NONLINEAR THEORY OF COLLECTIVE ACCELERATION BY IONS BY A RELATIVISTIC ELECTRON BEAM

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The problem of accelerating an ion beam by means of an electron beam is considered. The nonlinear analysis presented in the paper yields the characteristic parameters of the process, viz., the energy acquired by the ions, the acceleration time, and the maximum density of the accelerated ion beam.

1. Recently, in connection with the creation of powerful relativistic electron beams, it has become possible to realize in practice the ideas of Veksler[1,2], who proposed to accelerate heavy ions by using the inverse Cerenkov effect, which takes place in a moving medium. The gist of this phenomenon is that an electron beam moving relative to an ion bunch produces, in the reference frame connected with the ions, an electric field that is stationary with respect to the coordinates and has a geometry that depends on the shape of the bunch; according to the linear theory[1,2] of this effect, the field is proportional to the density of the electron beam and to the number of particles in the bunch. Therefore, by using strong-current beams for the acceleration and increasing the number of particles in the bunch, it is possible to accelerate ions to appreciable energies.

The linear theory, however, gives only a qualitative idea of the considered acceleration process, and does not make it possible to obtain rigorous quantitative estimates. In order to estimate the efficiency of the acceleration, i.e., to determine the energy to which the ions can be accelerated, and to find the characteristic time of the acceleration process, it is necessary to consider the problem in the nonlinear approximation, using the nonlinear equations of motion of both the electron beam and of the accelerated ion bunch. Since in general form the solution of this problem is a complicated task, we consider it in two limiting cases. In the first we assume that the energy density of the accelerating electron beam greatly exceeds the energy density of the accelerated ions. This approximation is valid in the case of a strongly relativistic electron beam and allows us to disregard the nonlinear terms in the equations of motion of the electrons. As will be shown below, the principal nonlinear effect limiting the acquisition of energy by the ions is in this case the phase shift of the accelerated ion bunches relative to the accelerating field, a shift connected with the acceleration of the ions.

The opposite limiting case, when the main nonlinear effect is the decleration of the accelerating electron beam, can take place in the acceleration of "heavy" bunches, when the energy density of the field produced by the waves is comparable with the energy density of the electron beam, and the dimensions of the bunch turn out to be of the order of the plasma wavelength. In analyzing this case, we shall assume the ion bunch to be at a standstill and estimate the momentum lost by the electron beam upon interacting with the field produced by the bunch. We shall use the "captured particle" model for the calculation[3].

2. Let a relativistic electron beam move with velocity v_0 relative to an infinite chain of ion bunches located at a distance a from each other. If the linear dimensions of the bunches δa are small compared with the distance between them, $\delta a \ll a$, then the nonlinear effects due to interaction with the electric field and connected with the motion of the bunches as a unit can be taken into account by replacing each bunch by a thin charged layer with a surface charge density σ[4,5]. The system of equations of this problem consists of the nonlinearized equations of motion of the electron beam, the Poisson equation for the field, and the equations of motion of each bunch. Combining the equations of motion of the beam with the equation for the field, we obtain

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial x}\right)^2 E+\omega_0^2 E\right]=4\pi q\sigma\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial x}\right)^2 \times\sum_{s=-\infty}^{\infty}\delta[x-sa-x_s(t)], \tag{1}$$

$$\frac{d}{dt}\frac{\dot{x}_s(t)}{(1-\dot{x}_s^2(t)/c^2)^{1/2}}=\frac{q}{M}\operatorname{Re}E[t,x_s(t)],$$

where $\omega_0^2 = 4\pi e^2 n_0 m^{-1}(1-\beta_0^2)^{3/2}$ is the plasma frequency of the beam, x_s is the coordinate of each bunch, and q and M are the charge and mass of the ion.

Taking into account the spatial periodicity of the system, we seek the electric field in the form

$$E=\varepsilon(t)\exp\left\{i\left[\vartheta(t)-\frac{2\pi}{a}x\right]\right\},\quad \omega_0=\frac{2\pi v_0}{a}, \tag{2}$$

where $\epsilon(t)$ and $\vartheta(t)$ are functions that vary slowly during the period of plasma oscillation. Substituting (2) in (1) and averaging the equation for the field over the spatial period of the system, we obtain a system of ordinary nonlinear differential equations

$$\dot{\varepsilon}=q\sigma\omega_0\cos\left(\vartheta-\frac{2\pi}{a}x\right),$$

$$\dot{\vartheta}=-q\sigma\omega_0\frac{1}{\varepsilon}\sin\left(\vartheta-\frac{2\pi}{a}x\right), \tag{3}$$

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2/c^2}}=\frac{q}{M}\varepsilon\cos\left(\vartheta-\frac{2\pi}{a}x\right).$$

Expressing $\dot{x}$ in terms of ϵ from the first and third equations:

$$\dot{x}=c\varepsilon^2/[(2c\sigma\omega_0 M)^2+\varepsilon^4]^{1/2} \tag{4}$$

and changing over to the variable $\Phi = 2\pi x/a - \vartheta$, we represent the system of equations (3) in the form

$$\dot{\varepsilon} = q\sigma\omega_0 \cos\Phi,$$

$$\dot{\Phi} = -q\sigma\omega_0 \frac{1}{\varepsilon}\sin\Phi + \frac{2\pi c}{a}\frac{\varepsilon^2}{\sqrt{(2c\sigma\omega_0 M)^2+\varepsilon^4}}. \quad (5)$$

Integrating (5), we can find the dependence of Φ on ϵ:

$$\sin\Phi = \frac{c}{v_0}\frac{1}{2q\sigma}\frac{1}{\varepsilon}[\sqrt{(2c\sigma\omega_0 M)^2+\varepsilon^4} - 2c\sigma\omega_0 M], \quad (6)$$

where $\Phi(0) = \epsilon(0) = 0$.

Since, according to the first equation of the system (5), $\dot{\epsilon} = 0$ when $\Phi = \pi/2$, it follows that the equation for the maximum field amplitude ϵ_m can be obtained by putting in (6) $\sin\Phi = 1$:

$$\left(\frac{\varepsilon_m}{2q\sigma}\right)^3 - \frac{v_0^2}{c^2}\frac{\varepsilon_m}{2q\sigma} - \frac{\omega_0^2}{\omega_1^2} = 0, \quad \omega_1^2 = \frac{2\pi\sigma q^2}{Ma}. \quad (6')$$

The solution of this equation at $\omega_0 > \omega_1$ can be represented with a high degree of accuracy in the form

$$\varepsilon_m = 2q\sigma\left(\frac{\omega_0}{\omega_1}\right)^{2/3}\left[1+\frac{1}{3}\frac{v_0^2}{c^2}\left(\frac{\omega_1}{\omega_0}\right)^{4/3}\right] \quad (7)$$

(when $\omega_0 = \omega_1$ and $v_0/c = 1$, the next higher-order addition to unity is equal to $1/81$).

The maximum beam velocity $\dot{x}_m$ is obtained by substituting ϵ_m in formula (4):

$$\dot{x}_m = v_0\frac{(\omega_1/\omega_0)^{2/3}}{\sqrt{1+v_0^2c^{-2}(\omega_1/\omega_0)^{4/3}}}. \quad (8)$$

The characteristic acceleration time is determined by formulas (5) and (7), and turns out to be $t_m \approx 2\omega_1^{-2/3}\omega_0^{-1/3}$. When $t > t_m$, the ions shift into the region of decelerating phases of the field, and a process opposite to acceleration begins, with the ions transferring energy to the electron beam.

Since we have assumed above that the characteristic time of variation of the amplitude and of the phase of the field, t_m, greatly exceeds the time of plasma oscillation, it follows that, strictly speaking, the formulas obtained above are valid only when $\omega_1^{2/3} \ll \omega_0^{2/3}$ and, in accord with (8), the energy acquired by the ion beam is small compared with the energy of the electron beam. Extrapolating the formulas obtained above into the region of higher ion densities $\omega_1 \sim \omega_0$, we see that the efficiency of acceleration increases. The accelerating field acting on the ions then turns out to be

$$\varepsilon_m \sim Mv_0\omega_0/q. \quad (9)$$

The approximation considered by us, in which the nonlinearity of the equations of motion of the beam can be disregarded, takes place when the following inequality is satisfied:

$$\frac{1}{2\pi}q^2\sigma^2\left(\frac{\omega_0}{\omega_1}\right)^{4/3}\frac{2}{n_0 m v_0^2}(1-\beta_0^2)^{3/2} \ll 1 \quad (10)$$

and holds for a strongly relativistic beam.

The acceleration process investigated above admits of a simple physical explanation. In the reference frame connected with the electron beam, the ions produce a modulated beam moving through the plasma with a velocity $-v_0$. As shown in[5], such a beam excites plasma oscillations and is decelerated, the maximum amplitude of the decelerating field, in accordance with (9), being determined by the momentum lost by the ion beam during the time of the Langmuir oscillation. In the laboratory frame, this effect is perceived as acceleration of the ions in the direction of motion of the electron beam.

3. Let us consider now the acceleration of a "heavy" bunch, when main nonlinear effect limiting the maximum energy of the ions is the deceleration of the electron beam in the field produced by the ions. We use for the calculation the model of "captured particles"[3], in which it is assumed that the bunch is in a potential well produced by the field of the wave and moves together with the wave.

The motion of the particles in the potential well, in a system where the bunch is at rest, can be described with the aid of a kinetic equation for the distribution function of the ions $f(t, x, p)$:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + qE\frac{\partial f}{\partial p} = 0, \quad p = \frac{Mu}{\sqrt{1-u^2/c^2}}. \quad (11)$$

The solution of this equation, in the case when the field is potential, $E = -\partial\varphi/\partial x$, is an arbitrary function of the parameter $\epsilon_0 = q\varphi + c(\sqrt{M^2c^2+p^2} - Mc)$. An analysis carried out by Bohm and Gross[3] shows that a distribution function that describes qualitatively correctly the motion of the ions captured by the field of the wave ($q > 0$, $\varphi < 0$) can be represented in the form

$$f(p,\varphi) = N_0\frac{(\delta+1-\sqrt{1+p^2/M^2c^2})^\alpha}{2\alpha\int_0^\Delta \xi^{\alpha-1}(\Delta-\xi)^{1/2}(\Delta+2-\xi)^{1/2}d\xi}, \quad \Delta = -\frac{q\Phi}{Mc^2}, \quad (12)$$

$$\delta = -q\varphi/Mc^2,$$

where α is an arbitrary positive number, and the normalization constant is expressed in terms of the ion density N_0 at the bottom of the well, i.e., at $\varphi = \Phi$.[1)]

The density of the captured particles as a function of the potential φ can be obtained by integrating $f(p, \varphi)$ with respect to the momenta. In the case when the field energy is sufficiently large $|\delta| \gg 1$, we obtain the simple formula

$$N(\varphi) = N_0(\varphi/\Phi)^{\alpha+1}. \quad (13)$$

Substituting $N(\varphi)$ in the Poisson equation, we obtain an equation for the potential $\varphi(t, x)$:

$$\frac{\partial^2\varphi}{\partial x^2} = 4\pi e(n-n_0) - 4\pi qN_0\left(\frac{\varphi}{\Phi}\right)^{\alpha+1} \quad (14)$$

which must be considered together with the nonlinear equations of motion of the electron beam.

One possible method of solving the problem, used in[6], is to investigate stationary solutions that depend on the variable $t - x/v_{ph}$. Such an analysis, however, cannot take into account the reflection of the beam electrons from the bunch, an effect arising when the density of the captured particles is sufficiently high, since the hydrodynamic equations of motion do not take into account the effect of particle "trajectory intersection". We therefore consider the problem in terms of the Lagrange variables

$$\frac{d}{d\tau} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}, \quad x = x(t), \quad (15)$$

1)In [3] they considered the particular case $\alpha = 1/2$, when the dependence of the density of the captured particles on the potential in the nonrelativistic case is linear, i.e., convenient for analytic calculations.

i.e., we follow the motion of each electron of the beam. The equation for the field acting on the particle along the trajectory is obtained by combining the equation

$$\frac{\partial^2\varphi}{\partial t\,\partial x}=4\pi(n_0v_0-nv) \tag{16}$$

with Eq. (14). The closed system of equations describing the interaction of the electron with the field of the ion plasmoid then takes the form

$$\frac{d}{d\tau}\frac{\partial\varphi}{\partial x}=4\pi en_0(v_0-v)-4\pi qN_0v\left(\frac{\varphi}{\Phi}\right)^{\alpha+1},\qquad \frac{d}{d\tau}\frac{mv}{\sqrt{1-\beta^2}}=e\frac{\partial\varphi}{\partial x}. \tag{17}$$

Integrating the second equation of (17), we express the potential φ in terms of the particle energy:

$$\varphi=\frac{mc^2}{e}(\gamma-\gamma_0),\qquad \gamma=\frac{1}{\sqrt{1-\beta^2}} \tag{18}$$

Relation (18) makes it possible to reduce the system (17) to a nonlinear second-order equation for the particle velocity:

$$\frac{d^2}{d\tau^2}(\beta\gamma)=\omega_0^2[\beta_0-\beta-\mu^2(\alpha+2)(\gamma_0-\gamma)^{\alpha+1}], \tag{19}$$

where

$$\omega_0^2=\frac{4\pi e^2n_0}{m},\qquad \mu^2=\frac{q}{e}\frac{N_1}{n_0}\left(\frac{mc^2}{e|\Phi|}\right)^{\alpha+1},\qquad N_1=\frac{N_0}{\alpha+2}.$$

Integrating (19) and choosing the integration constant from the condition $\beta=\beta_0$ and $\beta=0$, we get

$$\frac{d}{d\tau}(\beta\gamma)=-\omega_0\left[-(1-\beta\beta_0)\gamma+\frac{1}{\gamma_0}+\mu^2(\gamma_0-\gamma)^{\alpha+2}\right]^{1/2}. \tag{20}$$

The depth of the potential well Φ can be expressed in terms of the density of the captured particles, putting in Eq. (20) $\beta=\beta_{\min}$ and $\beta_{\min}=0$, and solving the obtained algebraic equation in conjunction with Eq. (18):

$$\Phi=-2\frac{q}{e^2}mv_0^2\gamma_0\frac{N_1/n_0}{(1+qe^{-1}N_1/n_0)^2-\beta_0^2}. \tag{21}$$

The energy lost by the electron is determined by substituting the function Φ from (21) into (18).

The accelerating field acting on the plasmoid is of the order of magnitude

$$E\sim 2\frac{q}{e^2}mc\gamma_0\beta_0^2\frac{N_1/n_0}{(1+qe^{-1}N_1/n_0)^2-\beta_0^2}\omega_0. \tag{22}$$

The energy acquired by the ions located at the bottom of the potential well is determined by putting $\varphi=\Phi$ in (12):

$$W_{i\,max}=-q\Phi. \tag{23}$$

The formulas obtained above make it possible to estimate the maximum density of the ions in the bunch and the value of Φ_m corresponding to this density. Putting $\beta=0$ in (18) and in (21), we get

$$\Phi_m=-\frac{mc^2}{e}(\gamma_0-1),\qquad N_{1m}=\frac{1}{\gamma_0}n_0. \tag{24}$$

Formula (24) determines the most effective acceleration regime, when the electron beam transfers its energy completely to the bunch. At ion density levels $N_1>N_{1m}$, the electrons are reflected from the field produced by the bunch, and become accelerated in the direction opposite to that of the beam, taking energy away from the ions. Substituting the value of N_{1m} from (24) in (22), we note that the maximum accelerating field acting on the ions coincides with the momentum lost by the electron beam during the time $1/\omega_0$ of the plasma oscillation.

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