EXPONENTIAL DEPENDENCE OF SURFACE IMPEDANCE ON THE AMPLITUDE OF AN ALTERNATING FIELD

L. B. DUBOVSKIĬ

Moscow State University

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A method is indicated for solving the Maxwell equations for metals; the nonlinear case is considered when $\omega t_0 \ll 1$, under conditions of a normal skin effect, for an arbitrary dependence of the magnetic moment and of the conductivity on the magnetic field strength. It is shown that the surface impedance depends exponentially on the alternating field amplitude and that in some cases this can be observed experimentally.

1. INTRODUCTION

NONLINEAR effects in metals placed in an electromagnetic field are usually small. However, as noted in[1], even in a relatively weak alternating field and in a constant quantizing magnetic field H, quantum oscillations of the magnetic moment M and of the conductivity σ give rise to a nonlinearity. An external-alternating-field amplitude h on the order of the period of the quantum oscillations, i.e., $h \gtrsim H(e\hbar H/cS)$, suffices for this purpose (S—area of extremal cross section of the Fermi surface).

Whereas M and σ in metals can depend strongly on the magnetic field, they do not depend on the electric field. This is connected with the fact that the metal becomes strongly heated even in very weak electric fields and melts prior to the appearance of a dependence on the electric field.

We note that the amplitude E_1 of an alternating electric field of frequency ω in metals is much smaller than the amplitude h of the electric field ($E_1 \sim \sqrt{\omega/\sigma}h$). Usually in metals $\omega/\sigma \lesssim 10^{-10}$ (ω/σ is equal to the ratio of the displacement current to the conduction current). Therefore, if for some reason M and σ were to depend on the electric field, this could be readily taken into account by perturbation theory, the small parameter of the expansion being $(\omega/\sigma)^{1/2}$.

In this paper we confine ourselves to the case when $\omega t_0 \ll 1$ (t_0 is the free path time) and when the conditions of the normal skin effect are satisfied, i.e., the current and the magnetic moment are connected with the alternating field locally in terms of the coordinate. This case is, on the one hand, the simplest to analyze and, on the other hand, the most interesting from the physical point of view: all the nonlinear effects reach a maximum value as a result of the local field dependence of the current and of the magnetic moment as functions of the coordinate, and also because the system can "follow" in time the variation of the external fields (the condition $\omega t_0 \ll 1$).

It turns out under these conditions (see Sec. 2) that the amplitude of the magnetic field on the surface of the metal at the fundamental frequency ω is much larger than the amplitude of the magnetic field on the surface at all other frequencies (see[2]). Physically the smallness of the multiple harmonics on the surface of the amplitude of a magnetic field is a consequence of the boundary conditions (the continuity of the tangential components of E and H on the boundary of the metal) and of the fact that inside the metal the amplitude of the electric field is much smaller than the amplitude of the magnetic field, while outside the metal they are equal. Therefore for multiple harmonics, when there is no incident wave of corresponding frequency, and there is only a reflected wave, the amplitudes of the electric and magnetic fields become equal to each other, owing to the "joining together" on the boundary, i.e., a factor $(\omega/\sigma)^{1/2}$ appears for the amplitude of the magnetic field of the multiple harmonic on the surface.

This brings about a situation wherein the reflection and absorption of electromagnetic waves in metals, in the main approximation in the parameter $(\omega/\sigma)^{1/2}$, is characterized in the nonlinear case by a single quantity, namely the surface impedance, and only in the next approximation in $(\sigma/\omega)^{1/2}$ does the transfer of energy from the fundamental harmonic to the multiple harmonics become important for the solution of the problem for the exterior of the metal.

We shall assume throughout that the amplitudes of the multiple harmonics are small compared with the amplitude of the magnetic field of the fundamental frequency, not only on the surface but also in the interior of the metal. This assumption, for example, is justified when $h\partial\chi/\partial H \ll 1$ (χ is the differential magnetic susceptibility). The result obtained under this assumption differs only by a factor on the order of unity from the general case, and it can be readily shown that this factor depends very little on the magnetic moment and on the conductivity. This assumption greatly simplifies the solution of the problem, because in this case Maxwell's equations reduce to an ordinary differential equation.

At a certain isotropy of M and σ, the obtained equation can be solved exactly (the most important consequence of isotropy is that M and σ depend only on one component of the alternating field; the formulation of the problem is described in detail in Sec. 2).

It was possible to solve the equation (see Sec. 3) with the aid of the following procedure. In the linear case, the magnetic field in the metal is given by

$$H(y,t)=h\exp\left\{-(1+i)\frac{y}{c}\sqrt{2\pi\omega\sigma}+i\omega t\right\}.$$

If we assume in this expression that ω is pure imaginary ($\omega = -i\epsilon$), then the phase of the magnetic field is independent of y:

$$H(y,t)=h\exp\left\{-\frac{y}{c}\sqrt{4\pi\epsilon\sigma}+i\omega t\right\}.$$

This property of the linear equation, namely that the phase of the magnetic field is independent of y at imaginary ω, remains in force also in the nonlinear case.

As a result we obtain for the magnetic-field amplitude, at imaginary ω, a nonlinear differential equation that reduces, with the aid of the standard change of variables, to a differential equation of first order relative to the square of a new unknown function. This unknown function is equal to the surface impedance, accurate to a certain multiplier.

The expression obtained for the surface impedance leads to a number of new effects (see Sec. 4), namely, to an exponential dependence of the impedance on the amplitude of the alternating field.

In Sec. 5 we consider surface impedance at multiple harmonics.

2. FORMULATION OF PROBLEM

Assume that an electromagnetic wave of frequency ω is normally incident on the surface of a half-space filled with metal and situated in a constant magnetic field[1]. We take the normal to the surface to be the y axis.

Let the metal be placed in a homogeneous magnetic field. Let the homogeneous magnetic-induction vector corresponding to this field be directed at an angle α to the surface of the metal, and let the projection of this vector on the surface be the z axis.

We confine ourselves to the case when the polarization of the magnetic field of the incident wave has only a projection along the z axis, and the vectors H and B in the metal lie at all time in the yz plane, while the conductivity tensor is diagonal. Then

$$B_x = H_x = E_z = E_y = 0,$$

and Maxwell's equations take the form

$$\frac{\partial H_z}{\partial y}=\frac{4\pi\sigma}{c}E_x,\qquad \frac{\partial E_x}{\partial y}=\frac{1}{c}\frac{\partial B_z}{\partial t},\qquad \frac{\partial B_y}{\partial t}=0, \tag{2.1}$$

while M and σ are connected with H by

$$B_z = H_z + 4\pi M(B_y, B_z)\ \text{or conversely}\ B_z = f(B_y, H_z), \tag{2.2}$$

$$\sigma = \sigma(B_y, B_z). \tag{2.2a}$$

Since it follows from (2.1) that

$$B_y = \text{const},$$

the system of Eqs. (2.1), (2.2), and (2.2a) contains B_y only as a parameter that can be fixed. The components B_z, H_z, and E_x which remain to be determined will henceforth be designated for brevity without subscripts[2].

It is necessary to calculate from (2.1), (2.2), and (2.2a) the surface impedance (for a definition see[5]), which assumes for this choice of axes the form

$$\zeta = -\left.\frac{E}{H}\right|_{y=0}. \tag{2.3}$$

We eliminate the fields E and B from the system (2.1), (2.2), and (2.2a), and obtain for H the equation

$$(1-4\pi\chi(H))\frac{\partial^2 H}{\partial y^2}=\frac{4\pi\sigma(H)}{c^2}\frac{\partial H}{\partial t}+\frac{\partial\ln|\sigma(H)|}{\partial B}\left(\frac{\partial H}{\partial y}\right)^2, \tag{2.4}$$

where

$$\chi(H)=\frac{\partial M}{\partial B}\{f(H)\}.$$

In (2.4) we used the fact that, as follows from (2.2),

$$\frac{\partial B}{\partial t}=\frac{1}{1-4\pi\chi(H)}\frac{\partial H}{\partial t}\qquad \frac{\partial B}{\partial y}=\frac{1}{1-4\pi\chi(H)}\frac{\partial H}{\partial y}$$

We represent the alternating part of the magnetic field H(y, t) in the form of a Fourier series

$$H(y,t)=\sum_{k\neq 0}H_k e^{ik\omega t},\qquad H_{-k}(y)=H_k^*(y). \tag{2.5}$$

We take the Fourier transform of (2.4):

$$(1-4\pi\chi(0))\frac{d^2 H_k}{dy^2}-\frac{4\pi\sigma(0)}{c^2}i\omega k H_k$$
$$=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}dt\,e^{-ik\omega t}\left\{4\pi[\chi(H)-\chi(0)]\frac{\partial^2 H}{\partial y^2}\right.$$
$$\left.+\frac{4\pi}{c^2}[\sigma(H)-\sigma(0)]\frac{\partial H}{\partial t}+\frac{\partial\ln|\sigma|}{\partial B}\left(\frac{\partial H}{\partial y}\right)^2\right\}. \tag{2.4a}$$

To determine the surface impedance, it is necessary to find not the general solution of (2.4), but the particular solution satisfying the boundary conditions

$$\text{a)}\ H(y,t)|_{y\to\infty}\to 0, \tag{2.6a}$$

$$\text{b)}\ H_k(0)=E_k(0)\ \text{for}\ |k|\neq 1. \tag{2.6b}$$

Here $E_k(y)$ is the Fourier component of E(y, t). E(y, t) is determined from (2.1):

$$E(y,t)=\frac{c}{4\pi\sigma}\frac{\partial H}{\partial y}. \tag{2.1a}$$

We note that the solution H(y, t) of Eqs. (2.4), satisfying the conditions (2.6a), (2.6b), and (2.1a), satisfies also the relation

$$|H_k(0)|\ll|H_1(0)|\ \text{for}\ |k|\neq 1. \tag{2.7}$$

Indeed, if we denote the right side of (2.4a) by $\delta_k^{-1}A(y)$, where $\delta_k^{-1} = (1 + i)(2\pi\omega\sigma(0)k)^{1/2}/c$, we get for the $H_k(y)$ satisfying (2.4a) and (2.6a):

$$H_k(y)=C_k\exp\left(-\frac{y}{\delta_k}\right)-\int_y^\infty d\eta A(\eta)\,\mathrm{sh}\frac{\eta-y}{\delta_k}, \tag{2.8a}$$

where C_k is an arbitrary constant.

Using (2.1a), we obtain

$$E_k(y)=\frac{c}{4\pi\sigma}\left(-\frac{1}{\delta_k}\right)\left\{C_k\exp\left(-\frac{y}{\delta_k}\right)+\int_y^\infty d\eta A(\eta)\,\mathrm{ch}\frac{\eta-y}{\delta_k}\right\}. \tag{2.8b}$$

Regarding (2.8a) and (2.8b) at y = 0 and (2.8b) as a system of three equations with three unknowns C_k, $E_k(0)$, and $H_k(0)$, we obtain

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[1]Non-orthogonality of wave incidence need not be taken into account at the fundamental frequency, because the surface impedance does not depend on the angle of incidence (see [3]). The proof given in [3] for this fact is not affected by the nonlinear dependence of M or σ on H, because this proof is actually based only on the fact that $(\omega/\sigma)^{1/2}\ll 1$. At multiple harmonics, on the other hand, the impedance depends on the angle of incidence (see Sec. 5).

[2]The writing down of (2.2) in the form of the second equation alone is inconvenient, since $\partial H/\partial B \cong 0$ near phase transitions (see [4]) and therefore $df/dH \to \infty$.

$$H_k(0) = -(1+i)\sqrt{\frac{\omega k}{8\pi\sigma}}\int_0^\infty d\eta A(\eta)\exp\left(-\frac{y}{\delta_k}\right). \qquad (2.9)$$

Since

$$H_1(0) \sim \int_0^\infty d\eta A(\eta)\exp\left(-\frac{y}{\delta_k}\right)$$

(see (2.8a)), it follows that $H_k(0) \sim (\omega/\sigma)^{1/2} H_1(0)$.

This proves (2.7).

To determine the impedance at the fundamental frequency we put for $H(y, t)$ in (2.4)

$$H(y,t) = H_1(y)e^{i\omega t} + H_1^*(y)e^{-i\omega t} = h\varphi(y)\sin(\omega t - \beta(y)), \qquad (2.10)$$

where, by definition, it is assumed that

$$\varphi(0) = 1, \qquad H_1(y) = \frac{1}{2i}h\varphi(y)e^{-i\beta(y)}.$$

Taking the Fourier transform in (2.4) and using (2.10), we get

$$\frac{d^2H_1}{dy^2} - \frac{4\pi i\omega}{c^2}[\sigma_0(h\varphi) + \sigma_2(h\varphi)]H_1 = 4\pi\chi_0(h\varphi)\frac{d^2H_1}{dy^2}$$
$$+ 4\pi\chi_2(h\varphi)\frac{d^2H_1^*}{dy^2}e^{-2i\beta(y)} - i\left(\frac{\partial \ln|\sigma(h\varphi)|}{\partial B}\right)_1\left[2\frac{dH_1}{dy}\frac{dH_1^*}{dy}e^{-i\beta(y)}\right.$$
$$\left. - \left(\frac{dH_1}{dy}\right)^2 e^{i\beta(y)}\right] - i\left(\frac{\partial \ln|\sigma(h\varphi)|}{\partial B}\right)_3\left(\frac{dH_1^*}{dy}\right)^2 e^{-3i\beta(y)}, \qquad (2.11)$$

where

$$a_k(h\varphi) = \begin{cases} \dfrac{\omega}{2\pi}\displaystyle\int_0^{2\pi/\omega} dt\, a(h\varphi\sin\omega t)\cos k\omega t, & k \text{ even} \\ \dfrac{\omega}{2\pi}\displaystyle\int_0^{2\pi/\omega} dt\, a(h\varphi\sin\omega t)\sin k\omega t, & k \text{ odd} \end{cases} \qquad (2.12)$$

The function a in (2.12) stands for σ or χ or $\partial \ln|\sigma|/\partial B$.

To obtain (2.11) we used the following properties of periodic functions:

1) The integral of a periodic function over the period does not depend on the integration limits;

2) $\int_0^{2\pi} A(\sin\xi)\cos\xi\, d\xi = 0$.

Thus, to determine the surface impedance (see (2.3)) and (2.1a) it is necessary to solve (2.11).

3. DERIVATION OF LINEAR EQUATION FOR THE SQUARE OF THE SURFACE IMPEDANCE AT THE FUNDAMENTAL FREQUENCY, AND CALCULATION OF THE SURFACE IMPEDANCE

Let us consider the solution of (2.11) at complex values of the parameter ω. We put $i\omega = \epsilon$, where ϵ is a real quantity. Since Eq. (2.11) is linear with respect to the phase factor $\exp[-i\beta(y)]$ ($H_1 \sim \exp[-i\beta(y)]$ and $H_1^* \sim \exp[(y)]$), the function β is independent of y at real values of ϵ, and the following equation is obtained for φ (see (2.10))

$$\frac{d^2\varphi}{dy^2} + a(\varphi)\left(\frac{d\varphi}{dy}\right)^2 - b(\varphi) = 0, \qquad (3.1)$$

where

$$a(\varphi) = -\frac{h}{2}\,\frac{3(\partial\ln|\sigma(h\varphi)|/\partial B)_1 - (\partial\ln|\sigma(h\varphi)|/\partial B)_3}{1 - 4\pi[\chi_0(h\varphi) - \chi_2(h\varphi)]}$$

$$b(\varphi) = \frac{4\pi[\sigma_0(h\varphi) + \sigma_2(h\varphi)]}{c^2}\,\frac{\epsilon\varphi}{1 - 4\pi[\chi_0(h\varphi) - \chi_2(h\varphi)]}.$$

From (3.1) we must determine ζ. Using (2.1a) and (2.10), as well as the definition (2.12), we transform (2.3) into

$$\zeta = -\frac{c}{4\pi}\left[\left(\frac{1}{\sigma(h)}\right)_0 + \left(\frac{1}{\sigma(h)}\right)_2\right]\frac{d\varphi}{dy}\bigg|_{y=0}. \qquad (3.2)$$

We introduce in (3.1) a new unknown function $p(\varphi) = d\varphi/dy$, $d^2\varphi/dy^2 = p\,dp/d\varphi$, and then (3.1) takes the form

$$dp^2/d\varphi + 2a(\varphi)p^2 - 2b(\varphi) = 0 \qquad (3.1a)$$

or

$$\frac{d}{d\varphi}\left[p^2\exp\left\{2\int_0^\varphi a(\varphi')d\varphi'\right\}\right] = 2b(\varphi)\exp\left\{\int_0^\varphi a(\varphi')d\varphi'\right\}.$$

From this we get

$$p^2(\varphi)\exp\left\{2\int_0^\varphi a(\varphi')d\varphi'\right\} - p^2(0) = \int_0^\varphi 2b(\varphi')\exp\left\{2\int_0^{\varphi'} a(\varphi'')d\varphi''\right\}d\varphi'.$$

Since $p^2(0) = 0$ (the condition at $y = \infty$), we get

$$p^2(\varphi) = \int_0 2b(\varphi')\exp\left\{2\int_\varphi^{\varphi'} a(\varphi'')d\varphi''\right\}d\varphi',$$

$$p(1) = -\left(2\int_0^1 d\varphi\, b(\varphi)\exp\left\{-2\int_\varphi^1 a(\varphi')d\varphi'\right\}\right)^{1/2}.$$

The minus sign is due to the fact that we chose a solution that decreases at infinity.

Changing over to real ω,[3)] we have for the impedance (see (3.2))

$$\zeta = (1+i)\sqrt{\frac{\omega}{8\pi}}\left[\left(\frac{1}{\sigma(h)}\right)_0 + \left(\frac{1}{\sigma(h)}\right)_2\right]\left(\int_0^1 \frac{2\varphi\, d\varphi[\sigma_0(h\varphi) + \sigma_2(h\varphi)]}{1 - 4\pi[\chi_0(h\varphi) - \chi_2(h\varphi)]}\right.$$
$$\left.\times\exp\left\{h\int_\varphi^1 \frac{d\varphi'[3(\partial\ln|\sigma(h\varphi')|/\partial B)_1 - (\partial\ln|\sigma(h\varphi')|/\partial B)_3]}{1 - 4\pi[\chi_0(h\varphi') - \chi_2(h\varphi')]}\right\}\right)^{1/2} \qquad (3.3)$$

The indices designating the functions σ, $1/\sigma$, $\partial \ln|\sigma|/\partial B$, and χ in formula (3.3) are defined in (2.12).

4. INVESTIGATION OF SURFACE IMPEDANCE

Formula (3.3) can be rewritten in a more compact form:

$$\zeta = (1+i)\sqrt{\frac{\omega}{8\pi}}\left(\int_0^1 \frac{2\varphi d\varphi}{\bar{\sigma}(h\varphi)[1 - 4\pi\bar{\chi}(h\varphi)]}\right.$$
$$\left.\times\exp\left\{h\int_\varphi^1 \frac{d\varphi'\partial\ln|\bar{\bar{\sigma}}(h\varphi')|/\partial B}{1 - 4\pi\bar{\chi}(h\varphi')}\right\}\right)^{1/2}, \qquad (4.1)$$

where $\bar{\chi}(h\varphi)$, $\bar{\sigma}(h\varphi)$, and $\bar{\bar{\sigma}}(h\varphi)$ are the time-averaged differential susceptibility χ and the conductivity σ. The exact meaning of the averaging is clear from (3.3) and (2.12).

We note first that the integrand in (4.1) is always positive, since only $1 - 4\pi\chi > 0$, corresponding to $\partial H/\partial B > 0$ can be realized thermodynamically (see[4]).

The factor $(1 + i)\sqrt{\omega/8\pi\bar{\sigma}}$ in (4.1) is the impedance in the linear case (see[5]). The factor $(1 - 4\pi\bar{\chi}(h\varphi))$ is due to the difference between B and H (see, for example,[4]), where its value is given for the case $\chi = \text{const}$ and as an estimate for the general case).

In addition, there arises in (4.1) an exponential dependence on the amplitude h of the alternating field. This exponential dependence is important when the co-

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3)See [5] concerning the analyticity of ζ with respect to ω.

efficient in front of the second derivative in (2.4) is smaller than the coefficient in front of the first derivative, i.e., it is perfectly analogous to the exponential function arising in the WKB method.

Let us stop to discuss in somewhat greater detail the oscillatory dependences of M and of σ on the field (the de Haas—van Alphen and Shubnikov—de Haas effects). In this case the dependence of σ on H can be neglected, but it is necessary to take into account the dependence of $\partial \ln|\sigma|/\partial B$ on H, because σ consists of a smoothly varying large part and a rapidly alternating small increment (see, for example,[4]).

We shall assume that the constant magnetic-induction vector **B** is parallel to the surface of the metal, the conductivity tensor is diagonal, and the Fermi surface is isotropic. Inasmuch as for the quantum oscillations M and σ depend only on $|\mathbf{B}|$, the component of the alternating field perpendicular to **B** has no significance in the first approximation in h/B under the sign of the rapidly alternating function: only the projection of the alternating field **H** on **B** will make a contribution. Let further χ and σ be given by

$$\chi=\chi_0\sum_l \mu_l\cos\left(nl\frac{H}{B}-\alpha_l\right), \tag{4.2}$$

$$\frac{\partial\ln|\sigma|}{\partial B}=\varkappa\sum_l \nu_l\cos\left(nl\frac{H}{B}-\beta_l\right),$$

where $\alpha_l = nl + \alpha_0$, $\beta_l = nl + \beta_0$, $n = cS/e\hbar B$, χ_0, μ_l, α_l, κ, and ν_l are expressed in terms of the coefficients of the Lifshitz-Kosevich formulas; $\chi_0 \sim n^{3/2}(v_F/c)^2$ and $\kappa \sim n^{1/2}/B$ (for details see[4]).

In (4.2), χ and $\partial \ln|\sigma|/\partial B$ are expressed in terms of the alternating magnetic field H, and not in terms of the alternating field of the magnetic-induction vector B_1 (see (2.2)). In the presence of only one conduction band, such a change of variables was made in explicit form in[6].

Using (4.2), (2.12), and (3.3) we get for ζ

$$\zeta=(1+i)\sqrt{\frac{\omega}{8\pi\sigma}}\left(\int_0^1\frac{2\varphi d\varphi}{1-4\pi\chi_0\sum_l\mu_l\cos\alpha_l(J_0-J_2)}\right.$$
$$\left.\times\exp\left\{\varkappa h\int_\infty^1\frac{\sum_l\nu_l\sin\beta_l(3J_1-J_3)\,d\varphi'}{1-4\pi\chi_0\sum_l\mu_l\cos\alpha_l(J_0-J_2)}\right\}\right)^{1/2}. \tag{4.3}$$

In this formula, the arguments of all the Bessel functions J_0, J_1, J_2, and J_4 are equal to $nlh\varphi/B$. In writing down (4.3) we used the Bessel formula, which is represented in the form of an integral of a Bessel function of integer order.

A more attentive analysis of the integrand in (4.1) and (4.3) leads to the following conclusions:

a) There is a factor $(1 - 4\pi\bar{\chi}(h\varphi))$ in the exponential and in front of the exponential in (4.1). The value of $(1 - 4\pi\chi(H))$ tends to zero near the magnetic phase transitions—see[4] (we emphasize that here $\chi(H)$, unlike $\bar{\chi}(h\varphi)$ is the magnetic susceptibility not averaged with respect to time.

The quantity $(1 - 4\pi\bar{\chi}(h\varphi))$ can also tend to zero near the phase transitions. To this end it is necessary to have $h \ll H_\chi$ (that this is indeed the case can be verified with the aid of (3.3) and (2.12)), where H_χ is a characteristic magnetic field, at which

$$\frac{\chi(H+\rho H_\chi)-\chi(H)}{\chi(H)}\sim 1 \tag{4.4}$$

for certain values of ρ satisfying the condition $0 < \rho < 1$ (for quantum oscillations, $H_\chi = B/n$).

b) In order for $\partial \ln|\bar{\bar{\sigma}}(h\varphi)|/\partial B$ not to be much smaller than $\partial \ln|\sigma(H)|/\partial B$ (non-averaged conductivity), it is necessary to have $h \gtrsim H_\sigma$ (this can be verified with the aid of (3.3) and (2.12)), where H_σ is a field characteristic of the conductivity and can be defined in analogy with (4.4).

c) For large alternating-field amplitudes $h \gg H_\chi$, H_σ we have in the case of quantum oscillations

$$|\bar{\chi}(h\varphi)|\ll|\chi(H)|,\qquad\left|\frac{\partial\ln|\bar{\bar{\sigma}}(h\varphi)|}{\partial B}\right|\ll\left|\frac{\partial\ln|\bar{\bar{\sigma}}(H)|}{\partial B}\right|.$$

This is expressed mathematically by

$$J_\nu(A)\,|_{A\to\infty}\to\sqrt{\frac{2}{\pi A}}\cos\left(A-\frac{\nu\pi}{2}-\frac{\pi}{4}\right). \tag{4.5}$$

The smallness of the time-average quantities is due to the fact that χ and $\partial \ln|\sigma|/\partial B$ are functions of H and oscillate rapidly about zero.

Thus, for large h all the effects will be greatly weakened as a result of averaging with respect to time.

d) Time averaging for large h will not lead to a decrease of the effect if phase transitions occur periodically in time as a result of variation of χ.[4)] As a result, although the rapid oscillations still remain, their mean value will not be small, namely, the oscillations will be not about a mean "zero" but about some constant different from zero. This is analogous mathematically to

$$\lim_{A\to\infty}\int_0^1\sin Ax\,dx=0,\quad\text{но}\ \lim_{A\to\infty}\int_0^1\sin^2Ax\,dx=\frac{1}{2}.$$

The properties a)—d) of formulas (4.1) and (4.3) lead to the following effects:

1) Let

$$1-4\pi\chi_{max}(H)>0. \tag{4.6}$$

Here χ, unlike $\bar{\chi}$, is the magnetic susceptibility not averaged with respect to time, and depends not on $h\varphi$ but on $H(y, t)$. The maximum in (4.6) is taken at fixed B and h for all y and t.

It follows from (4.3) that when $h \lesssim B/n$ the impedance ζ oscillates with changing B. As soon as we get $h \gtrsim B/n$, the oscillations disappear and are replaced by absorption—an increase in the modulus of the impedance. The transition from oscillations to absorption occurs when the integrand in (4.3) begins to oscillate with changing φ.

The oscillations in the absorption are of interest in the region of amplitudes $h \sim B/n$. When $h \ll B/n$ the oscillations are small to the extent that h is small. When $h \gg B/n$, the impedance ζ barely differs from

4)Thermodynamically stable states correspond to $4\pi\chi(H) < 1$. At large χ there occurs stratification into domains. Transitions from a homogeneous state to a state with inhomogeneous **B** are possible also when $4\pi\chi < 1$. This gives rise to periodic structures (see [4]). It is of no importance to us whether a periodic or a domain structure is produced as a result of the transition. It is only important that a phase transition be actually realized, rather than, say, a metastable state.

the usual impedance because of the averaging of the alternating field with respect to time (see c)).

2) Let now

$$1-4\pi\chi_{max}(H) < 0, \text{but } 1-4\pi\bar{\chi}(0) > 0.$$

As in the preceding case 1), at first there are oscillations, which then give way to absorption with increasing h. The difference lies in the fact that because of the property d) the impedance ζ does not decrease with increasing h. It is therefore possible to detect experimentally the appearance of a domain or periodic structure, owing to the difference between ζ in this case and ζ in the case 1) $h \gg B/n$.

3) We call attention to an effect predicted in[7], which arises when $1 - 4\pi\chi_{max}(H) \ll 1$.

The essential feature of this effect is that at arbitrarily small alternating-field amplitudes ($h \ll B/n$), the value of ζ oscillates at a function of B, and the smaller h the more pronounced the oscillations, their amplitude being proportional to

$$\left[\frac{1}{4\pi\chi_0(nh/B)^2}\ln\left\{1+\frac{4\pi\chi_0(nh/B)^2}{1-4\pi\chi_0}\right\}\right]^{1/2}.$$

4) Besides the effects considered so far, which are connected with the pre-exponential factor in (4.1) and (4.3), there are effects in which there appears an exponential dependence of ζ on h. For such effects it is necessary that, first, the factor $1 - 4\pi\bar{\chi}(h\varphi)$ be close to zero, i.e., a smooth dependence of χ on H is required (see a)), and second, at the same values of H as in the first case, the value of σ should vary rapidly as a function H, so as to be able to realize the condition b). Therefore, it is necessary to have

$$H_\chi \gg H_\sigma, \quad (4.7)$$

and then in the range h of the alternating field

$$H_\sigma \lesssim h \ll H_\chi \quad (4.7a)$$

it is possible to observe an exponential dependence of ζ on h.

The condition (4.7) can be realized, for example, in the case when there are two bands in the metal, one with a large number of electrons and the other with a small one.

Another example of the possibility of realizing (4.7) is as follows: We have a ferromagnet. The value of χ is determined by the ferromagnetic properties. In addition, there are at low temperatures also quantum oscillations of σ, which determine the value of $\partial \ln|\sigma|/\partial B$.

When conditions (4.7a) are satisfied, there will be observed an anomalously large increase of ζ. Depending on whether the quantity $[1 - 4\pi\chi_{max}(H)]$ is larger or smaller than zero, the dependence will be different—see c) and d). When $1 - 4\pi\chi_{max}(H) < 0$, the exponential growth will be faster.

With increasing ζ, the normal skin effect ($\delta \ll r$) gradually becomes anomalous ($r \gg \delta$) (δ—depth of skin layer and r—radius of the electron orbit in the magnetic field). In constructing a consistent theory in the nonlinear case of anomalous skin effect, it turns out that for the square of the surface impedance $p^2(\varphi)$ (see Sec. 3) one obtains a linear equation, but unlike the case of the normal skin effect (see (3.1a)), this equation is integro-differential.

5) If $h > B/n^{1/2}$, there occurs in the case of quantum oscillations (see (4.3)) an exponential growth of ζ in the presence of only one conduction band (and not two with greatly differing carrier numbers, as was required for effects in case 4) under the condition

$$1-4\pi\chi_{max}(H) < 0 \text{ (cf. d)}.$$

6) Besides the exponential growth of the absorption, an exponential decrease of absorption is also possible. It can occur, for example, under condition (4.7a) in thin plates. The thickness of the plate is determined by the fact that the exponential of (4.1) contains a negative quantity at all φ.

7) For the quantum oscillations upon change of the constant field B, superposition of the effects of cases 4) and 6) leads to oscillations of ζ as a function of B, with the amplitude of the oscillations being exponentially dependent on h.

DEPENDENCE OF THE SURFACE IMPEDANCE AT MULTIPLE HARMONICS ON THE ANGLE OF INCIDENCE. NONLINEAR QUANTUM PSEUDORESONANCE

Let us consider the surface impedance at higher harmonics. We assume first that the electromagnetic wave is incident perpendicularly on the surface of the metal.

To find the impedance at multiple harmonics, it is necessary to substitute in (2.4) an alternating magnetic field H(y, t) in the form of a sum of two terms. The first term is the field at the fundamental harmonic (see (2.10)). The second term is a field representing a superposition of all the multiple harmonics (see also (2.5)). Recognizing that the second term, by assumption, is small compared with the first, we can linearize Eq. (2.4) relative to the second term. As a result we obtain a linear partial differential equation with variable coefficients in terms of the field of the multiple harmonics. The highest derivative is of second order with respect to the coordinate and of first with respect to time.

This equation cannot be solved exactly. A solution of this equation in the form (2.9) is not more than an estimate, since the right-hand side of (2.9) contains the field of the multiple harmonics. The method described in Sec. 2 can only be used to prove that the field of the multiple harmonics is small on the surface, but t is value cannot be determined by this method.

The nonlinear problem for multiple harmonics can be easily solved in the limiting case of weak nonlinearity ($4\pi\chi_{max}(H)) \ll 1$) for the quantum oscillations. In this case the dependence of σ and $\partial\sigma/\partial B$ on H(y, t) can be neglected, and the method described in Sec. 2 and leading to (2.9) yields a solution of the problem[5)].

The most interesting region in this case is $h \gg B/n$. The amplitude of the reflected wave at multiple harmonics H_k first increases slowly with increasing num-

[5)]A preliminary report of the effect described below is contained in [2].

ber of harmonics, and then at $k \cong nh/B$ there is a sharp increase of the amplitude H_k, by a factor $(nh/B)^{1/5}$, followed by a rapid decrease. The width of the growth region is $\sim(nh/B)^{-2/3}$ (measured along the $k(nh/B)^{-1}$ axis, on which the growth occurs at the "unity" point). This increase of the amplitude of the nonlinear harmonic is called a pseudoresonance (see[2]).

From the data given in this section and in Sec. 2 it is clear that inasmuch as the main approximation in $(\omega/\sigma)^{1/2}$ of Eq. (2.4) for the multiple harmonics vanishes as a result of the boundary conditions, it is necessary to take into account the dependence of the surface impedance on the incidence angle of the wave, since this yields corrections of the same order $(\sim(\omega/\sigma)^{1/2})$.

6. CONCLUSIONS

1. We have shown that in the main approximation in $(\omega/\sigma)^{1/2}$ in the nonlinear case the reflection and absorption of electromagnetic waves in metals is characterized, just as in the linear case, by a single quantity—the surface impedance. This is always valid for metals in the frequency region $\omega \ll \sigma$, whether the skin effect is normal or anomalous, and for domain as well as spatially-periodic structures.

2. In the linear case, the surface impedance depends exponentially on the amplitude of the alternating field, as can be observed experimentally (see cases 4)—7) of Sec. 4).

3. It is shown in Sec. 4, that the absorption of electromagnetic waves depend strongly on whether the metal is homogeneous or breaks up into domains.

4. It is shown in Sec. 5 that at multiple harmonics the surface impedance depends on the angle of incidence of the electromagnetic waves.

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