-\frac{2\varphi}{c^2}+\frac{\Omega^2 y^2}{c^2} & -\frac{\Omega^2 xy}{c^2} & 0 \\ -\frac{\Omega x}{c}\left(1-\frac{2\varphi}{c^2}\right) & -\frac{\Omega^2 xy}{c^2} & -1-\frac{2\varphi}{c^2}+\frac{\Omega^2 x^2}{c^2} & 0 \\ 0 & 0 & 0 & -1-\frac{2\varphi}{c^2} \end{pmatrix} \tag{5}$$

where x, y, and z are the coordinates in the rotating reference frame.

We introduce three-dimensional electromagnetic-field vectors in accordance with the scheme[4]:

$$D_i=\sqrt{-g}H^{i0},\quad H_i=-{}^1/{}_2\varepsilon_{ikl}\sqrt{-g}H^{kl},$$
$$E_i=F_{0i},\qquad B_i=-{}^1/{}_2\varepsilon_{ikl}F_{kl}, \tag{6}$$

where ϵ_{ikl} is a completely antisymmetric unit pseudotensor of third rank. With the aid of (6) we can write the covariant Maxwell's equations in vector form

$$\operatorname{rot}\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=0,\quad \operatorname{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0,$$
$$\operatorname{div}\mathbf{D}=0,\quad \operatorname{div}\mathbf{B}=0. \tag{7}$$

Substituting in the first relation of (2) the values of the components of the tensors $H^{\alpha\beta}$ and $F_{\alpha\beta}$ from (6), of the dielectric and magnetic tensors $S^{\alpha\beta}_{\gamma\nu}$ from (3), and of the metric tensor from (5), we obtain material vector equations describing the vectors of the electromagnetic field in a medium that is at rest in a reference frame rotating in the static gravitational field. In the first approximation in $(-2\varphi/c^2)$ and $c^{-1}|\mathbf{\Omega}\times\mathbf{r}|$, the material equations take the form

$$\mathbf{D}=\left(1-\frac{2\varphi}{c^2}\right)\varepsilon\mathbf{E}+\frac{1}{\mu}\left[\frac{[\mathbf{r}\Omega]}{c}\mathbf{B}\right],$$

$$\mathbf{H}=\left(1+\frac{2\varphi}{c^2}\right)\frac{1}{\mu}\mathbf{B}+\varepsilon\left[\frac{[\mathbf{r}\Omega]}{c}\mathbf{E}\right]. \tag{8*}$$

Maxwell's vector equations (7) together with the material equations (8) describe completely the propagation of electromagnetic waves in an isotropic medium rotating in a gravitational field, in a reference frame connected with the medium.

2. INFLUENCE OF GRAVITATIONAL FIELD ON THE NATURAL FREQUENCIES OF A ROTATING RING RESONATOR

Let us consider a plane monochromatic electromagnetic wave propagating in an isotropic medium rotating in a gravitational field. Maxwell's equations for such a wave are:

$$\mathbf{D}=[\mathbf{H}\mathbf{n}^*],\quad \mathbf{B}=[\mathbf{n}^*\mathbf{E}], \tag{9}$$

where n* is the effective refractive index of the medium.

Eliminating the vectors **E**, **B**, **D**, and **H** from relations (8) and (9), we arrive at an equation for n*. Solving the obtained equation accurate to first order in $(-2\varphi/c^2)$ and $c^{-1}|\mathbf{\Omega}\times\mathbf{r}|$, we obtain the effective refractive index in the reference frame connected with the medium

$$n^*=\left(1-\frac{2\varphi}{c^2}\right)n+\frac{n^2+1}{2n}\left(\mathbf{n}\frac{[\Omega\mathbf{r}]}{c}\right), \tag{10}$$

where $n=\sqrt{\epsilon\mu}$ and **r** is the radius vector drawn from the origin of the rotating reference frame to the point under consideration.

If, in addition to the foregoing, the mass m producing the gravitational field rotates with constant angular velocity about its own axis, then the components $g^{0i}=g^{i0}$ in the metric tensor (4) also differ from zero[1]

$$g^{0i}=g^{i0}=g_i=-\frac{2k}{c^3R^3}[\mathbf{RM}]_i,$$

where **M** is the angular momentum of the body. Relation (10) for the effective refractive index assumes the form

$$\tilde{n}^*=\left(1-\frac{2\varphi}{c^2}\right)n+\frac{n^2+1}{2n}\left(\mathbf{n}\frac{[\Omega\mathbf{r}]}{c}\right)-\frac{n^2+1}{2n}(\mathbf{ng}).$$

Let L be the perimeter of the resonator loop. Part of the resonator loop, with length l, is filled with the active medium. The refractive index of the active medium in the inertial reference frame, in the absence of a gravitational field, is n_a. Part of the resonator loop, of length d_0, is filled with a "nonreciprocal" medium, the refractive index of which in the inertial reference frame and in the absence of a gravitational field is equal to n_1 for a wave in one direction and n_2 for a wave traveling in the opposite direction, and ω_0 is the natural frequency of the ring resonator with active medium at rest in the inertial reference frame in the absence of the gravitational field. The change of the frequency of the traveling wave can be obtained by equating the phase shift following a single circuit of the electromagnetic waves around the resting ring resonator and the resonator rotating in the gravitational field:

$$\frac{\omega_{1(2)}}{c}\oint n^*_{1(2)}\,dl=\frac{\omega_0}{c}\oint n\,dl.$$

The integral in the left side of the equation determines the effective optical path of the electromagnetic wave in a ring resonator with nonreciprocal medium, rotating in the gravitational field, and the integral in the right side corresponds to a resonator at rest in the absence of a gravitational field and of the nonreciprocating medium.

The frequency shift of the wave, in first order in $(-2\varphi/c^2)$ and $c^{-1}|\mathbf{\Omega}\times\mathbf{r}|$, is

$$\Delta\omega_{1(2)}=\frac{\omega_0}{\mathscr{L}_{1(2)}}\left\{d_0(n_{1(2)}-1)-\frac{2\varphi}{c^2}[L+l(n_a-1)]\pm\Phi_0\pm\Phi_{1(2)}\right\}, \tag{11}$$

where

$$\Phi_0=\frac{n_a^2+1}{2}\frac{\Omega}{c}\int_0^l[\mathbf{r}\,d\mathbf{l}]+\frac{\Omega}{c}\int_{l+d_0}^{L}[\mathbf{r}\,d\mathbf{l}],\quad \Phi_{1(2)}=\frac{n^2_{1(2)}+1}{2}\frac{\Omega}{c}\int_l^{l+d_0}[\mathbf{r}\,d\mathbf{l}],$$
$$\mathscr{L}_{1(2)}=L+l(n_a-1)+d_0(n_{1(2)}-1).$$

The index 1 (2) and the sign + (–) pertain to the wave propagating in the direction (against the direction) of rotation of the resonator. The frequency difference of the opposing waves in the ring resonator rotating in the rotational field is

$$\Delta\omega=\frac{\omega_0}{\mathscr{L}_1\mathscr{L}_2}[L+l(n_a-1)]\left\{\left(1+\frac{2\varphi}{c^2}\right)d_0(n_1-n_2)\right. \tag{12}$$
$$\left.+(2\Phi_0+\Phi_1+\Phi_2)+\frac{d_0(n_1-n_2)(\Phi_0+\Phi_2)}{\mathscr{L}_2}-\frac{d_0(n_1-n_2)(\Phi_0+\Phi_1)}{\mathscr{L}_1}\right\}.$$

*$[\mathbf{r}\Omega\equiv\mathbf{r}\times\Omega]$.

If the nonreciprocity of the optical ring resonator is produced in such a way that $n_{1(2)} - n_0 \pm \Delta n$ (for example, by a Faraday cell[5] or by a medium moving in the resonator[6]), then the frequency shift of the opposing wave is, in first approximation

$$\Delta\omega = 2\omega_0 \frac{(1 + 2\varphi/c^2) d_0 \Delta n + \Phi_0 + \Phi_{1,2}}{[L + l(n_a - 1) + d_0(n_0 - 1)]}, \qquad (13)$$

where

$$\Phi_{1,2} = \frac{n_0^2 + 1}{2} \frac{\Omega}{c} \int_l^{l+d_0} [\mathbf{r}\, d\mathbf{l}].$$

Relations (11), (12), and (13) show that even in nonrotating ring resonators the resonant frequencies of the traveling electromagnetic waves change under the influence of the gravitational field. In a resonator with a nonreciprocal element, the frequency difference of the opposing waves also depend on the gravitational field.

In the case when the gravitational field is produced by a rotating mass, it is necessary to take Φ_0, Φ_1, and Φ_2 in formulas (11), (12), and (13) to mean the quantities

$$\Phi_0 = \frac{n_a^2 + 1}{2} \frac{\Omega}{c} \int_0^l [\mathbf{r}\, d\mathbf{l}] + \frac{\Omega}{c} \int_{l+d_0}^{L} [\mathbf{r}\, d\mathbf{l}] + \frac{n_a^2 + 1}{2} \frac{2kM}{c^3} \int_0^l \frac{[\mathbf{R}\, d\mathbf{l}]}{R^3} + \frac{2kM}{c^3} \int_{l+d_0}^{L} \frac{[\mathbf{R}\, d\mathbf{l}]}{R^3},$$

$$\Phi_{1(2)} = \frac{n_{1(2)}^2 + 1}{2} \frac{\Omega}{c} \int_l^{l+d_0} [\mathbf{r}\, d\mathbf{l}] + \frac{n_{1(2)}^2 + 1}{2} \frac{2kM}{c^3} \int_l^{l+d_0} \frac{[\mathbf{R}\, d\mathbf{l}]}{R^3}$$

Let us estimate the contribution of each of the quantities to the frequency shift of the traveling wave in the ring resonator. The relative frequency shift due to rotation of a resonator with angular velocity on the order of 0.01–0.001 deg/hr, for typical resonator dimensions, is $\sim \Omega r/c \approx 10^{-16}$–$10^{-17}$. The relative frequency shift due to the earth's static rotational field is determined by the quantity $(-2\varphi/c^2) \approx 10^{-9}$. This corresponds to resonator rotation with velocity $\Omega = 10^5$ deg/hr. The relative frequency difference of the opposing waves in the ring resonator with nonreciprocal elements (for $n_1 - n_2 \approx 10^{-3}$–$10^{-4}$) is 10^{-12}–10^{-13}. The effect of mass rotation produces a relative frequency shift $\sim 2kR \times M/c^3R^3 \approx 10^{-15}$–$10^{-16}$ (for the earth). The frequency shift under the influence of the gravitational field changes with increasing or decreasing distance from the resonator to the center of the mass. For heavier planets, the gravitational effect becomes stronger. Thus, the need for taking into account the influence of the gravitational field on the natural frequencies of the ring resonator is obvious. This influence turns out to be most significant when working with single beam ring lasers.

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