ANOMALOUS SKIN EFFECT IN A CYLINDRICAL CONDUCTOR LOCATED IN A MAGNETIC FIELD

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The effect of the curvature of the sample surface on the skin effect in metals located in a magnetic field is studied theoretically. A general expression for the current density in a cylindrical conductor in a magnetic field parallel to the axis is found for an arbitrary law of conduction-electron scattering by the sample surface. The case of a strong anomalous skin effect, when the skin depth is small compared with the mean path, cylinder radius, and Larmor radius, is considered in detail. Expressions are obtained for the surface impedance in the region of specular [condition (3.4)] and nonspecular [condition (3.5)] scattering of the current carriers by the sample surface. The size effect in a cylindrical sample, due to the appearance of new electron trajectories in the case when the Larmor radius exceeds the cylinder radius, is considered.

1. INTRODUCTION

WE develop in this paper a theory of the anomalous skin effect in a cylindrical metal sample in the presence of a constant and homogeneous magnetic field parallel to the cylinder axis, at arbitrary scattering of the conduction electrons from the surface.

Theoretical studies of the anomalous skin effect have been carried out as a rule for samples with flat surfaces (see, for example,[1-3]), and the experiments were performed on samples whose linear dimensions are frequently comparable with the mean free path of the carriers. In this connection, it is of interest to ascertain the degree to which the final results are effected by the curvature of the surface of the investigated samples. The influence of the surface curvature is conveniently investigated with cylindrical samples. The anomalous skin effect in thin cylindrical conductors without a magnetic field, assuming a large mean free path (compared with the radius of the cylinder), was investigated by the author earlier[4]. In particular, it was shown in[4] that under conditions of a strong skin effect and nonspecular reflection of the electrons from the surface of the sample, the surface impedance does not depend at all on the radius of the cylinder, and is apparently also independent of the shape of the cross section. In the case of specular reflection, this dependence is relatively weak. There arises, however, an appreciable dependence of the surface impedance on the law governing the scattering of the electrons from the surface, the impedance decreasing strongly when the reflection is close to specular.

No less important is the question of the influence of the magnetic field on the anomalous skin effect in metals. The dependence of the surface impedance on the magnetic field was calculated only for samples with flat surfaces, as a rule in connection with cyclotron resonance[5-9]. It is physically clear, however, that in pure metals with a mean free path on the order of the dimensions of the sample the curvature of the surface can greatly influence the electromagnetic properties, particularly the conductivity, since, depending on the ratio of the Larmor radius to the dimensions of the sample, the electron orbits either lie entirely inside the metal, or are bounded by the surface. It will be shown below that in the highly interesting region of strong skin effect (when the depth of penetration of the field in the metal is small compared with the radius of the cylinder), the curvature of the surface has a strong influence only when the Larmor radius is of the order of or larger than the cylinder radius. This circumstance makes it possible to study separately the influences of the surface curvature and of the magnetic field on the anomalous skin effect. The curvature of the surface is important in the region of the strongly anomalous skin effect. On the other hand in a strong magnetic field (when the radii of the orbits are of the order of the skin-layer thickness, so that the conditions for strong anomaly of the skin effect is violated) the surface curvature does not enter in the final results. It is therefore sufficient to investigate the influence of the magnetic field on the degree of anomaly of the skin effect by using a half-space as an example.

In this paper, a general expression is derived for the conductivity kernel of cylindrical conductors in a magnetic field for an arbitrary law of scattering of the electrons by the sample surface. This is followed by a detailed study of the most significant region of small skin-layer thickness (compared with the radius of the cylinder, the Larmor radius, and the mean free path). It is shown that in the case of nonspecular carrier scattering from the sample surface the surface impedance depends on the radius R of the cylinder only in the region $R < r_F$ ($r_F = v_0/\Omega$ is the Larmor radius, v_0 is the Fermi velocity, and Ω is the Larmor frequency of the electrons). On the other hand, if $R > r_F$, then the surface impedance does not depend at all on the radius of *the cylinder* and has the same form as in the case of a sample with a flat surface. In the case of specular reflection from the surface, the impedance depends most strongly on the cylinder radius in the region $R \sim r_F$.

In the region $r \approx r_F$, there takes place a size effect connected with the appearance of electron orbits of a new type in the region $r_F > R$ (see Fig. 4b below). In a narrow interval of $|r_F - R|$ of the order of the depth of penetration, the size-effect line shape is strongly influenced by electrons moving on orbits located to a considerable extent in the skin layer.

It is assumed in the calculations that the Fermi surface of the metal is a sphere and that the electron gas is degenerate. The calculations are carried out within the framework of classical mechanics[1)], and the collision integral in the kinetic equation is written in the relaxation-time approximation.

2. CURRENT DENSITY IN CYLINDER

We consider a sufficiently long cylindrical nonmagnetic metal sample, placed in a constant inhomogeneous magnetic field **H** parallel to the cylinder axis, and in an axially symmetrical alternating field (of frequency ω), the electric vector **E** of which is parallel to the vector **H** and depends only on the distance r to the cylinder axis. Let R be the radius of the cylinder. Eliminating from Maxwell's equations the ac component of the magnetic field and neglecting the displacement current compared with the conduction current, we obtain for the electric vector **E** the equation

$$\Delta \mathbf{E} = \frac{4\pi i\omega}{c^2}\mathbf{j}. \tag{2.1}$$

Here **j** is the current density vector in the metal, and c is the speed of light. The current density in the cylinder, in turn, depends on the field and on the electronic properties of the metal. To find this dependence, we use Boltzmann's kinetic equation linearized with respect to the field **E**. Introducing a cylindrical reference frame in coordinate space and a spherical frame in velocity space:

$$v_r = v\sin\theta\cos\varphi, \quad v_\varphi = v\sin\theta\sin\varphi, \quad v_z = v\cos\theta,$$
$$0 \leqslant \theta \leqslant \pi, \quad -\pi/2 \leqslant \varphi \leqslant 3\pi/2, \tag{2.2}$$

we reduce the kinetic equation to the form

$$\left(i\omega + \frac{1}{\tau}\right) f_1 + v_r \frac{\partial f_1}{\partial r} + \left(\Omega - \frac{v_\varphi}{r}\right)\frac{\partial f_1}{\partial \varphi} = e\frac{\partial f_0}{\partial \varepsilon} v_z E_z. \tag{2.3}$$

We introduced the following notation:

$$f_0 = \{\exp[(\varepsilon - \mu)/T] + 1\}^{-1}$$

is the equilibrium Fermi distribution function of the electrons in the metal; f_1 is the correction, linear in the field E, to the equilibrium distribution function, ϵ is the electron energy, τ is the relaxation time, $\Omega = eH/mc$ is the Larmor frequency of the electrons, m is the effective mass, T is the temperature, and μ is the chemical potential.

Just as in[4], it is convenient to solve the kinetic equation by introducing in place of f_1 two functions f_+ and f_- corresponding to electrons moving away from and towards the center of the cylinder:

$$f_1 = \begin{cases} e\dfrac{\partial f_0}{\partial \varepsilon} f_+, & |\varphi| < \dfrac{\pi}{2} \\ e\dfrac{\partial f_0}{\partial \varepsilon} f_-, & \dfrac{\pi}{2} < \varphi < \dfrac{3\pi}{2} \end{cases} \tag{2.4}$$

1)The most significant error occurring here is apparently connected with neglect of the quantum properties of the surface electrons, i.e., with the surface levels. The magnetic surface levels in a cylindrical sample were investigated by Prange [10]. The conditions under which the surface levels can be regarded quasiclassically can be found in the author's earlier papers [14,11].

In the equation for the function f_-, we make the change of variable $\varphi = \pi - \varphi'$, after which we omit the prime sign. As a result the interval $|\varphi| < \pi/2$ becomes the region where both functions f_+ and f_- are defined. The current density can be expressed in terms of f_+ and f_- in the following manner:

$$j_z(r) = -\frac{2e^2m^3}{h^3}\int_0^\infty v^3 \frac{\partial f_0}{\partial \varepsilon} dv \int_0^\pi \sin\theta\cos\theta\, d\theta \int_{-\pi/2}^{\pi/2} (f_+ + f_-)\, d\varphi. \tag{2.5}$$

Putting

$$\beta = (1 + i\omega\tau)/\Omega\tau \tag{2.6}$$

and

$$r_H = (v/\Omega)\sin\theta \tag{2.7}$$

(r_H is the projection of the Larmor radius on the plane z = 0), we reduce the equations for the functions f_+ and f_- to the form

$$\beta f_\pm \pm r_H \cos\varphi \frac{\partial f_\pm}{\partial r} \pm \left[1 - \frac{r_H}{r}\sin\varphi\right]\frac{\partial f_\pm}{\partial \varphi} = r_H \operatorname{ctg}\theta E_z(r), \quad |\varphi| < \pi/2. \tag{2.8}$$

The functions $f_\pm$ depend on the arguments r and φ, and also on the parameters θ and r_H. Equation (2.8) is a linear partial differential equation of first order. Its solution reduces to finding the characteristics on which the functions f_+ and f_- are constant. The solution of the characteristic equations

$$d\alpha = \frac{dr}{r_H\cos\varphi} = \frac{d\varphi}{1 - (r_H/r)\sin\varphi}, \quad |\alpha| < \frac{\pi}{2} \tag{2.9}$$

can be represented in the form

$$r = (r_H^2 + r_0^2 + 2r_0r_H\sin\alpha)^{1/2} \equiv r(\alpha), \tag{2.10}$$

$$r_0 = (r^2 + r_H^2 - 2rr_H\sin\varphi)^{1/2} \equiv r_0(r, \varphi). \tag{2.11}$$

The integration constant r_0 is the coordinate of the center of the Larmor circle. Relations (2.10) and (2.11) represent the well known geometrical cosine law for the triangle made up of the segments r, r_H, and r_0 (Fig. 1). The quantity α introduced in (2.9) is the angle variable in the reference frame connected with the center of the electron trajectory. The characteristic (2.10) enables us to write down the general solution of (2.8) in the following form:

$$f_\pm = e^{\mp\beta\alpha}\left\{A_\pm(r_0) \pm r_H \operatorname{ctg}\theta \int^\alpha E_z(r(\alpha'))\, e^{\pm\beta\alpha'} d\alpha'\right\}. \tag{2.12}$$

Here $A_+(r_0)$ and $A_-(r_0)$ are arbitrary functions. The variable α can be expressed with the aid of (2.10) and (2.11) in terms of r and φ, while r_0 is given by expression (2.11).

Just as in the case of a flat surface (see, for example,[8]), the functions $A_\pm(r_0)$ are determined by the character of motion of the electrons. The trajectories

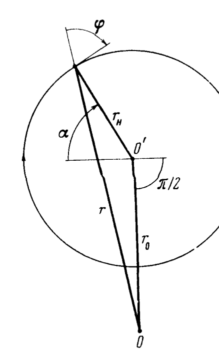

FIG. 1. Larmor trajectory of electron. O—cylinder axis. O′—center of Larmor circle.

of electrons whose Larmor radius and center coordinates satisfy the condition

$$r_0 + r_H < R, \tag{2.13}$$

lie entirely inside the cylinder (Fig. 2a). At the points B ($\alpha = \pi/2$) and C ($\alpha = -\pi/2$), the radial velocity of the electron reverses sign. The conditions that the distribution function be continuous at the points B and C determine uniquely the functions A_+ and A_- in the region (2.13). As a result we get

$$f_+ + f_- = \frac{r_H \operatorname{ctg}\theta}{\operatorname{sh}\beta\pi} \int\limits_{-\pi/2}^{\pi/2} E_z(r(\alpha'))\,[\operatorname{ch}\beta(\alpha+\alpha') + \operatorname{ch}\beta(\pi - |\alpha-\alpha'|)]\,d\alpha',$$

$$r_0 + r_H < R. \tag{2.14}$$

The electrons whose parameters r_0 and r_H are such that

$$r_0 + r_H > R, \tag{2.15}$$

collide with the surface (Fig. 2b). In this case reversal of the sign of the radial velocity of the electrons takes place both inside the metal (at the point C, $\alpha = -\pi/2$) and upon collision with the surface of the (at the point D, $\alpha = \alpha_R$, where

$$\alpha_R = \arcsin\frac{R^2 - r_H^2 - r_0^2}{2r_0 r_H} \tag{2.16}$$

is determined from the condition $r(\alpha) = R$). At the point C, located inside the metal, there takes place again the condition for continuity of the distribution function. The scattering of the electrons by the surface of the cylinder will be taken into account by means of the phenomenological boundary condition of Fuchs[12]:

$$f_- = pf_+, \quad \alpha = \alpha_R. \tag{2.17}$$

The only parameter p (specularity coefficient) in condition (2.17) must be regarded as dependent, generally speaking, on the angle with which the electron is incident on the surface. The condition for the continuity of the distribution function at the point C and the boundary condition (2.17) determine uniquely the functions A_+ and A_- in the region (2.15). Putting

$$\chi \equiv \frac{1}{2\beta}\ln p,$$

we get[2)]

$$f_+ + f_- = \frac{r_H \operatorname{ctg}\theta}{\operatorname{sh}\beta(\pi/2 + \alpha_R - \chi)} \int\limits_{-\pi/2}^{\alpha_R} E_z(r(\alpha'))\left[\operatorname{ch}\beta\left(\frac{\pi}{2} - \alpha_R + \alpha + \alpha' + \chi\right)\right.$$

$$\left. + \operatorname{ch}\beta(\pi/2 + \alpha_R - |\alpha - \alpha'| - \chi)\right] d\alpha', \quad r_0 + r_H > R. \tag{2.18}$$

Changing over in (2.5) from integration with respect to v and φ to integration with respect to r_H and r_0 with

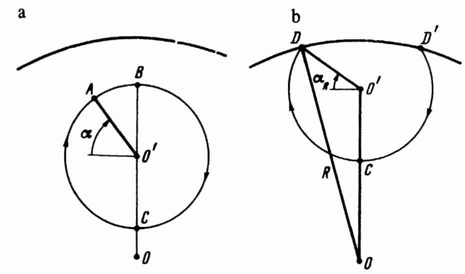

FIG. 2. Trajectory of an electron that does not collide (a) and collides (b) with the surface of the cylinder.

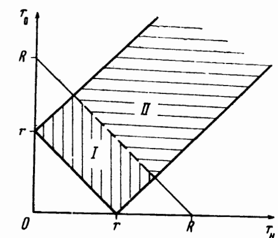

FIG. 3. Region of integration with respect to r_0 and r_H in formula (2.19).

the aid of relations (2.7) and (2.11), we obtain for the current density the expression

$$j_z(r) = -\frac{8e^2m^2\Omega^2}{h^3}\int\limits_0^{\pi/2}\operatorname{ctg}\theta\,d\theta\int\limits_0^{\infty} r_H^2\frac{\partial f_0}{\partial r_H}\,dr_H \int\limits_{|r-r_H|}^{r+r_H}\frac{(f_+ + f_-)\,r_0\,dr_0}{[4r^2r_H^2 - (r^2 + r_H^2 - r_0^2)^2]^{1/2}}. \tag{2.19}$$

The region of integration with respect to r_0 and r_H in (2.19) is shown in Fig. 3. In region I, the expression for $f_+ + f_-$ is given by formula (2.14), and in region II by formula (2.18).

Expression (2.19) for the current density in the cylinder is valid for an arbitrary ratio of the mean free path of the carriers, the Larmor radius, the cylinder radius, and the depth of penetration of the field into the metal, and also for an arbitrary law of electron scattering from the surface of the sample. In the limiting case $r_F \gg R$, expression (2.19) for the current density coincides with that obtained earlier ([4], formula (3.10)). On the other hand, if the cylinder radius is much larger than all the other parameters with the dimension of length, the sample surface can be regarded as flat. In this case formula (2.19) reduces to the corresponding expression for the current density in a flat sample (see, for example, formula (2.14) in[8]).

3. STRONGLY ANOMALOUS SKIN EFFECT

We confine ourselves below to the case of a strongly anomalous skin effect. We assume that the depth of penetration of the field into the metal δ is much smaller than the radius of the cylinder, the Larmor radius, and the electron mean free path l:

$$\delta \ll R, \quad \delta \ll r_F, \quad \delta \ll l. \tag{3.1}$$

Within the framework of conditions (3.1), the relations between R, r_F, and l can be arbitrary. The smallness of the depth of penetration makes it possible to use, when solving the problem, the Fourier integral transformation with the function E(r) continued into the region $r > R$ in even fashion. Changing over with the aid of (2.10) from integration with respect to r to integration with respect to α, we obtain the following expression for

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2)We present also expressions for the functions f_+ and f_- themselves. In the region (2113) we have

$$f_\pm = \frac{r_H \operatorname{ctg}\theta e^{\mp\beta\alpha}}{e^{\beta\pi} - e^{-\beta\pi}}\left[\int\limits_{-\pi/2}^{\pi/2} e^{\mp\beta\alpha'} + \int\limits_{\alpha}^{\pi/2} e^{\mp\beta(\alpha'-\pi)} + \int\limits_{\pi/2}^{\alpha} e^{\pm\beta(\alpha'+\pi)}\right] E_z(r(\alpha'))\,d\alpha',$$

and in the region (2.15)

$$f_+ = \frac{r_H \operatorname{ctg}\theta e^{-\beta\alpha}}{e^{\beta\pi} - pe^{-2\beta\alpha_R}}\left[\int\limits_{-\pi/2}^{\alpha_R} e^{-\beta\alpha'} + \int\limits_{-\pi/2}^{\alpha} e^{\beta(\alpha'+\pi)} + p\int\limits_{\alpha}^{\alpha_R} e^{\beta(\alpha'-2\alpha_R)}\right] E_z(r(\alpha'))\,d\alpha';$$

$$f_- = \frac{r_H \operatorname{ctg}\theta e^{\beta\alpha}}{e^{\beta\pi} - pe^{-2\beta\alpha_R}}\left[pe^{-2\beta\alpha_R}\left(\int\limits_{-\pi/2}^{\alpha_R} e^{\beta(\alpha'+\pi)} + \int\limits_{-\pi/2}^{\alpha} e^{-\beta\alpha'}\right) + \int\limits_{\alpha}^{\alpha_R} e^{-\beta(\alpha'-\pi)}\right] E_z(r(\alpha'))\,d\alpha'.$$

When $r_0 + r_H = R$, the distribution function is discontinuous if $p \neq 1$.

the Fourier component of the current density:

$$j(k)=\int_0^\infty \mathscr{E}(k')\zeta(k,k')\,dk', \qquad (3.2)$$

where

$$\mathscr{E}(k)=2\int_0^R E_z(r)\cos k(R-r)\,dr$$

is the Fourier component of the field in the metal and $\zeta(k, k')$ is the conductivity kernel in k-space, given by

$$\zeta(k,k')=-\frac{8e^2m^2\Omega^2}{\pi h^3}\int_0^{\pi/2}\operatorname{ctg}^2\theta\,d\theta\left\{\int_0^R dr_H\int_0^{R-r_H}dr_0\right.$$

$$\times\int_{-\pi/2}^{\pi/2}d\alpha\int_{-\pi/2}^{\pi/2}d\alpha'\frac{\operatorname{ch}\beta(\alpha+\alpha')+\operatorname{ch}\beta(\pi-|\alpha-\alpha'|)}{\operatorname{sh}\beta\pi}+\int_0^\infty dr_H\int_{|R-r_H|}^{R+r_H}dr_0\int_{-\pi/2}^{\alpha_R}d\alpha$$

$$\left.\times\int_{-\pi/2}^{\alpha_R}d\alpha'\frac{\operatorname{ch}\beta(\alpha+\alpha'-\alpha_R+\chi+\pi/2)+\operatorname{ch}\beta(\alpha_R-|\alpha-\alpha'|-\chi+\pi/2)}{\operatorname{sh}\beta(\alpha_R-\chi+\pi/2)}\right\}$$

$$\times\frac{r_0r_H{}^3}{r(\alpha)}\frac{\partial f_0}{\partial r_H}\cos[k(R-r(\alpha))]\cos[k'(R-r(\alpha'))]. \qquad (3.3)$$

Conditions (3.1) enable us to change over to an asymptotic expression for the conductivity kernel, which can be calculated by the saddle-point method. The main contribution to the integrals, just as in the case of a half-space, is made by small sections of the electron trajectories near the saddle points, which are determined from the condition that the radial velocity vanish. The computational difficulties are connected with the fact that in specular reflection of the electrons from the metal surface the integrand in (3.3) has an additional singularity at the saddle point $\alpha_R = -\pi/2$. This singularity is due to the "hopping" electrons in the skin layer, (also called "surface" electrons) (Fig. 4). The main difference from the flat case lies in the fact that when $r_F > R$ there exists, besides the "convex" surface trajectories whose centers lie on the opposite side of the surface (Fig. 4a) also "concave" trajectories, shown in Fig. 4b. Such electrons are continuously located in the skin layer, as a result of which their contribution to the conductivity becomes dominant if the reflection from the surface is close to specular.

The asymptotic expression for the conductivity kernel (3.3) depends in a radical fashion on the relation between the scattering of the "surface" electrons inside the metal (by impurities, phonons, lattice defects, etc.) and scattering by the surface. If the reflection is so close to specular that

$$1-p\ll|\beta|(\delta/r_F)^{1/2}, \qquad (3.4)$$

then, in spite of frequent collisions with the surface, the "surface" electrons are scattered predominantly in the interior of the metal. With increasing degree of diffuseness of the scattering by the surface, the condition (3.4) is violated, and in the region

$$1-p\gg|\beta|(\delta/r_F)^{1/2} \qquad (3.5)$$

the electrons "hopping" in the skin layer are scattered predominantly by the surface inhomogeneities. (It is assumed that the Larmor radius and the radius of the cylinder are of the order of the electron mean free path. Therefore, by virtue of the condition (3.1) we have

$$|\beta|(\delta/r_F)^{1/2}\ll 1,\quad |\beta|(\delta/R)^{1/2}\ll 1, \qquad (3.6)$$

so that inequality (3.4) takes place only for values of p

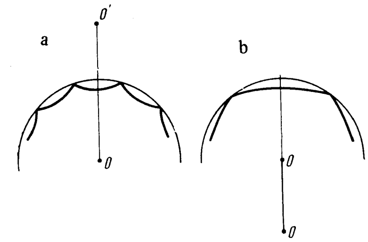

FIG. 4. "Convex" (a) and "concave" (b) trajectories of surface electrons.

very close to unity. To the contrary, condition (3.5) holds in a considerable range of variation of the specularity coefficient.)

Omitting the intermediate steps, we write down first the asymptotic expression (under conditions (3.1) and (3.6)) for the conductivity kernel in the region of nonspecular scattering (3.5):

$$\zeta(k,k')=\frac{2\pi^2e^2m^2v_0{}^2}{h^3}\left\{r\frac{\delta(k-k')}{k}-\frac{2s}{(k+k')(kk')^{1/2}}-2t\frac{\ln(k/k')}{k^2-k'^2}\right\}. \qquad (3.7)$$

Here

$$r=(1-\Gamma_z)\frac{1+p}{1-p}+\Gamma_z\frac{1-p\lambda^2}{(1-\lambda)(1-p\lambda)},\quad s=\frac{\Gamma_z}{8\pi}\frac{(1-p)(1+\lambda)^2}{(1-\lambda)(1-p\lambda)}$$

$$t=\frac{1}{\pi^2}\frac{1+p}{1-p}\left[1-\frac{\Gamma_z}{4}\frac{(1+p)(1-\lambda)}{1-p\lambda}\right],\quad \lambda=e^{-2\pi\beta},$$

$$\Gamma_z=\begin{cases}1, & r_F<R\\ \dfrac{2}{\pi}\left[\arcsin\dfrac{R}{r_F}+\dfrac{R}{r_F}\sqrt{1-\left(\dfrac{R}{r_F}\right)^2}\right], & r_F>R.\end{cases} \qquad (3.8)$$

With the aid of (2.1), (3.2), and (3.7) we obtain the following equation for the Fourier component of the field in the cylinder:

$$\left(k^2+ir\frac{k_0{}^3}{k}\right)\mathscr{E}(k)$$

$$-2ik_0{}^3\int_0^\infty\mathscr{E}(k')\left[\frac{s}{(k+k')(kk')^{1/2}}+t\frac{\ln(k/k')}{k^2-k'^2}\right]dk'=-2E_z'(R), \qquad (3.9)$$

where

$$k_0=\left(\frac{e^2m^2v_0{}^2\omega}{c^2\hbar^3}\right)^{1/3}. \qquad (3.10)$$

We note that if the Larmor radius r_F is smaller than the radius of the cylinder R, the latter drops out completely from Eq. (3.9), which coincides in this case with the equation for a bulky sample with a flat surface[8]. The influence of the surface curvature becomes manifest only in the region $r_F > R$.

An equation of the type of (3.9) was obtained by Azbel' and Kaner in the theory of cyclotron resonance in metals[5]. Its exact solution was obtained by Hartman and Luttinger[13]. For the surface impedance in the region of nonspecular scattering, we obtain the following expression:

$$Z=-\frac{4\pi i\omega}{c^2}\frac{E_z(R)}{E_z'(R)}=Z_d\frac{4}{r^{1/3}}\left[\frac{\sin(\pi z_1/3)\sin(\pi z_2/3)}{\sin(\pi z_1/2)\sin(\pi z_2/2)}\right]^2 \qquad (3.11)$$

where $Z_d = 2\sqrt{3}\pi\omega e^{i\pi/3}/c^2k_0$ is the surface impedance of a bulky sample with a flat diffuse surface, and the quantities z_1 and z_2 ($0 \le \operatorname{Re} z_i \le 1$, i – 1, 2) are determined from the equation

$$(\cos\pi z)^2-(1-2\pi s/r-\pi^2t/r)\cos\pi z-2\pi s/r=0.$$

In the region $r_F < R$, expression (3.11) for the surface impedance does not depend on the radius of the cylinder and coincides with formula (3.7) of[8]. In the limiting case $R \ll r_F$, the surface impedance depends on neither the magnetic field nor on the cylinder radius:

$$Z = Z_d\left(\frac{1-p}{1+p}\right)^{1/3}. \qquad (3.12)$$

Formula (3.12) was obtained in[4] by assuming a large electron mean free path (compared with the radius of the cylinder). From (3.6) it follows, however, that expression (3.12) is valid in fact in a wider range

$$l \gg (\delta R)^{1/2} \qquad (3.13)$$

($l = v_0/|i\omega + 1/\tau|$ is the electron mean free path and $\delta = k_0^{-1}$ is the depth of penetration of the field into the metal). The meaning of condition (3.13) becomes clear if it is noted that $(\delta R)^{1/2}$ is of the order of magnitude of the electron path in the skin layer between two successive collisions with the surface.

In region (3.4) of specular reflection from the surface, expression (3.3) for the conductivity kernel reduces to the form

$$\zeta(k, k') = \frac{2\sqrt{\pi}\, e^2 m^2 v_0^2}{\beta h^3} \times r_F^{1/2}\left[Y_1\left(\frac{r_F}{R}\right) + Y_2\left(\frac{r_F}{R}\right)\right]\frac{|k-k'|^{-1/2} - (k+k')^{-1/2}}{(kk')^{1/2}}. \qquad (3.14)$$

The functions

$$Y_1(x) = \int_0^1 [y(1-y^2)(xy+1)]^{1/2}\, dy \qquad (3.15)$$

and

$$Y_2(x) = \begin{cases} 0, & x < 1 \\ \int_{1/x}^{1} [y(1-y^2)(xy-1)]^{1/2}\, dy, & x > 1 \end{cases} \qquad (3.16)$$

are proportional to the contribution made to the conductivity by the electrons whose trajectories are represented in Figs. 4a and 4b, respectively. Using the asymptotic form (3.14) of the conductivity kernel, we obtain the following expression for the Fourier component of the field in a cylinder with specular surface:

$$k^2\mathscr{E}(k) + i\Lambda k_0^3 \int_0^\infty \mathscr{E}(k')\frac{|k-k'|^{-1/2} - (k+k')^{-1/2}}{(kk')^{1/2}}\, dk' = -2E'(R), \qquad (3.17)$$

where

$$\Lambda = \frac{r_F^{1/2}}{\pi^{3/2}\beta}\left[Y_1\left(\frac{r_F}{R}\right) + Y_2\left(\frac{r_F}{R}\right)\right].$$

Reducing (3.17) to dimensionless form, we obtain the following expression for the surface impedance:

$$Z = Z_3 f(r_F/R), \qquad (3.18)$$

where

$$f(x) = \left\{\frac{4x}{9[Y_1(x) + Y_2(x)]^2}\right\}^{1/5},$$

$$Z_3 = \frac{A}{c}\left[\frac{R\omega}{c}\left(\frac{\omega}{\omega_0}\right)^4\left(1 + \frac{1}{\omega^2\tau^2}\right)\right]^{1/5} \exp\left\{i\left(\frac{\pi}{2} - \frac{2}{5}\operatorname{arctg}\frac{1}{\omega\tau}\right)\right\} \qquad (3.19)$$

is the surface impedance of a cylinder with specular walls in the absence of a magnetic field, calculated in[4]. The numerical factor A is determined by the solution of the dimensionless equation. For the case of a flat surface, this equation was solved by Kaner and Makarov[7]. In accordance with[7], we get

$$A = \frac{2^{16/5}\pi}{5}e^{-I}, \qquad I = \int_0^\infty\left[\frac{e^{3t/2} - e^t}{e^{5t/2} - 1} - \frac{1}{5}e^{-t}\right]\frac{dt}{t} \approx 0{,}215.$$

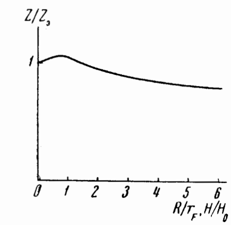

FIG. 5.

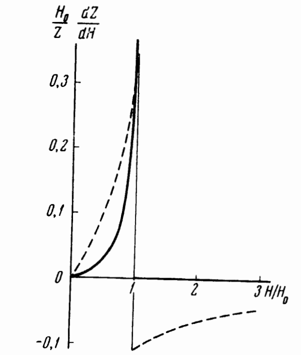

FIG. 6

FIG. 5. Dependence of the surface impedance on the cylinder radius and on the magnetic field in specular reflection of the electrons from the surface, when current flows along the cylinder ($H_0 = mv_0c/eR$).

FIG. 6. Dependence of the derivative of the surface impedance in the azimuthal direction on the magnetic field. The dashed curve corresponds to specular reflection and the solid one to diffuse reflection ($H_0 = mv_0c/eR$).

A plot of the function (3.19) is shown in Fig. 5. With increasing magnetic field in the region $r_F \ll R$, we have $f(x) \sim x^{1/5}$ so that the radius of the cylinder drops out and its place is taken by the Larmor radius. In this case we obtain the same expression for the surface impedance as in the case of a sample with a flat surface[7,8].

4. SIZE EFFECT IN CYLINDRICAL SAMPLE

The theory of anomalous skin effect in a cylindrical conductor in a magnetic field, developed in this paper, makes it possible to present a simple and lucid interpretation of the behavior of the size effect, observed in cylindrical samples of potassium[14]. By measuring the dependence of the derivative of the surface impedance in the azimuthal direction (when the currents flow around the cylinder), Blaney[14] observed, in particular, a monotonic growth of dZ/dH with the field in the region $r_F > R$.

We shall calculate below the surface impedance (and its derivative with respect to the magnetic field) in the approximation of a strongly anomalous skin effect. In the case when the current flows around the cylinder, the calculation is carried out in exactly the same manner as in the case when **H** is parallel to **E**. The only difference is that it is necessary to substitute $v_\varphi E_\varphi$ for $v_Z E_Z$ in the right side of the kinetic equation (2.3) and in the subsequent formulas. Omitting all the intermediate steps, we turn directly to the asymptotic expressions for the conductivity kernel in the case when the skin layer contains a small fraction of the trajectory of the electrons making an appreciable contribution to the conductivity. Both in the case of nonspecular (3.5) and specular scattering (3.4) of the electrons by the surface, we obtain for the asymptotic value of the conductivity kernel the same expressions (3.7) and (3.14), the only difference being that now Γ_Z in (3.8) is replaced by

$$\Gamma_\varphi = \begin{cases} 1, & r_F < R, \\ \frac{2}{\pi}\left[\arcsin\frac{R}{r_F} - \frac{R}{r_F}\sqrt{1 - \left(\frac{R}{r_F}\right)^2}\right], & r_F > R \end{cases} \qquad (4.1)$$

and the functions Y_1 and Y_2 are determined by the formulas

$$Y_1(x)=\int_0^1 [y^5(xy+1)/(1-y^2)]^{1/2}\,dy, \qquad (4.2)$$

$$Y_2(x)=\begin{cases} 0, & x<1 \\ \int_{1/x}^1 [y^5(xy-1)/(1-y^2)]^{1/2}\,dy, & x>1 \end{cases} \qquad (4.3)$$

The integrals (4.2) and (4.3), just like (3.15) and (3.16), can be reduced to elliptic integrals.

Figure 6 shows the dependence of the derivative of the surface impedance in the azimuthal direction on the magnetic field for the cases of diffuse (solid curve) and specular (dashed curve) scattering of the electrons by the surface. The solid curve in Fig. 6 corresponds to the case $\lambda = 0$, i.e., the mean free path is assumed to be small compared with the Larmor radius. It is curious that even in this case a size effect takes place at $r_F \approx R$. In view of the weak dependence of the expression in the square brackets in (3.11) on z_1 and z_2, the derivative of the surface impedance in diffuse scattering is well approximated by the formula

$$\frac{dZ}{dH} \sim \begin{cases} \dfrac{\text{const}\cdot H^2}{\sqrt{H_0^2-H^2}}, & H<H_0 \\ 0, \quad H>H_0, & H_0=\dfrac{mv_0c}{eR}. \end{cases} \qquad (4.4)$$

In the case of specular reflection of the electrons from the surface, the surface impedance and its derivative with respect to the magnetic field do not depend at all on the ratio of the mean free path to the Larmor radius.

It is important to note that when the current flows in the azimuthal direction the asymptotic expressions (3.7) and (3.14) are valid under the additional condition

$$|R-r_F| \gg \delta, \qquad (4.5)$$

i.e., when only small segments of the Larmor circles of the electrons making the main contribution at the conductivity fall in the skin layer. In the small region

$$|R-r_F| \sim \delta \qquad (4.6)$$

our results do not hold, since we do not take into account the contribution made to the conductivity by the electrons moving along circles that are concentric with the surface of the cylinder and are located in the skin layer. In the case when the current-density vector is parallel to the cylinder axis, the limitation (4.5) is of no importance, for when $r_F \approx R$ the electrons corresponding to the maximum intersection of the Fermi surface with the plane $p_z = 0$ stay in the skin layer for a long time. These electrons make an appreciable contribution only to the axial current.

Thus, when $H = H_0$ ($H_0 = mv_0c/eR$), a jump takes place (at $\delta \to 0$) in the derivative of the surface impedance with respect to the magnetic field. This jump is connected with the appearance in the region $r_F > R$ of a new type of electron trajectories, namely, "concave" trajectories, located in the skin layer (Fig. 4b). In the case of a flat surface, there are no such trajectories. Therein lies the main difference between the size effect in a cylinder and the size effect in a plane-parallel plate[15], connected with the cutoff of the electron orbits by the boundaries of the sample. In the small region (4.6) (the region of the jump), an appreciable contribution to the conductivity is made by electrons which are not accounted for in our calculation, and have an appreciable part of their trajectories situated in the skin layer. Allowance for these electrons is essential in the calculation of the exact form of the size-effect line in the region of the jump.

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