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CALCULATION OF THE CRITICAL CURRENT FOR PARTICLES DECAYING INTO NONPARALLEL EXCITATIONS

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The current of foreign particles interacting with a Bose system at absolute zero is calculated in the case when the main dissipation mechanism is decay into two non-parallel excitations. It is shown that at small field strengths the current possesses a singularity of the $|\ln E|^{-1}$ type and can either increase or decrease with increasing field strength.

IN[1] there was considered the question of the motion of extraneous particles (ions) in a superfluid liquid under the influence of an electric field at zero temperature and at low particle density. It was assumed there that the main mechanism of dissipation is the decay of the ion excitation, when the threshold momentum p_c is reached, into an ion excitation and a phonon[1)] with parallel momenta, one of the three possible types of decay indicated by Pitaevskiĭ[2]. The dependence of the critical current of the particles on the field was obtained as a result of an investigation of the exact (in the limit of low particle density) equation for the Green's function of the ion in a weak field.

In this paper we solve a similar problem for the decay into excitations that are emitted at an angle to each other. In this case, according to[2], the momentum of the ion excitation after the decay, p_m, corresponds to the minimum of the spectrum of the ionic excitations, and momentum q_m of the emitted phonon corresponds to the minimum of the phonon spectrum (i.e., a roton). The angle θ_0 between $\mathbf{p}_m$ and $\mathbf{p}_c$ is determined from the condition $\mathbf{p}_m + \mathbf{q}_m = \mathbf{p}_c$.

Near the end point, the spectrum of the ionic excitations is of the form

$$\varepsilon(p) = \varepsilon(p_c) - \alpha \exp\{a/(p - p_c)\}, \quad \alpha, a > 0. \tag{1}$$

There are grounds for assuming that this form of decay can be realized for ions in He II at a pressure above 12 atm. Indeed, Rayfield[3] has shown that the critical current of the ions of $P > 12$ atm is close in value to the current given by the Landau criterion: $e\omega(q_m)/q_m$. The roton momentum q_m and its energy $\omega(q_m)$ were measured directly in experiments on neutron scattering. Apparently, at P = 12 atm a transition takes place from a decay into parallel excitations to a decay into nonparallel excitations.

In Secs. 1 and 2 of this paper we solve the exact equation for the Green's function of the ion in a weak field. We use here the procedure developed in[1]. In Sec. 3 we calculate the current and generalize the results obtained in this paper and in[1], and also advance considerations that make it possible, without solving the exact equation, to obtain the qualitative dependence of the current on the field and on the pressure. These considerations can be used in those cases when it is difficult to carry out exact calculations, for example in determining the pressure dependence of the current, if a transition from one type of decay to another takes place as a result of the change of the pressure.

1. INVESTIGATION OF SINGLE-PARTICLE GREEN'S FUNCTION IN A WEAK FIELD

In the low-density limit, the ion current can be calculated with the aid of the Green's function

$$G^+(x, x') = \langle \psi^+(x')\psi(x) \rangle, \tag{1.1}$$

where the averaging is carried out over the stationary state of the system, containing one ion and located in a weak electric field. The two-particle Green's function G^+ satisfies a certain linear equation (see[4]), which contains the single-particle Green's function G(x, x′).

The function G is determined from the Dyson equation, the differential form of which is (see[1])

$$i\frac{\partial G(\mathbf{p}, t'-t)}{\partial t'} - \varepsilon_0(\mathbf{p} + \mathbf{E}(t'-t))G(\mathbf{p}, t'-t) = \int_t^{t'} d\tau'\Sigma(\mathbf{p} + \mathbf{E}(\tau'-t), t'-\tau')G(\mathbf{p}, \tau'-t) + \delta(t'-t), \tag{1.2}$$

where $G(\mathbf{p}, t'-t)$ is the Fourier transform of G(x, x′) with respect to the spatial coordinates, and **p** is the momentum of the line end, corresponding to the instant t.

We are interested in the behavior of G at $t' - t \gg 1$. When $t' - t \lesssim 1$, the function G differs from the value it has in the absence of a field by an amount[2)] $E \ll 1$ and therefore introduces in the current a correction that is linear in the field and can be neglected compared with the correction resulting from the presence of the decay point.

We shall solve Eq. (1.2) in the quasiclassical approximation, i.e., we shall seek the solution for G in the form

$$G(\mathbf{p}, t'-t) = A(\mathbf{p}, t'-t)\exp\left\{-i\int_0^{t'-t} \varepsilon_1(\mathbf{p} + \mathbf{E}\tau)\,d\tau\right\}, \tag{1.3}$$

[1)]We used the term "phonon" for any excitation of the Bose system, i.e., a phonon, a roton, or a vortex ring.

[2)]In the dimensional estimates we use the system of units $\hbar = \epsilon(p_c) = p_c = 1$, the unit field intensity being the combination $\epsilon(p_c)p_c/\hbar e$.

where A is a slowly varying function.

In this approximation we obtain for $\epsilon_1(\mathbf{p})$ the equation (see[1])

$$\varepsilon_1(\mathbf{p}) - \varepsilon_0(p) - \Sigma_F(\mathbf{p}, \varepsilon_1(p)) = 0, \tag{1.4}$$

where $\Sigma_F(\mathbf{p}, \omega)$ is the Fourier transform of the function $\Sigma(\mathbf{p}, t)$ with respect to t.

The quantity $\epsilon_1(\mathbf{p})$ coincides with the ion spectrum $\epsilon(\mathbf{p})$ everywhere with the exception of the vicinity of the point p_c, where the explicit dependence of $\Sigma_F(\mathbf{p}, \omega)$ on the field becomes appreciable and causes a small addition to $\epsilon(\mathbf{p})$ to appear in $\epsilon_1(\mathbf{p})$; the imaginary part of this addition determines the damping of the ionic excitation.

For A we obtain the following expression

$$A(\mathbf{p}, t'-t) = -i\Theta(t'-t)\,(1 - \partial\Sigma_F/\partial\varepsilon)_{\mathbf{p}}^{-1/2}\,(1 - \partial\Sigma_F/\partial\varepsilon)_{\mathbf{p}'}^{-1/2} \tag{1.5}$$

where $\mathbf{p}' = \mathbf{p} + \mathbf{E}(t' - t)$ is the momentum of the start of the line, and $(\Sigma_F)_\mathbf{p} \equiv \Sigma_F(\mathbf{p}, \epsilon(\mathbf{p}))$.

When $p' \to p_c$, the derivative of A with respect to time becomes infinite like $E\exp\{a/(p_c - p')\}$ (see[2]). We shall show, however, that the essential region for the determination of the current is the region of the momenta satisfying the condition

$$\varepsilon(p_c) - \varepsilon(p) \gg E^{2/3}. \tag{1.6}$$

It can be shown that within the limits of this region A changes little during times that are characteristic for the integral in (1.2), and therefore the quasiclassical approximation is valid for the determination of the current.

To investigate $\Sigma_F(\mathbf{p}, \omega)$ near p_c, we consider first one loop, with which, as shown by Pitaevskiĭ[2], the singularity resulting from the presence of a decay threshold is connected. We designate the expression for this loop by $\Sigma_{1F}(\mathbf{p}, \omega)$:

$$\Sigma_{1F}(\mathbf{p}, \omega) = g^2 \int_0^\infty dt \int \frac{d^3q}{(2\pi)^3} A(\mathbf{p} - \mathbf{q}, t) \tag{1.7}$$

$$\times \exp\left\{-i\int_0^t [\varepsilon(\mathbf{p} - \mathbf{q} + \mathbf{E}\tau) + \omega(q) - \omega]\,d\tau\right\}.$$

We have retained here the pole part of G, i.e., the part of G which does not vanish at large $t' - t$.

The singularity of the loop is connected with the fact that the effective region of integration with respect to t becomes of the order of $(\epsilon(p_c) - \omega)^{-1}$ at $\omega \sim \epsilon(p_c)$, $|\mathbf{p} - \mathbf{q}| \sim p_m$, and $q \sim q_m$ as a result of the fact that under these conditions the arguments of the exponentials cancel each other (we are interested in the region of frequencies satisfying the inequality (1.6), and therefore $\mathbf{p}$ and $\mathbf{p} + \mathbf{E}\tau$ differ from each other by an amount $\sim E^{1/3}$).

For the integration with respect to $\mathbf{q}$, we introduce cylindrical coordinates with an axis directed along $\mathbf{p}$, and with an azimuthal angle φ (see Fig. 1). We introduce also the vectors $\mathbf{q}_m(\varphi)$ satisfying the conditions $|\mathbf{p} - \mathbf{q}_m(\varphi)| = p_m$, and $|\mathbf{q}_m(\varphi)| = q_m$, near which we expand the argument of the exponential. We have

$$\omega(q) + \varepsilon(\mathbf{p} - \mathbf{q} + \mathbf{E}\tau) = \varepsilon(p_c) + M\frac{(\mathbf{q} - \mathbf{q}_m(\varphi), \mathbf{q}_m(\varphi))^2}{q_m^2}$$
$$+ L\frac{(\mathbf{E}\tau - \mathbf{q} + \mathbf{q}_m(\varphi), \mathbf{p}_m(\varphi))^2}{p_m^2},$$

$$M = \frac{1}{2}\frac{\partial^2\omega}{\partial q^2}\bigg|_{q_m} > 0, \quad L = \frac{1}{2}\frac{\partial^2\varepsilon(p)}{\partial p^2}\bigg|_{p_m} > 0. \tag{1.8}$$

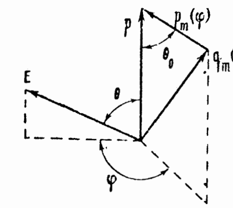

FIG. 1

We take the pre-exponential functions outside the integral sign and integrate the remaining part with respect to $q_\perp dq_\perp dq_\parallel$. As a result we obtain for the singular part of $\Sigma_{1F}(\mathbf{p}, \omega)$ the following expression:

$$\Sigma_{1F}(\mathbf{p}, \omega) = -\frac{g^2}{4\pi}\left(1 - \frac{\partial\Sigma_F}{\partial\varepsilon}\right)^{-1}_{p_m} \int_0^{2\pi} \frac{d\varphi}{2\pi}\frac{p_m q_m}{p\sqrt{ML}}$$
$$\times \int_0^\infty \frac{dt}{t}\exp\left\{-i(\varepsilon(p_c) - \omega)t - it^3\frac{LE^2\cos^2\chi}{12}\right\}, \tag{1.9}$$

where χ is the angle between $\mathbf{p}_m(\varphi)$ and $\mathbf{E}$ and is determined from the formula

$$\cos\chi = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos\varphi. \tag{1.10}$$

The integrand has an integrable singularity at t = 0, resulting from the fact that we have neglected the non-pole part of the Green's function in (1.7). However, as already noted earlier, small times can result in only a current correction that is linear in the field, and therefore it can be assumed that integrals with respect to t in (1.9) is suitably regularized. Calculating this integral under the condition $\epsilon(p_c) - \omega \gg E^{2/3}$, we obtain

$$\Sigma_{1F}(\mathbf{p}, \omega) = \frac{g^2}{4\pi}\frac{p_m q_m}{p\sqrt{ML}}\left(1 - \frac{\partial\Sigma_F}{\partial\varepsilon}\right)^{-1}_{p_m}\left\{\ln(\varepsilon(p_c) - \omega)\right. \tag{1.11}$$
$$\left. + i\sqrt{\frac{\pi}{6}}\int_0^{2\pi} d\varphi\left[\frac{4}{3}\frac{(\varepsilon(p_c) - \omega)^{3/2}}{L^{1/2}E|\cos\chi|}\right]^{-1/2}\exp\left[-\frac{4}{3}\frac{(\varepsilon(p_c) - \omega)^{3/2}}{L^{1/2}E|\cos\chi|}\right]\right\}.$$

We have neglected here small real terms compared with the logarithm.

The singular part of Σ_F is obtained, in accordance with[2], from the singular part of Σ_{1F} by replacing g with the renormalized vertex $\Gamma(\mathbf{p}, \omega) = P/(1 + Q\Sigma_{1F}(\mathbf{p}, \omega))$, where P and Q are real functions. Therefore the total self-energy part of Σ_F can be represented in the form

$$\Sigma_F(\mathbf{p}, \omega) = \Sigma_F(\mathbf{p}, \omega)|_{E=0} - i\frac{\Gamma^2}{g^2}\operatorname{Im}\Sigma_{1F}(\mathbf{p}, \omega). \tag{1.12}$$

We substitute this equation in expression (1.4) for $\epsilon_1(\mathbf{p})$ and solve it by regarding $\operatorname{Im}\Sigma_{1F}$ as a perturbation

$$\varepsilon_1(\mathbf{p}) = \varepsilon(p) - i\operatorname{Im}\Sigma_{1F}(\mathbf{p}, \varepsilon(p))\left(\frac{\partial\Sigma_{1F}}{\partial\varepsilon}\right)^{-1}_p. \tag{1.13}$$

We have thus obtained the damping, due to the electric field, of the ionic excitations near the threshold momentum. As seen from (1.11) and (1.13), the main contribution to the damping is made by diagrams describing a decay process in which the projection of the ion momentum on the field direction after the decay, $p_m|\cos\chi|$, has the maximum of all possible values at a specified initial ion momentum. Consequently, it can be stated that after the decay the ion acquires a momentum lying in the plane made up by the field vector and the initial momentum $\mathbf{p}_c$. This, as we shall see in the study of the two-particle Green's function G^+, leads to a unique distribution of the ions in momentum space.

The maximum of the function $|\cos\chi|$ (see (1.10)) can be realized at $\varphi = 0$ if $|\theta_0 - \theta| < \pi/2$, in which case $|\cos\chi_{1m}| = \cos|\theta_0 - \theta|$, and also at $\varphi = \pi$, if $|\theta_0 + \theta| > \pi/2$, in which case $|\cos\chi_{2m}| = |\cos(\theta_0 + \theta)|$.

We integrate with respect to φ in (1.11) and substitute (1.13) in (1.3). After integration with respect to τ in (1.3) we obtain for G the expression

$$G(\mathrm{p}, t'-t) = A(\mathrm{p}, t'-t)\frac{\Phi(\mathrm{p}')}{\Phi(\mathrm{p})}\exp\left\{-i\int_0^{t'-t}\varepsilon(\mathrm{p}+\mathrm{E}\tau)\,d\tau\right\}, \quad (1.14)$$

where

$$\Phi(\mathrm{p}) = \exp\left\{-E^{1/3}\ln^{-2}E\sum_k \varphi_k(\theta)\,r_k^{-4/3}e^{-r_k}\right\},$$

$$r_k = \frac{4}{3}\frac{(\varepsilon(p_c)-\varepsilon(p))^{3/2}}{L^{1/2}E\,|\cos\chi_{mk}|}, \quad (1.15)$$

The summation in $\Phi(\mathrm{p})$ is over the maxima of $|\cos\chi|$, the sum containing either one or two terms.

The function $\Phi(\mathrm{p})$ changes from 1 to 0 in a region having a dimension $\sim |\ln E|^{-2}$ near the point

$$p_0 = p_c + \frac{3}{2}\frac{a}{\ln(E|\ln E|)}. \quad (1.16)$$

Outside this region, $\Phi = 1$ at $p < p_0$ and $\Phi = 0$ at $p > p_0$. Thus, $\Phi(p')$ cuts off G at $p' \sim p_0$. Condition (1.16) is still satisfied in this case. Consequently, expression (1.14) for G is valid for all values of the arguments, with the exception of a case of no importance for the determination of the current, namely, when simultaneously $\epsilon(p_c) - \epsilon(p) < E^{2/3}$ and $\epsilon(p_c) - \epsilon(p') < E^{2/3}$.

We note that expression (1.14) has been calculated with a relative error $\sim |\ln E|^{-1}$.

2. INVESTIGATION OF TWO-PARTICLE GREEN'S FUNCTION

The exact expression for G^+ is

$$G^+(\mathrm{p}, t'-t) = -\int_{-\infty}^{t'} d\tau_1' \int_{-\infty}^{\tau_1'} d\tau_2' \int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdot$$
$$\times\int\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}G(\mathrm{p}+\mathrm{E}(\tau_1'-t),\ t'-\tau_1')G^*(\mathrm{p}+\mathrm{E}(\tau_1-t),\ t-\tau_1)$$
$$\times G^+(\mathrm{p}+\mathrm{q}_1+\mathrm{q}_2+\mathrm{E}(\tau_2-t), \tau_2'-\tau_2)\cdot$$
$$\times\gamma(\mathrm{p}+\mathrm{q}_1+\mathrm{q}_2+\mathrm{E}(\tau_2-t),\ \tau_2'-\tau_2;\ \mathrm{q}_1,\mathrm{q}_2;\ \tau_1'-\tau_2',\ \tau_2-\tau_1), \quad (2.1)$$

where γ is an irreducible four-point diagram (see[4]).

To investigate this equation, there has been developed in[1] a special procedure, according to which one first determines the pole part of G^+ at $p \neq p_m$, i.e., when the term describing the arrival of the ions in the state p due to the decay is small. The pole part of G^+, henceforth designated $G_n^+(\mathrm{p}, t'-t)$, is then represented in the form of a product

$$G_n^+(\mathrm{p}, t'-t) = G_n(\mathrm{p}')G_n^*(\mathrm{p})n(\mathrm{p}_\perp), \quad (2.2)$$

where $G_n(\mathrm{p}')$ is a solution of (1.2) that depends only on p', i.e.,

$$G_n(\mathrm{p}') = \Phi(\mathrm{p}')\left(1-\frac{\partial\Sigma_F}{\partial\varepsilon}\right)_{\mathrm{p}'}^{-1/2}\exp\left\{-i\int_0^{p_x'}\frac{\varepsilon(x, \mathrm{p}_\perp)}{E}dx\right\} \quad (2.3)$$

where $\mathrm{p}_x' \parallel \mathrm{E}$ and $\mathrm{p}_\perp' \perp \mathrm{E}$. $n(\mathrm{p}_\perp)$ is the distribution function with respect to $\mathrm{p}_\perp$ and is determined after substituting G_n^+ in (2.1), the latter being the balance equation for the influx and outflow of ions in the given state p; $n(\mathrm{p}_\perp)$ is different in different regions obtained by cutting up the momentum space by means of the surfaces $|\mathrm{p}| = p_m$, $|\mathrm{p}| = p_c$, and $|\mathrm{p}_\perp| = p_m$ (Fig. 2).

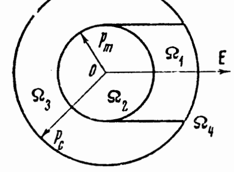

FIG. 2

When $p > p_c$ there are no ionic excitations, and therefore $n_4(\mathrm{p}_\perp) = 0$. In addition, after the decay the ion acquires a momentum $p = p_m$, and then moves in the direction of the field, so that it cannot enter the region Ω_3, i.e., $n_3(\mathrm{p}_\perp) = 0$. It remains to consider the functions $n_1(\mathrm{p}_\perp)$ and $n_2(\mathrm{p}_\perp)$, defined in regions Ω_1 and Ω_2 respectively.

From an analysis of the right-hand side of (2.1) we can determine which of the diagrams that enter in γ contribute to G_n^+. Contributions to G_n^+ are made only by those diagrams which contain stationary points of the oscillating exponentials. In the remaining diagrams the integrations with respect to time are limited to a band of the order of unity near the upper limit, and therefore we do not obtain the pole part of G^+ from them.

Stationary points can result from the pole parts of G, G^+, and γ when the decay conditions are satisfied. As a result, the only diagrams that matter for γ are those containing not more than one phonon line joining the upper and lower branches of the diagram (see Fig. 1[4]). In the opposite case, stationary points are produced when the conditions are satisfied for the decay with emission of two or more phonons; this, however, can occur only when $p > p_c$, where G_n^+ vanishes.

Thus, that part of γ which contributes to G_n^+ has the following form:

$$\gamma(\mathrm{p}+\mathrm{q};\ t'-t) = -\Gamma^2(\mathrm{p}+\mathrm{q},\ \varepsilon(\mathrm{p}+\mathrm{q}))\exp(\{i\omega(q)(t'-t)\}. \quad (2.4)$$

We substitute γ from (2.4), G from (1.14), and G_n^+ from (2.2) and (2.3) in the right side of (2.1). We denote the obtained expression by $G_1^+(\mathrm{p}, t'-t)$. When $p \neq p_m$ and $|t'-t| \gg 1$, the function G_1^+ goes over from G_n^+. From this we obtain for $n(\mathrm{p}_\perp)$ the following equations:

$$n_1(\mathrm{p}_\perp) = \int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau'\int\frac{d^3q}{(2\pi)^3}(K_1+K_2)\,n_1(\mathrm{p}_\perp+\mathrm{q}_\perp),$$

$$n_2(\mathrm{p}_\perp) = \int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau'\int\frac{d^3q}{(2\pi)^3}K_2\,n_1(\mathrm{p}_\perp+\mathrm{q}_\perp). \quad (2.5)$$

We introduce $\boldsymbol{\pi} = \mathrm{p} + \mathrm{E}(\tau - t)$. The integral kernels have the same analytic form

$$K(\pi, \tau, \tau', q) = -g^2\left(1-\frac{\partial\Sigma_F}{\partial\varepsilon}\right)_{p_m}^{-1}\left(\frac{\partial\Sigma_1 F}{\partial\varepsilon}\right)_{\pi+\mathrm{q}}^{-1}$$
$$\times\exp\left\{-i\int_0^{\tau'-\tau}\left[\varepsilon(\pi+\mathrm{q}+\mathrm{E}\eta)-\varepsilon(\pi+\mathrm{E}\eta)-\omega(q)\right]d\eta\right\}\Phi^2(\pi+\mathrm{q}),$$

and differ only in that for the kernel K_1 the value of $\boldsymbol{\pi}$ must be taken on the boundary S_{12} between the regions Ω_1 and Ω_2, while for K_2 it must be taken on the boundary S_{23}.

We introduce in the integrals (2.5) new variables for each of the kernels: φ, $|\pi + q| - p_c$, $\pi'_x - \pi_x$, $\pi_x - p_{mx}(\varphi)$, $(q - q_m(\varphi)$, and $q_m(\varphi))$, where φ is the angle between the (πq) and (πE) planes. The characteristic regions of integration of the kernels over these variables are respectively $\sim |\ln E|^{-1}$, $|\ln E|^{-2}$, $E^{1/3}$, and $E^{1/3}$. We assume therefore that the characteristic region of variation of $n_1(p_\perp)$ is comparable with the largest of these regions, and then $n_1(p_\perp)$ can be taken outside the sign of integration with respect to all the variables, with the exception of φ:

$$n_1(p_m \sin\chi) = \int_0^{2\pi} \frac{d\varphi}{2\pi} [K_1(\varphi, \chi) + K_2(\varphi, \chi)] n_1(p_c \sin\theta),$$

$$n_2(p_m \sin\chi) = \int_0^{2\pi} \frac{d\varphi}{2\pi} K_2(\varphi, \chi) n_1(p_c \sin\theta), \tag{2.6}$$

where χ is the angle between π and $\mathbf{E}$, θ is the angle between $\pi + q_m(\varphi)$ and $\mathbf{E}$, with

$$\cos\theta = \cos\chi\cos\theta_0 + \sin\chi\sin\theta_0\cos\varphi, \tag{2.7}$$

$K_1(\varphi, \chi)$ and $K_2(\varphi, \chi)$ are the results of the integration of the kernels with respect to all the variables with the exception of φ.

To find the form of the functions $n(p_\perp)$ we use the statement made in the analysis of (1.13), namely that the ion remains in the plane passing through $\mathbf{E}$ and $\mathbf{p_c}$ after the emission. This statement makes it possible to construct the momentum distribution function of the ionic excitations in the zeroth approximation in the field. To this end we consider two distribution functions that are actually symmetrical with respect to the field direction, $f_{01}(\mathbf{p})$ and $f_0(\mathbf{p})$ (see Fig. 3), each of which has the form of a cylindrical surface constructed on the vectors $p_c - p_m$, which are parallel to the field. The radius of each of the cylinders is chosen such that after the decay the ion remains on the wall of the cylinder, i.e.,

$$\rho_k \equiv p_m \sin\chi_{0k} = p_c |\sin(\chi_{0k} - \theta_0)|, \tag{2.8}$$

where k takes on the values 1 and 2.

Let us check these distribution functions for stability. Let the ion be located at a distance $\delta \ll 1$ from the cylinder wall, and then after the decay the ion will be located at a distance

$$\delta' = \delta \frac{1 - (-1)^k (p_m/p_c)\cos\theta_0}{1 - (-1)^k (p_c/p_m)\cos\theta_0}.$$

The distribution function is stable when $\delta' < \delta$, and therefore $f_{01}(\mathbf{p})$ is stable when $\theta_0 < \pi/2$ and $f_{02}(\mathbf{p})$ is stable when $\theta_0 > \pi/2$. Consequently, the significant contributions in Eq. (2.6) is made by a small vicinity of the point $p_\perp = \rho_1$ when $\theta_0 < \pi/2$, the vicinity of the point $p_\perp = \rho_2$ at when $\theta_0 > \pi/2$, and the vicinity of $\rho_1 = \rho_2$ at $\theta_0 = \pi/2$.

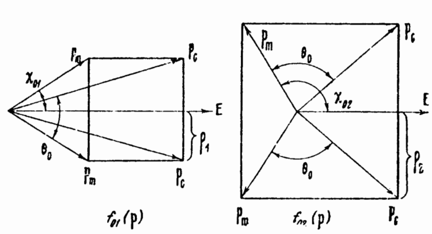

FIG. 3

We introduce the variable $z = (p_\perp - \rho)|\ln E|/b$, where

$$b = 3p_c \frac{|\cos\chi_{0k}|\cos(\chi_{0k} - \theta_0)}{\sin\chi_{0k}} > 0, \quad \rho = \rho_k,$$

and $u(z) = n(p_\perp)$, with k = 1 when $\theta_0 < \pi/2$ and k = 2 when $\theta_0 > \pi/2$.

Expanding the kernels near the point ρ, we obtain for $n(p_\perp)$ the following equations:

$$u_1(z) = \frac{\alpha}{\sqrt{\pi}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} u_1(-\alpha z + y)\, dy, \tag{2.9}$$

where

$$u_2(z) = u_1(z)\{1 + \exp[-\beta(\theta_0 - \pi/2)|\ln E|]\}^{-1},$$

$$\alpha = \frac{p_c}{p_m} \frac{\cos(\chi_{0k} - \theta_0)}{|\cos\chi_{0k}|} > 1, \quad \beta = \frac{2}{3} \frac{\sin\chi_{0k}}{\cos^2\chi_{0k}} > 0,$$

and $K(\varphi, \chi)$ are calculated with a relative error $\sim |\ln E|^{-1}$.

Changing over to Fourier components $u_{1F}(\omega)$, we obtain

$$u_{1F}(\omega) = \left(1 - \frac{i\omega}{\alpha}\right)^{-1/2} u_{1F}(-\omega/\alpha). \tag{2.10}$$

With the aid of this functional equation we can find the asymptotic form of $u_1(z)$ at large values of z:

$$u_1(z) = \frac{\text{const}}{\sqrt{|z|}} \begin{cases} e^{-\alpha z}, & z \to +\infty \\ e^{\alpha^2 z}\sqrt{\alpha/(1+\alpha)}, & z \to -\infty \end{cases}. \tag{2.11}$$

The asymptotic form of $u_1(z)$ shows that $n_1(p_\perp)$ is concentrated near ρ with a distribution width $\sim |\ln E|^{-1}$, thus justifying the assumptions made in the derivation of (2.9).

When $\theta_0 < \pi/2$ we have $u_2 \to 0$ and we obtain a distribution function $f_0(\mathbf{p})$ with a cylinder-wall width $|\ln E|^{-1}$; when $\theta_0 > \pi/2$ we have $u_2 \to u_1$ and we obtain a distribution function $f_{02}(\mathbf{p})$ with the same cylinder-wall width.

We shall need in what follows the derivative of $u_{1F}(\omega)$ at $\omega = 0$; it is obtained by expanding (2.10) near zero:

$$u_{1F}'(0) = u_{1F}(0) \frac{i}{2(1+\alpha)}. \tag{2.12}$$

We now construct the complete solution of (2.1). According to[1], G^+ is obtained by an iteration method, starting with $G^+ = G_1^+$. We then obtain for G^+ the following equation in symbolic form:

$$G^+ = G_1^+ + \sum_{s=1}^{\infty} (-GG^+\gamma)_0^s (G_1^+ - G_n^+). \tag{2.13}$$

Here the operators G, G^+, and γ are taken at E = 0, since the pole part of the difference $G_1^+ - G_n^+$ vanishes, and consequently there occur no points where the exponentials are stationary. It is therefore possible to change over in (2.13) to Fourier components with respect to $(t' - t)$, i.e., to assume that this equation is written in terms of Fourier components.

Using expression (2.2), we obtain for $G_n^+(p, \omega)$ accurate to $|\ln E|^{-1}$ inclusive, the expression

$$G_n^+(p, \omega) = 2\pi\delta(\omega - \varepsilon_0(p) - \Sigma_F(p, \omega)|_{E=0}) f(p), \tag{2.14}$$

where

$$f(p) = n_1(p_\perp)\Theta(p_0 - p) + n_2(p_\perp) \tag{2.15}$$

is the distribution function of the quasiparticles (p_0 is taken from (1.16)).

We can now use the formulas of[4] to calculate the current.

3. CALCULATION OF THE CURRENT

According to[4], the current and the particle density are given by

$$\mathbf{I}=e\int\frac{\partial\varepsilon}{\partial\mathbf{p}}f(\mathbf{p})\frac{d^3p}{(2\pi)^3},\quad \frac{N}{V}=\int f(\mathbf{p})\frac{d^3p}{(2\pi)^3} \tag{3.1}$$

To calculate these quantities, we first consider the expression $\int_0^\infty n_1(p_\perp)\varphi(p_\perp)dp_\perp$, where $\varphi(p_\perp)$ is any one of the functions obtained by integration with respect to dp_x in (3.1).

Recognizing that $n_1(p_\perp)$ is concentrated near ρ ($\rho=\rho_1$ when $\theta_0<\pi/2$ and $\rho=\rho_2$ when $\theta_0>\pi/2$), we expand $\varphi(p_\perp)$ in terms of the small quantities $p_\perp-\rho$, retaining the first two terms of the expansion. Then

$$\int_0^\infty n_1(p_\perp)\varphi(p_\perp)dp_\perp=bu_{1F}(0)\,|\ln E|^{-1}\left\{\varphi(\rho)+\varphi'(\rho)\frac{b\rho}{2(1+\alpha)}|\ln E|^{-1}\right\}. \tag{3.2}$$

We have used here Eq. (2.12).

The factor in front of the curly bracket is common to I and n/V, and therefore cancels out when the current per particle is determined. Thus, in spite of the fact that $n_1(p_\perp)$ has been determined with a relative error $\sim|\ln E|^{-1}$, it is possible to obtain the current particle with accuracy to the term $\sim|\ln E|^{-1}$ inclusive. This is explained by the fact that $n_1(p_\perp)$ enters in the calculation of the correction to the current only in the ratio $u_{1F}'(0)/u_{1F}(0)$, which is multiplied by the small quantity $|\ln E|^{-1}$.

Using (3.2), (2.15), (2.9), and (1.16), we obtain the final result for the current per particle:

$$j=e\frac{\varepsilon(p_c)-\varepsilon(p_m)}{p_{cx}-p_{mx}+2p_{mx}\xi(\theta_0)}\left\{1-\frac{3}{2}|\ln E|^{-1}\right.$$
$$\left.\times\left[\frac{p_{mx}}{p_{cx}+p_{mx}}\frac{p_{cx}-p_{mx}-2p_{cx}\xi(\theta_0)}{p_{cx}-p_{mx}+2p_{mx}\xi(\theta_0)}-\frac{p_{cx}}{p_c}\frac{a}{p_{cx}-p_{mx}+2p_{mx}\xi(\theta_0)}\right]\right\}, \tag{3.3}$$

where

$$\xi(\theta_0)=\left\{1+\exp\left[-\frac{2}{3}\frac{\rho p_m}{n^2}\left(\theta_0-\frac{\pi}{2}\right)|\ln E|\right]\right\}^{-1},$$
$$p_{mx}=\sqrt{p_m^2-\rho^2},\quad p_{cx}=\sqrt{p_c^2-\rho^2},$$
$$\rho^2=\frac{p_m^2p_c^2\sin^2\theta_0}{p_m^2+p_c^2+2p_mp_c|\cos\theta_0|},$$

and a characterizes the behavior of the spectrum near the end point (see (1)).

When $\theta_0>\pi/2$, Eq. (3.3) changes into

$$j=e\frac{\omega(q_m)}{q_m}\left\{1+\frac{3}{2}|\ln E|^{-1}\left[\frac{p_{cx}a}{p_cq_m}+\frac{p_{mx}}{q_m}\right]\right\}. \tag{3.4}$$

In this case the current always increases with the field and satisfies the Landau criterion when $E\to0$.

When $\theta_0<\pi/2$ we have

$$j=e\frac{\omega(q_m)}{\sqrt{q_m^2-4\rho^2}}\left\{1+\frac{3}{2}|\ln E|^{-1}\left[\frac{p_{cx}a}{p_c(p_{cx}-p_{mx})}-\frac{p_{mx}}{p_{cx}+p_{mx}}\right]\right\}. \tag{3.5}$$

The current can either increase or decrease with increasing field, depending on the value of a. When $E\to0$ the current is larger than that given by the Landau criterion.

In the region $\theta_0\sim\pi/2$ the current depends essentially on θ_0. If it is assumed that θ_0 varies linearly with pressure, then the rate of change of the current with changing pressure is anomalously large in this region:

$$\partial j/\partial P\sim|\ln E|.$$

The results obtained here for the current differ from the results of[1] for the case of decay into parallel excitations in having a stronger dependence on the field ($|\ln E|^{-1}$ in place of $E^{1/3}$). This difference can be explained qualitatively in the following manner.

We represent the current in the form of a sum

$$j=j_0+\alpha(E).$$

The term j_0 is determined by the anisotropy of the distribution function in the zeroth approximation in the field, since the field produces a preferred direction in momentum space. It is easy to obtain the distribution function by considering the equilibrium establishment process.

The term $\alpha(E)$ is determined by the behavior of the spectrum near the end point. It is a result of this fact that quantum effects cause the ion to decay not when p_c is reached, but when a certain momentum p(E) smaller than p_c is reached. The distribution function is then smeared out by an amount $p_c-p(E)$, hence $\alpha(E)\sim p_c-p(E)$.

Let us estimate this difference. The state of the system with one ion at a momentum **p** is separated from the state with an ion and a phonon with the same total momentum by an energy gap $\bar\epsilon(p)-\epsilon(p)$, where $\bar\epsilon(p)=\min_q[\epsilon(p-q)+\omega(q)]$.

In the presence of a field, such a state of the system is not stationary the lifetime of this state is $\tau\sim(p_c-p)/E$, and the associated energy uncertainty is $\sim E/(p_c-p)$. As soon as the energy uncertainty exceeds the value of the gap, the ion decays. Consequently, p(E) is determined from the equation

$$E/(p_c-p)\approx\bar\varepsilon(p)-\varepsilon(p). \tag{3.6}$$

In the case of decay into parallel excitations $\bar\epsilon(p)-\epsilon(p)\sim(p_c-p)^2$ (see[2]), and consequently $\alpha(E)\sim p_c-p(E)\sim E^{1/3}$. In the case of decay of non-parallel excitations we have $\bar\epsilon(p)-\epsilon(p)\sim\exp[a/(p-p_c)]$ and $\alpha(E)\sim|\ln E|^{-1}$.

With this reasoning we can determine the dependence of the current on the pressure, if a transition from decay to non-parallel excitations to a decay into parallel excitations is possible. The dependence of j_0 on P is of no interest, since j_0 changes continuously, whereas $\alpha(E)$ changes from $|\ln E|^{-1}$ to $E^{1/3}$. To calculate $\alpha(E)$ we use the dependence of the spectrum on the pressure as given in[5]. We then find that a change in the form of $\alpha(E)$ from $|\ln E|^{-1}$ to $E^{1/3}$ occurs in the region of decay into parallel excitations between $P_0+2P_1/|\ln(E\ln^2E)|$ and $P_0+6P_1/|\ln E|$ in accordance with the formula

$$\alpha(E)\sim E^{1/3}\exp(P_1/(P-P_0)), \tag{3.7}$$

where P_0 is the pressure at which the transition takes place and P_1 is a certain constant. Near $P_0+2P_1|\ln(E\ln^2E)|^{-1}$ the rate of change of the current with pressure is anomalously large:

$$\frac{\partial j}{\partial P}\sim\frac{\partial\alpha}{\partial P}\sim|\ln E|.$$

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