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RADIATION TRANSFER IN ANISOTROPIC MEDIA

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Formulas are obtained for the angular distribution and polarization of photons scattered in an optically thick anisotropic medium. It is shown that the photon propagation can be characterized by a diffusion tensor, an explicit expression for which has been obtained in terms of the polarizability tensor of the particles of the medium. The scattering of light in a gyrotropic medium, in a medium composed of axially oriented particles, and in an aligned medium is considered in detail. Analytic formulas for linear and circular polarization, and for the angular distribution of photons emitted from a thick plane layer of matter, are derived for a weakly gyrotropic medium.

1. STATEMENT OF THE PROBLEM

LET us consider the problem of the intensity and polarization of electromagnetic radiation passing through an optically thick anisotropic (in particular, gyrotropic) medium which consists of individual scatterers with dimensions much smaller than the wavelength of the radiation. Such media are frequently encountered in nature. For example, the dust particles in cosmic dust clouds are frequently oriented by magnetic fields or gas flows; colloidal particles are oriented by the flow of a solution; dipole molecules in a gas can be oriented by external electric fields; a rarefied electron-proton plasma possesses gyrotropic properties in a magnetic field, and so on. The theory of the passage of radiation through such a medium has been worked out with sufficient completeness only for the case of small optical thickness.[1-3] The problem of the emergence of synchrotron radiation from a thick layer of equilibrium magneto-active plasma has also been considered,[3-5] as well as the problem of the passage of radiation through a thick layer of an isotropic medium.[6,7] Recently, a number of new phenomena have been discovered in astrophysics, and their analysis requires a detailed theory of the passage of electromagnetic radiation through various types of anisotropic media. We have in mind, for example, the discovery of discrete x-ray sources, many of which evidently have a thick envelope of magneto-active plasma,[8,9] and also the discovery of powerful sources of infrared radiation, which is most likely due to heat radiation of clouds consisting of oriented dust particles and gas molecules.[10,11]

In the present work, we find the characteristics of radiation both inside an optically thick homogeneous anisotropic medium, and also passing through such a medium. We shall assume that a fixed set of coordinates is connected with the medium, with the z axis directed along **h**—one of the physically distinct directions in the anisotropic medium. The transport equation for the density matrix $\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h})$ (erg/cm^2-sec-sr), which is defined in the usual way in terms of the components of the vector of the electric field intensity of a wave propagating in the direction $\mathbf{n}_1$ ($\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h}) = (8\pi)^{-1}cE_\alpha(\mathbf{n}_1\mathbf{r})E^*_\beta \times \times (\mathbf{n}_1\mathbf{r})$) has the form[5,12]

$$(\mathbf{n}_1\nabla)\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h}) = -N_0\lambda i[\rho_{\alpha\gamma}(\mathbf{n}_1\mathbf{r}\mathbf{h})\langle t_{\gamma\beta}{}^+(\mathbf{n}_1\mathbf{n}_1\mathbf{h})\rangle - \langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_1\mathbf{h})\rangle\rho_{\gamma\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h})]$$
$$+ N_0\int d\mathbf{n}_x\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\rho_{\gamma\nu}(\mathbf{n}_x\mathbf{r}\mathbf{h})t_{\nu\beta}{}^+(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\rangle + Q_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h}). \quad (1)$$

Here N_0 is the concentration of scattering particles in the medium, $Q_{\alpha\beta}$ the density matrix of the sources, $t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})$ the scattering matrix of the photon from the direction $\mathbf{n}_x$ to the direction $\mathbf{n}_1$, which is connected with the polarizability tensor of the scattering particle $\alpha_{nm}(\mathbf{h})$ by the relation

$$t_{\nu\gamma}(\mathbf{n}_1\mathbf{n}_x\mathbf{h}) = k^2M_{\nu n}(\mathbf{n}_1\mathbf{n}_x)\alpha_{n\gamma}(\mathbf{n}_x) \equiv k^2M_{\nu n}(\mathbf{n}_1\mathbf{h})\alpha_{nm}(\mathbf{h})M_{m\gamma}{}^+(\mathbf{n}_x\mathbf{h}),$$

where $k = 2\pi/\lambda = \omega/c$ is the wave number of the photon, the matrix $M_{km}(\mathbf{n}_1\mathbf{n}_0) = \mathbf{e}_k^{(1)*}, \mathbf{e}_m^{(0)}$ connects the components of the arbitrary vector $A_k(\mathbf{n}_1)$ with the components $A_m(\mathbf{n}_0)$, i.e., $A_k(\mathbf{n}_1) = M_{km}(\mathbf{n}_1\mathbf{n}_0)A_m(\mathbf{n}_0)$; the index **n** on the components of the vectors or tensors indicates that these components are connected in the set of coordinates with z axis along the unit vecotr **n**; $\mathbf{e}_k^{(1)}$ and $\mathbf{e}_m^{(0)}$ are the unit vectors of the corresponding set of coordinates, possessing orthonormality: $\mathbf{e}_k^{(1)}, \mathbf{e}_m^{(1)*} = \delta_{km}$, $\mathbf{e}_k^{(0)}\cdot\mathbf{e}_m^{(0)*} = \delta_{km}$ (these unit vectors can be complex). The Greek indices can take on two values (−1, 1) and the Latin, three (−1, 0, 1). Summation is indicated by repeated indices. The brackets ⟨ ⟩ denote averaging over the initial and summation over the final states of the scattering particle. In what follows, we shall make use of the orthonormalized cyclic unit vectors, which are most convenient in the theory of transport of polarized radiation:

$$\mathbf{e}_{-1} = -(\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}, \quad \mathbf{e}_0 = \mathbf{e}_z, \quad \mathbf{e}_1 = (\mathbf{e}_x - i\mathbf{e}_y)/\sqrt{2}.$$

In these coordinates, the matrix $M_{km}(\mathbf{n}_1\mathbf{n}_0)$ is equal to the well-known Wigner rotation matrix[13] $D^{(1)}_{km}(\Omega_{01})$. Ω_{01} is the set of Eulerian angles $(\varphi_{01}, \beta_{01}, \gamma_{01})$ in the transition from the set of coordinates with the z axis along $\mathbf{n}_0$ to the set of coordinates with the x axis along $\mathbf{n}_1$. The explicit form of $D^{(1)}_{kn}(\Omega_{01})$ is:

$$D^{(1)}_{kn}(\varphi, \beta, \gamma) = d^{(1)}_{kn}(\beta)\exp[i(k\varphi + n\gamma)],$$

where

$$d^{(1)}_{-1-1} = d^{(1)}_{11} = (1 + \cos\beta)/2, \quad d^{(1)}_{00} = \cos\beta, \quad d^{(1)}_{-11}$$
$$= d^{(1)}_{1-1} = (1 - \cos\beta)/2, \quad d^{(1)}_{-10} = d^{(1)}_{01} = -d^{(1)}_{0-1}$$
$$= -d^{(1)}_{10} = -\sin\beta/\sqrt{2}.$$

Equation (1) has the following form in cyclic coordinates:

$$(\mathbf{n}_1\nabla)\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h}) = -N_0 2\pi i k[\rho_{\alpha\gamma}(\mathbf{n}_1\mathbf{r}\mathbf{h})D^{(1)+}_{\gamma n}(\Omega_{h1})\langle\alpha_{nm}{}^+(\mathbf{h})\rangle D^{(1)}_{m\beta}(\Omega_{h1})$$
$$- D^{(1)+}_{\alpha n}(\Omega_{h1})\langle\alpha_{nm}(\mathbf{h})\rangle D^{(1)}_{m\gamma}(\Omega_{h1})\rho_{\gamma\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h})] + N_0 k^4 D^{(1)+}_{\alpha k}(\Omega_{h1})D^{(1)}_{n\beta}(\Omega_{h1})$$
$$\times\langle\alpha_{kr}(\mathbf{h})\alpha_{sn}{}^+(\mathbf{h})\rangle\int d\mathbf{n}_x D^{(1)}_{r\gamma}(\Omega_{hx})\rho_{\gamma\nu}(\mathbf{n}_x\mathbf{r}\mathbf{h})D^{(1)+}_{\nu s}(\Omega_{hx}) + Q_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h}). \quad (2)$$

In a homogeneous medium, under the conditions $\lambda \ll l$ and $N_0 a_0^3 \ll 1$, where l is the free path length of the photon and a_0 the characteristic dimension of the scattering particle, plane waves of two types can be propagated, as is well known:

$$\mathbf{E}^{(1)}(\mathbf{n}) = \mathbf{f}^{(1)}(\mathbf{n})\mathscr{E}_1\exp\left[i\left(\frac{\omega}{c}\varkappa_1(\mathbf{n})\mathbf{r}\mathbf{n} - \omega t\right)\right],$$

$$\mathbf{E}^{(2)}(\mathbf{n}) = \mathbf{f}^{(2)}(\mathbf{n})\mathscr{E}_2\exp\left[i\left(\frac{\omega}{c}\varkappa_2(\mathbf{n})\mathbf{r}\mathbf{n} - \omega t\right)\right].$$

Their indices of refraction $\kappa_1(\mathbf{n})$ and $\kappa_2(\mathbf{n})$ and the unit polarization vectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ are determined by the properties of the forward scattering matrix

$$\langle t_{\alpha\gamma}(\mathbf{nnh})\rangle f^{(n)}_\gamma(\mathbf{nh}) = t^{(n)}(\mathbf{nh})f^{(n)}_\alpha(\mathbf{nh}),$$
$$\varkappa_i(\mathbf{n}) = 1 + 2\pi N_0 t^{(i)}(\mathbf{nh})/k^2.$$

We note that the unit vectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ (the normalized eigenvectors of the matrix $\langle\hat{t}(\mathbf{nnh})\rangle$) can also be non-orthogonal in the general case ($\mathbf{f}^{(1)}\cdot\mathbf{f}^{(2)*} \neq 0$). They are orthogonal only for the so-called normal matrices ($\hat{t}\hat{t}^+ = \hat{t}^+\hat{t}$). The eigenvalues $t^{(i)}(\mathbf{nh})$ and the eigenvectors $f^{(i)}(\mathbf{nh})$ are expressed in corresponding fashion in terms of the elements $\langle t_{\alpha\gamma}(\mathbf{nnh})\rangle$:

$$2t^{(1,2)}(\mathbf{nh}) = \mathrm{Sp}\langle\hat{t}(\mathbf{nnh})\rangle \pm \sqrt{[\mathrm{Sp}\langle\hat{t}(\mathbf{nnh})\rangle]^2 - 4\det\langle\hat{t}(\mathbf{nn})\rangle},$$
$$f_1{}^{(i)}(\mathbf{nh}) = f^{(i)}_{-1}(\mathbf{nh})\,(t^{(i)}(\mathbf{nh}) - \langle t_{-1-1}(\mathbf{nnh})\rangle)/\langle t_{-11}(\mathbf{nnh})\rangle. \quad (3)$$

By knowing $f^{(i)}_\alpha(\mathbf{nh}) \equiv u_{\alpha i}(\mathbf{nh})$, we can connect the components of the field intensity of the wave $E_\alpha(\mathbf{nr})$ and the density matrix $\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h})$ with the components $\mathscr{E}_i(\mathbf{nr})$ and the density matrices $\mathscr{R}_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h}) \equiv (c/8\pi)\mathscr{E}_i(\mathbf{n}_1\mathbf{r})\mathscr{E}^*_k(\mathbf{n}_1\mathbf{r})$ (here i, k = 1, 2):

$$E_\alpha(\mathbf{n}) = u_{\alpha i}(\mathbf{nh})\mathscr{E}_i(\mathbf{n}), \quad \rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h}) = u_{\alpha i}(\mathbf{n}_1\mathbf{h})\mathscr{R}_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h})u_{k\beta}{}^+(\mathbf{n}_1\mathbf{h}). \quad (4)$$

Using (4), we can obtain the transport equation for $\mathscr{R}_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h})$ from Eq. (1):

$$(\mathbf{n}_1\nabla)\mathscr{R}_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h}) = -N_0 g_{ik}(\mathbf{n}_1\mathbf{h})\mathscr{R}_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h})$$
$$+ N_0\int d\mathbf{n}_x\langle s_{in}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\mathscr{R}_{nm}(\mathbf{n}_x\mathbf{r}\mathbf{h})s_{mk}{}^+(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\rangle + q_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h}), \quad (5)$$

where $s_{ik}(\mathbf{n}_1\mathbf{n}_x\mathbf{h}) = u^{-1}_{i\alpha}(\mathbf{n}_1\mathbf{h})t_{\alpha\beta}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})u_{\beta k}(\mathbf{n}_x\mathbf{h})$ is the scattering matrix in the new unit vectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$,

$$q_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h}) = u^{-1}_{i\alpha}(\mathbf{n}_1\mathbf{h})Q_{\alpha\gamma}(\mathbf{n}_1\mathbf{r}\mathbf{h})[u^+(\mathbf{n}_1\mathbf{h})]^{-1}_{\gamma k},$$
$$g_{nk}(\mathbf{n}_1\mathbf{h}) = i\lambda[t^{(k)*}(\mathbf{n}_1\mathbf{h}) - t^{(n)}(\mathbf{n}_1\mathbf{h})], \quad g_{nk} = g_{kn}{}^*. \quad (6)$$

In the first term on the right side of (5), there is no summation over i and k. The propagation function $g_{nk}(\mathbf{n}_1\mathbf{h})$ describes the attenuation of the radiation in its propagation through the medium, both at the expense of scattering and by pure absorption, as well as by the interference of the waves as a consequence of the difference of their phase velocities $v_k(\mathbf{n}_1\mathbf{h})$. Using the optical theory

$$\sigma_0^{(k)}(\mathbf{nh}) = \sigma_s^{(k)}(\mathbf{nh}) + \sigma_a^{(k)}(\mathbf{nh}) = 2\lambda\,\mathrm{Im}\,\langle s_{kk}(\mathbf{nnh})\rangle,$$

we can put Eq. (6) in the form

$$2g_{nm}(\mathbf{n}_1\mathbf{h}) = \sigma_0^{(n)}(\mathbf{n}_1\mathbf{h}) + \sigma_0^{(m)}(\mathbf{n}_1\mathbf{h}) + 2i\omega N_0^{-1}[v_m{}^{-1}(\mathbf{n}_1\mathbf{h}) - v_n{}^{-1}(\mathbf{n}_1\mathbf{h})]. \quad (7)$$

Here $\sigma_0^{(n)}(\mathbf{n}_1\mathbf{h})$, $\sigma_s^{(n)}(\mathbf{n}_1\mathbf{h})$ and $\sigma_a^{(n)}(\mathbf{n}_1\mathbf{h})$ are respectively the total cross section, the elastic scattering cross section and the cross section of pure absorption for a photon whose polarization has the direction $\mathbf{f}^{(n)}(\mathbf{n}_1\mathbf{h})$, incident on the scatterer in the direction $\mathbf{n}_1$. In what follows, we need a separate expression for $\sigma_s^{(i)}(\mathbf{n}_1\mathbf{h})$, connected with the tensor $\alpha_{nm}(\mathbf{h})$ in the following way:

$$\sigma_s^{(i)}(\mathbf{n}_1\mathbf{h}) = \frac{8\pi}{3}k^4\langle\alpha_{nm}(\mathbf{h})\alpha_{rn}{}^+(\mathbf{h})\rangle$$
$$\times D^{(1)}_{m\gamma}(\Omega_{h1})u_{\gamma i}(\mathbf{n}_1\mathbf{h})u_{i\nu}{}^+(\mathbf{n}_1\mathbf{h})D^{(1)+}_{\nu r}(\Omega_{h1}). \quad (8)$$

Summation over i is not carried out in (8).

2. THE RADIATION DENSITY MATRIX INSIDE A THICK LAYER OF MATTER AND THE EQUATION OF RADIATION DIFFUSION.

It is very difficult to solve the transport equation (2) in the case of an anisotropic medium. However, for optically thick media, one can develop a method of approximate solution of the problem. This is seen to be possible for the reason that the very physics of the process of radiation transport inside a thick target is materially simplified.

In the case of an optically thick isotropic medium, the radiation becomes diffuse in the interior of the medium, as is well known. That is to say, the radiation becomes unpolarized and isotropic. The radiation becomes diffuse at a distance of several mean path lengths from the boundary of the medium and from delta-like sources, if the true absorption in the medium is small: $p = \sigma_a/\sigma_0 \ll 1$. Diffuse radiation varies comparatively little with the distance $\sim r^{-1}\exp(-\sqrt{3p}N_0\sigma_0 r)$ (for the case of a point source located at the origin). Here the term $(\mathbf{n}_1\nabla)\rho_{\alpha\beta}$ in the transport equation is a quantity $\sim(\sqrt{p} + l/r) \ll 1$ relative to the first two terms on the right hand side of the equation. So far as the source is concerned, it is still less in the interior of the medium: $Q_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) \sim \exp(-N_0\sigma_0 r)$. Thus, in the interior of an isotropic medium, the approximate equation

$$N_0\sigma_s\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = N_0\int d\mathbf{n}_x\langle t_{\alpha\gamma}(\mathbf{n}_1\mathbf{n}_x)\rho_{\gamma\nu}(\mathbf{n}_x\mathbf{r})t_{\nu\beta}{}^+(\mathbf{n}_1\mathbf{n}_x)\rangle$$
$$+ O(l/r + \sqrt{p}) + O(e^{-N_0\sigma_0 r}). \quad (9)$$

should be satisfied.[12] Equation (9) permits us to determine the polarization and the angular dependence of the diffuse radiation: $\rho^{\mathrm{diff}}_{\alpha\beta}(\mathbf{n}_1\mathbf{r}) = \delta_{\alpha\beta}R(\mathbf{r})$. The dependence of radiation on the coordinates is determined by the diffusion equation for the function $R(\mathbf{r})$, as is well known. We note that one should have σ_s, and not σ_0, in the left hand side of (9), since account of σ_a exceeds the accuracy of Eq. (9) (this term is $\sim p$).

In the deep layers of an anisotropic medium, some state of radiation should be established, which we shall henceforth call "quasi-equilibrium." This radiation, by virtue of the anisotropy of the medium, will generally be neither isotropic nor unpolarized. The quasi-equilibrium density matrix, which we shall denote by $\mathscr{R}^{(0)}_{ik}(\mathbf{n}_1\mathbf{r}\mathbf{h})$ (or $\rho^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{r}\mathbf{h})$, will satisfy an equation similar to (9), and with the same estimates of applicability, if we assume smallness of the parameters $(l/r + \sqrt{p})$ and $\exp(-N_0\sigma_0 r)$ for all directions and polarizations:

$$g_{ik}^{(s)}(\mathbf{n}_1\mathbf{h})\mathscr{R}_{ik}^{(0)}(\mathbf{n}_1\mathbf{rh}) = \int d\mathbf{n}_x \langle s_{in}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\mathscr{R}_{nm}^{(0)}(\mathbf{n}_x\mathbf{rh})s_{mk}^{+}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\rangle. \quad (10)$$

Here $g_{ik}^{(s)}(\mathbf{n}_1\mathbf{h})$ is the part of the propagation function $g_{ik}(\mathbf{n}_1\mathbf{h})$ due to elastic scattering: there is no summation over i and k in (10).

We denote the integral term in Eq. (2) by $B_{\alpha\beta}(\mathbf{n}_1\mathbf{rh})$. It has the meaning of radiation density (erg/cm³-sec-sr). The matrix $B_{\alpha\beta}(\mathbf{n}_1\mathbf{rh})$, according to (2), has the form

$$B_{\alpha\beta}(\mathbf{n}_1\mathbf{rh}) = N_0 D_{\alpha n}^{(1)+}(\Omega_{h1})K_{nm}(\mathbf{rh})D_{m\beta}^{(1)}(\Omega_{h1}),$$

$$K_{nm}(\mathbf{rh}) = k^4\langle\alpha_{nr}(\mathbf{h})\,\alpha_{sm}^{+}(\mathbf{h})\rangle \int d\mathbf{n}_x D_{r\gamma}^{(1)}(\Omega_{hx})\rho_{\gamma\nu}(\mathbf{n}_x\mathbf{rh})D_{\nu s}^{(1)+}(\Omega_{h1}). \quad (11)$$

Carrying out the transformation (4) in (10), and using (11), we obtain the following expression for $\rho_{\alpha\beta}^{(0)}(\mathbf{n}_1\mathbf{rh})$:

$$\rho_{\alpha\beta}^{(0)}(\mathbf{n}_1\mathbf{rh}) = u_{\alpha i}(\mathbf{n}_1\mathbf{h})\,u_{i\gamma}^{-1}(\mathbf{n}_1\mathbf{h})D_{\gamma n}^{(1)+}(\Omega_{h1})K_{nm}^{(0)}(\mathbf{rh})D_{m\nu}^{(1)}(\Omega_{h1})$$
$$\times[u^{+}(\mathbf{n}_1\mathbf{h})]_{\nu k}^{-1}u_{k\beta}^{+}(\mathbf{n}_1\mathbf{h})\;[g_{ik}^{(s)}(\mathbf{n}_1\mathbf{h})]^{-1}. \quad (12)$$

By using (11) and (12), we get an algebraic set of equations for the components of the Hermitian tensor $K_{nm}^{(0)}(\mathbf{rh})$:

$$K_{nm}^{(0)}(\mathbf{rh}) = k^4\langle\alpha_{ni}(\mathbf{h})\,\alpha_{lm}^{+}(\mathbf{h})\rangle B_{ilrs}(\mathbf{h})K_{rs}^{(0)}(\mathbf{rh}),$$

$$B_{ilrs}(\mathbf{h}) = \int d\mathbf{n}_1 D_{i\alpha}^{(1)}(\Omega_{h1})u_{\alpha p}(\mathbf{n}_1\mathbf{h})u_{p\gamma}^{-1}(\mathbf{n}_1\mathbf{h})D_{\gamma r}^{(1)+}(\Omega_{h1})D_{s\nu}^{(1)}(\Omega_{h1})$$
$$\times[u^{+}(\mathbf{n}_1\mathbf{h})]_{\nu q}^{-1}u_{q\beta}^{+}(\mathbf{n}_1\mathbf{h})D_{\beta l}^{(1)+}(\Omega_{h1})\,[g_{pq}^{(s)}(\mathbf{n}_1\mathbf{h})]^{-1}. \quad (13)$$

The solution of the homogeneous set (13) and Eqs. (11) and (12) allow us to determine the angular and polarization dependences of the quasi-equilibrium radiation. In a number of important cases, Eq. (13) is greatly simplified, since here, depending on the specific meaning of the operation of averaging ⟨⟩ and the explicit form of the polarizability tensor of a system of 9th order, (13) breaks up into a series of independent homogeneous systems of lower order. In the present paper, an anisotropic medium is defined as one whose anisotropy is due to some outside causes (magnetic or electric fields, directed flow of particles, etc.), thus excluding crystalline media from consideration. By eliminating the cause of the anisotropy of the medium, we should evidently arrive at the case of an isotropic medium. If we make the limiting transition in all the formulas to the case of an isotropic medium ($\alpha_{ik} \to \alpha\delta_{ik}$), we can be confident that only one of these independent systems has a nonzero solution (its determinant is equal to zero). From physical considerations (the existence of a continuous transition to the isotropic case), it is evident that even in the anisotropic case all other independent systems have only a null solution. Thus, the solution of (13) has the form

$$K_{nm}^{(0)}(\mathbf{rh}) = a_{nm}(\mathbf{h})R(\mathbf{rh}), \quad (14)$$

where the tensor $a_{nm}(\mathbf{h})$ is completely determined from (13) and the real scalar function $R(\mathbf{rh})$ satisfies the equation which we shall now derive.

The function $R(\mathbf{rh}) \equiv R(\mathbf{r})$ in an isotropic medium satisfies the well-known diffusion equation with a single scalar diffusion coefficient. It must be expected that in the case of an anisotropic medium the function $R(\mathbf{rh})$ satisfies the diffusion equation with the diffusion tensor. We obtain this equation by using (12) and the exact transport equations (2) and (5). Integrating (2) over the angles and substituting the quasi-equilibrium value ρ_{α}^{β} for the exact density matrix $\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{rh})$ on the right hand side of the resultant expression, we obtain the formula for the divergence of the radiation flux vector:

$$\mathbf{F}(\mathbf{rh}) = \int d\mathbf{n}_1\mathbf{n}_1\rho_{\alpha\alpha}(\mathbf{n}_1\mathbf{rh}), \qquad Q(\mathbf{rh}) = \int d\mathbf{n}_1 Q_{\alpha\alpha}(\mathbf{n}_1\mathbf{rh}),$$

$$\operatorname{div}\mathbf{F}(\mathbf{rh}) = -N_0\int d\mathbf{n}_1\sigma_{\alpha}^{(\alpha)}(\mathbf{n}_1\mathbf{h})\rho_{\alpha\alpha}^{(0)}(\mathbf{n}_1\mathbf{rh}) + Q(\mathbf{rh}). \quad (15)$$

We divide Eq. (5) by the quantity $N_0 g_{ik}(\mathbf{n}_1\mathbf{h})$ and subject it to the transformation (4). We multiply the resultant expression by the vector $\mathbf{n}_1$, calculate the sum of the diagonal elements, integrate over $\mathbf{n}_1$ and take the divergence of the whole expression. Using (15) and replacing everywhere $\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{rh})$ and $\mathscr{R}_{ik}(\mathbf{n}_1\mathbf{rh})$ by their quasi-equilibrium values $\rho_{\alpha\beta}^{(0)}$ and $\mathscr{R}_{ik}^{(0)}$, we get the following sought diffusion equation:

$$(-1)^k\chi_{ik}(\mathbf{h})\frac{\partial^2 R(\mathbf{rh})}{\partial x_i(\mathbf{h})\,\partial x_{-k}(\mathbf{h})} = \mu R(\mathbf{rh}) + d_i(\mathbf{h})\frac{\partial R(\mathbf{rh})}{\partial x_i(\mathbf{h})} - Q(\mathbf{rh}). \quad (16)$$

The absorption coefficient μ is determined by the expression

$$\mu R(\mathbf{rh}) = N_0\int d\mathbf{n}_1\sigma_{\alpha}^{(\alpha)}(\mathbf{n}_1\mathbf{h})\rho_{\alpha\alpha}^{(0)}(\mathbf{n}_1\mathbf{rh}) = R(\mathbf{rh})N_0 a_{nm}(\mathbf{h})\int d\mathbf{n}_1\sigma_{\alpha}^{(\alpha)}(\mathbf{n}_1\mathbf{h})$$
$$\times u_{\alpha i}(\mathbf{n}_1\mathbf{h})u_{i\gamma}^{-1}(\mathbf{n}_1\mathbf{h})D_{\gamma n}^{(1)+}(\Omega_{h1})D_{m\nu}^{(1)}(\Omega_{h1})\,[u^{+}(\mathbf{n}_1\mathbf{h})]_{\nu k}^{-1}u_{k\alpha}^{+}(\mathbf{n}_1\mathbf{h})\,[g_{ik}^{(s)}(\mathbf{n}_1\mathbf{h})]^{-1}. \quad (17)$$

The diffusion tensor $\chi_{ik}(\mathbf{h})$ has the form

$$\chi_{pq}(\mathbf{h}) = N_0^{-1}a_{nm}(\mathbf{h})\int d\mathbf{n}_1 D_{p0}^{(1)}(\Omega_{h1})D_{q0}^{(1)*}(\Omega_{h1})u_{\alpha i}(\mathbf{n}_1\mathbf{h})u_{i\gamma}^{-1}(\mathbf{n}_1\mathbf{h})D_{\gamma n}^{(1)+}(\Omega_{h1})$$
$$\times D_{m\nu}^{(1)}(\Omega_{h1})\,[u^{+}(\mathbf{n}_1\mathbf{h})]_{\nu k}^{-1}u_{k\alpha}^{+}(\mathbf{n}_1\mathbf{h})\,[g_{ik}^{(s)}(\mathbf{n}_1\mathbf{h})]^{-2}, \quad (18)$$

$$R(\mathbf{rh})d_i(\mathbf{h}) \equiv \int d\mathbf{n}\,\mathbf{n}_i\rho_{\alpha\alpha}^{(0)}(\mathbf{nrh}) \equiv F_i^{(0)}(\mathbf{rh}) =$$
$$= R(\mathbf{rh})\int d\mathbf{n}_1 D_{i0}^{(1)}(\Omega_{h1})u_{\alpha i}(\mathbf{n}_1\mathbf{h})u_{i\gamma}^{-1}(\mathbf{n}_1\mathbf{h})D_{\gamma n}^{(1)+}(\Omega_{h1})a_{nm}(\mathbf{h})D_{m\nu}^{(1)}(\Omega_{h1})$$
$$\times[u^{+}(\mathbf{n}_1\mathbf{h})]_{\nu k}^{-1}u_{k\alpha}^{+}(\mathbf{n}_1\mathbf{h})\,[g_{ik}^{(s)}(\mathbf{n}_1\mathbf{h})]^{-1}. \quad (19)$$

In obtaining (16), we have discarded terms $\sim l/r \ll 1$ and $p \ll 1$ in comparison with the remaining terms. The term $(\mathbf{d}\nabla)R(\mathbf{rh})$ describes the additional radiation flux in an anisotropic medium under the condition that the quasi-equilibrium radiation has the nonzero flux $\mathbf{d} \sim \int \mathbf{n}\rho_{\alpha\alpha}^{(0)}(\mathbf{nrh})d\mathbf{n}$. It is obvious that the power of this source should be proportional to $\nabla R(\mathbf{rh})$.

Equation (16) is written in the set of coordinates with the z axis along h. Since its form is invariant, it will be the same in any other set of coordinates, but with different χ_{ik} and d_i, which are obtained through the transformation formulas of vectors and tensors. In all the formulas, the matrices $u_{\alpha i}(\mathbf{n}_1\mathbf{h})$ enter in combinations of the type $u_{\alpha i}u_{i\gamma}^{-1}$, so that everywhere the unnormalized eigenvectors $f_{\alpha}^{(i)}(\mathbf{n}_1\mathbf{h})$ can be used, which simplifies the calculations. Equation (16) transforms into the usual diffusion equation in the case of an isotropic medium if we make the limiting transition: $\alpha_{ik} \to \alpha\delta_{ik}$.

In the following sections, we shall study several types of anisotropic media, for which the general formulas of this section are greatly simplified.

3. RADIATION TRANSFER IN A GYROTROPIC MEDIUM

First of all, let us consider the passage of radiation in the depths of a so-called gyrotropic medium, in which the polarizability tensor of the particle has the following form:

$$\alpha_{lk}(\mathbf{h}) = \alpha_l\delta_{kl}, \quad \alpha_l = \alpha + lg, \quad l = 0, \pm 1. \tag{20}$$

A special case of such a medium is a weakly magneto-acoustic plasma. The anisotropy of such a medium is due to the gyration vector $\mathbf{g} = g\mathbf{h}$. In the case of a gyrotropic medium, the vectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ in (3) are equal to the cyclic unit vectors $\mathbf{e}_{-1}$ and $\mathbf{e}_1$; therefore, it is necessary to set $u_{\alpha i}(\mathbf{n}\mathbf{h}) = \delta_{\alpha i}$ in all the formulas. The propagation functions $g^{(s)}_{ik}(\mathbf{n}_1\mathbf{h})$ in this case has the form

$$g^{(s)}_{nm}(\mathbf{n}_1\mathbf{h}) \equiv g^{(s)}_{\alpha\beta}(\mathbf{n}_1\mathbf{g}) = {}^1\!/_2[\sigma_s^{(\alpha)}(\mathbf{n}_1\mathbf{g}) + \sigma_s^{(\beta)}(\mathbf{n}_1\mathbf{g})] + 2\pi ik(\beta - \alpha)g\cos\beta_{g1},$$

$$\sigma_s^{(\gamma)}(\mathbf{n}_1\mathbf{g}) = {}^8\!/_3\pi k^4[|\alpha|^2 + g^2(1+\cos^2\beta_{g1})/2 + 2g\gamma\,\mathrm{Re}\,\alpha\cdot\cos\beta_{g1}]. \tag{21}$$

According to (13), the coefficients $B_{nmrs}(\mathbf{g})$ have the form

$$B_{nmrs}(\mathbf{g}) = \int d\mathbf{n}_1 D^{(1)}_{n\alpha}(\Omega_{g1})D^{(1)+}_{\alpha r}(\Omega_{g1})D^{(1)}_{s\beta}(\Omega_{g1})D^{(1)+}_{\beta m}(\Omega_{g1})$$
$$\cdot[g^{(s)}_{\alpha\beta}(\mathbf{n}_1\mathbf{g})]^{-1} = \delta_{n-m,r-s}b_{nmrs}(\mathbf{g}). \tag{22}$$

Substituting (22) in Eq. (13), we establish the fact that the set of equations (13) break up into several independent systems, connecting the elements $K^{(0)}_{nm}(\mathbf{rg})$ with the same $(n-m) = 0, \pm 1, \pm 2$. Of these, only one set, which connects the diagonal elements $K^{(0)}_{nm}$, has a determinant equal to zero. Thus, for a gyrotropic medium,

$$K^{(0)}_{nm}(\mathbf{rg}) = \delta_{nm}a_n(\mathbf{g})R(\mathbf{rg}). \tag{23}$$

Inasmuch as the coefficients $b_{nnrr}(\mathbf{g})$ entering into the algebraic set of third order for the quantities a_n are tabular integrals (see the explicit form of the D functions), we do not need to write out here the explicit form of the coefficients of a_n.

The angular and polarization dependences of the quasi-equilibrium matrices $B^{(0)}_{\alpha\beta}$ and $\rho^{(0)}_{\alpha\beta}$ are determined from (11), (12), and (23):

$$B^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{rg}) = N_0D^{(1)+}_{\alpha n}(\Omega_{g1})a_n(\mathbf{g})D^{(1)}_{n\beta}(\Omega_{g1})R(\mathbf{rg}),$$
$$\rho^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{rg}) = B^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{rg})[N_0g^{(s)}_{\alpha\beta}(\mathbf{n}_1\mathbf{g})]^{-1}. \tag{24}$$

The tensor $\chi_{ik}(\mathbf{g})$ is diagonal in a gyrotropic medium: $\chi_{ik}(\mathbf{g}) = \delta_{ik}\chi_k(\mathbf{g})$, and $\chi_{-1}(\mathbf{g}) = \chi_1(\mathbf{g})$:

$$\chi_1(\mathbf{g}) = {}^1\!/_2N_0^{-1}a_n(\mathbf{g})\int d\mathbf{n}_1\sin^2\beta_{g1}|D^{(1)}_{n\gamma}(\Omega_{g1})|^2[\sigma_s^{(\gamma)}(\mathbf{n}_1\mathbf{g})]^{-2},$$
$$\chi_0(\mathbf{g}) = N_0^{-1}a_n(\mathbf{g})\int d\mathbf{n}_1\cos^2\beta_{g1}|D^{(1)}_{n\gamma}(\Omega_{g1})|^2[\sigma_s^{(\gamma)}(\mathbf{n}_1\mathbf{g})]^{-2}. \tag{25}$$

In a gyrotropic medium, $d_i(\mathbf{g}) = 0$, which is easily proved by using the relation $g^{(s)}_{\alpha\beta}(-\mathbf{n}_1\mathbf{g}) = g_{-\alpha-\beta}(\mathbf{n}_1\mathbf{g})$.

The constant μ, which takes into account the absorption, is equal to

$$\mu = N_0a_n(\mathbf{g})\int d\mathbf{n}_1\sigma_a^{(\gamma)}(\mathbf{n}_1\mathbf{g})|D^{(1)}_{n\gamma}(\Omega_{g1})|^2[\sigma_s^{(\gamma)}(\mathbf{n}_1\mathbf{g})]^{-1}. \tag{26}$$

For a weakly gyrotropic medium, discarding terms that are quadratic in the small parameter $\eta = 2g\,\mathrm{Re}\,\alpha/|\alpha|^2$, we can easily find

$$a_n(\mathbf{g}) = \sigma(1 + n\eta)/8\pi, \quad \sigma = {}^8\!/_3\pi k^4|\alpha|^2,$$
$$B^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{rg}) = \delta_{\alpha\beta}(1 + \beta\eta\cos\beta_{g1})N_0\sigma R(\mathbf{rg})/8\pi, \quad \rho^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{rg}) = \delta_{\alpha\beta}R(\mathbf{rg})/8\pi,$$
$$\chi_{ik}(\mathbf{g}) = \delta_{ik}\chi, \quad \chi = (3N_0\sigma)^{-1}, \quad \mu = \frac{N_0}{8\pi}\int d\mathbf{n}_1[\sigma_a^{(1)}(\mathbf{n}_1\mathbf{g}) + \sigma_a^{(-1)}(\mathbf{n}_1\mathbf{g})]. \tag{27}$$

We see that, although the radiation density $B^{(0)}_{\alpha\beta}(\mathbf{n}_1\mathbf{rg})$ is anisotropic and circularly polarized, in all the remaining relations, the radiation in the interior of the weakly gyrotropic medium behaves as in the ordinary isotropic medium.

4. LIGHT SCATTERING BY AXIALLY ORIENTED PARTICLES

If the scattering particles in the medium are oriented in an axially symmetric fashion relative to the direction $\mathbf{h}$, then the averaging symbol $\langle\,\rangle$ includes averaging over the angle φ_{h1} with the weighting function $w(\varphi_{h1}) = 1/2\pi$. In this case, the general formulas of Sec. 3 are greatly simplified. Carrying out the averaging over φ_{h1}, we get

$$\langle\alpha_{ik}(\mathbf{h})\rangle = \delta_{ik}\langle\alpha_i(\mathbf{h})\rangle, \quad \langle\alpha_i(\mathbf{h})\rangle = \langle\alpha_{ii}(\mathbf{h})\rangle,$$
$$\langle\alpha_{ik}(\mathbf{h})\alpha_{rs}^+(\mathbf{h})\rangle = \delta_{k-i,r-s}\langle\alpha_{ik}(\mathbf{h})\alpha_{rs}^+(\mathbf{h})\rangle. \tag{28}$$

We leave the averaging symbol $\langle\,\rangle$ on the right sides of (28), understanding by it all the remaining averaging. According to (3), (5) and (28),

$$g_{ik}(\mathbf{n}_1\mathbf{h}) = g_{ik}(\cos^2\beta_{h1}), \quad u_{\alpha i}(\mathbf{n}_1\mathbf{h}) = u_{\alpha i}(\beta_{h1};\gamma_{h1}).$$

Therefore the integral over φ_{h1} in (13) is calculated in elementary fashion and the expression for $B_{nmrs}(\mathbf{h})$ is simplified: $B_{nmrs}(\mathbf{h}) = \delta_{n-m,r-s}T_{nmrs}(\mathbf{h})$. Also, taking (28) into account, we obtain independent sets of equations for the elements $K^{(0)}_{nm}$ with the same $(n-m) = 0, \pm 1, \pm 2$, just as for the case of a gyrotropic medium. As before, we get $K^{(0)}_{nm}(\mathbf{rh}) = \delta_{nm}r_n(\mathbf{h})R(\mathbf{rh})$, where the coefficients r_n are found from the equation

$$r_n(\mathbf{h}) = k^4\langle|\alpha_{nm}(\mathbf{h})|^2\rangle T_{mmkk}(\mathbf{h})r_k(\mathbf{h}). \tag{29}$$

The solution of this homogeneous equation of third degree presents no difficulty. By using the properties of $K^{(0)}_{nm}$, $g_{ik}(\mathbf{n}_1\mathbf{h})$ and $u_{\alpha i}(\mathbf{n}_1\mathbf{h})$, we have no difficulty in noting that $\chi_{pq}(\mathbf{h}) = \delta_{pq}\chi_p(\mathbf{h})$, and $\chi_{-1}(\mathbf{h}) = \chi_1(\mathbf{h})$, and also $d_i(\mathbf{h}) = \delta_{i0}d_0(\mathbf{h})$.

The results of this section refer to a very broad class of anisotropic media with axially symmetric orientations of the scattering particles.

5. RADIATION TRANSFER IN AN ALIGNED MEDIUM

On the basis of the results of the previous section, we shall consider the radiation transfer in a so-called aligned medium. For an aligned medium, the averaging symbol $\langle\,\rangle$ denotes averaging over the orientations of particles with the following distribution functions:

$$w(\varphi_{h1};\beta_{h1};\gamma_{h1}) = \frac{1}{4\pi^2}w(\cos^2\beta_{h1}), \quad \int_0^\pi d\beta\sin\beta\, w(\cos^2\beta) = 1. \tag{30}$$

Carrying out the additional averaging over the angles β_{h1} and γ_{h1} in the formulas of the previous section, we get

$$2\langle\alpha_0(\mathbf{h})\rangle = (1 - A_2)[\alpha_{11}(\mathbf{n}_c) + \alpha_{-1-1}(\mathbf{n}_c)] + 2A_2\alpha_{00}(\mathbf{n}_c),$$
$$4\langle\alpha_1(\mathbf{h})\rangle = (1 + A_2)[\alpha_{11}(\mathbf{n}_c) + \alpha_{-1-1}(\mathbf{n}_c)] + 2(1 - A_2)\alpha_{00}(\mathbf{n}_c) = 4\langle\alpha_{-1}(\mathbf{h})\rangle, \tag{31}$$
$$A_n = \int_0^\pi d\beta\sin\beta\, w(\cos^2\beta)\cos^n\beta, \quad A_0 \equiv 1.$$

Here and in what follows, the index n_c in $\alpha_{ik}(n_c)$ indicates that the components of the polarizability tensor are taken in a set of coordinates rigidly attached to the scattering particle itself. Under the condition (31), we obtain the result that the unit vectors $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ in an aligned medium are equal to the Cartesian unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$, i.e., the matrix $u_{\alpha i}(\mathbf{n}_1\mathbf{h}) = u_{\alpha i}(\gamma_{h1})$ is the well-known unitary transformation matrix from the Cartesian unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ to the cyclic unit vectors $\mathbf{e}_{-1}$ and $\mathbf{e}_1$. The eigenvalues $t^{(1)} \equiv t^{(x)}$ and $t^{(2)} \equiv t^{(y)}$ are the following in this case:

$$t^{(1)}(\mathbf{n}_1\mathbf{h}) = k^2[\langle\alpha_1(\mathbf{h})\rangle\cos^2\beta_{h1} + \langle\alpha_0(\mathbf{h})\rangle\sin^2\beta_{h1}],\qquad t^{(2)}(\mathbf{n}_1\mathbf{h}) = k^2\langle\alpha_1(\mathbf{h})\rangle. \tag{32}$$

The elastic scattering cross sections $\sigma_s^{(i)}(\mathbf{n}_1\mathbf{h})$ are, according to (8), equal to

$$\sigma_s^{(1)}(\mathbf{n}_1\mathbf{h}) = {}^8/_3\pi[B_1\cos^2\beta_{h1} + B_0\sin^2\beta_{h1}],\quad \sigma_s^{(2)}(\mathbf{n}_1\mathbf{h}) = {}^8/_3\pi B_1, \tag{33}$$

where the following notation has been introduced:

$$B_1 = k^4\sum_n\langle|\alpha_{n1}(\mathbf{h})|^2\rangle = {}^1/_4(b_{-1}+b_1)(1+A_2) + {}^1/_2 b_0(1-A_2),$$

$$B_0 = k^4\sum_n\langle|\alpha_{n0}(\mathbf{h})|^2\rangle = {}^1/_2(b_{-1}+b_1)(1-A_2) + b_0A_2, \tag{34}$$

$$b_i = k^4\sum_r|\alpha_{ri}(\mathbf{n}_c)|^2.$$

It follows from (32) that the real absorption cross sections have the form

$$\sigma_a^{(1)}(\mathbf{n}_1\mathbf{h}) = C_1\cos^2\beta_{h1} + C_2\sin^2\beta_{h1},\quad \sigma_a^{(2)}(\mathbf{n}_1\mathbf{h}) = C_1.$$

The constants C_1 and C_2 depend on the specific form and constitution of the particle. The coefficients $T_{mmkk}(\mathbf{h})$ entering into Eq. (29) have the form $T_{mmkk}(\mathbf{h}) \equiv T_{mk}(\mathbf{h})$, $T_{-1-1} = T_{11} \equiv T_1$, $T_{-10} = T_{0-1} = T_{01} = T_{10} \equiv T_2$, $T_{-11} = T_{1-1} \equiv T_3$, $T_{00} \equiv T_4$ in this case. In the following, we need the explicit expressions for T_n given below:

$$T_1 + T_3 = 2\pi\int_0^1 dx\{[\sigma_s^{(2)}]^{-1} + x^4[\sigma_s^{(1)}(x^2)]^{-1}\}, \tag{35}$$

$$T_2 = 2\pi\int_0^1 dx\,x^2(1-x^2)[\sigma_s^{(1)}(x^2)]^{-1},\quad T_4 = 4\pi\int_0^1 dx(1-x^2)^2[\sigma_s^{(1)}(x^2)]^{-1}.$$

The integrals (35) are tabulated. Using (33) and (35), we can establish the fact that $(T_1 + T_2)B_1 + T_2B_0 = 1$, $2T_2B_1 + T_4B_0 = 1$.

The matrix elements $\langle|\alpha_{nl}(\mathbf{h})|^2\rangle$ are given by the expression

$$\langle|\alpha_{nl}(\mathbf{h})|^2\rangle = \frac{1}{4\pi^2}\alpha_{mk}(\mathbf{n}_c)\alpha_{rs}^+(\mathbf{n}_c)\int d\mathbf{n}_1 w(\cos^2\beta_{h1})D_{nm}^{(1)}(\Omega_{h1})\cdot \times D_{kl}^{(1)+}(\Omega_{h1})D_{lr}^{(1)}(\Omega_{h1})D_{sn}^{(1)+}(\Omega_{h1}). \tag{36}$$

From (36) it is easy to find the symmetry properties of this matrix:

$$\langle|\alpha_{-1-1}(\mathbf{h})|^2\rangle = \langle|\alpha_{11}(\mathbf{h})|^2\rangle,\quad \langle|\alpha_{-10}(\mathbf{h})|^2\rangle = \langle|\alpha_{10}(\mathbf{h})|^2\rangle,$$
$$\langle|\alpha_{0-1}(\mathbf{h})|^2\rangle = \langle|\alpha_{01}(\mathbf{h})|^2\rangle,\quad \langle|\alpha_{-11}(\mathbf{h})|^2\rangle = \langle|\alpha_{1-1}(\mathbf{h})|^2\rangle.$$

We note that $\langle|\alpha_{10}(\mathbf{h})|^2\rangle = \langle|\alpha_{01}(\mathbf{h})|^2\rangle$ if the condition

$$\Delta = |\alpha_{0-1}(\mathbf{n}_c)|^2 + |\alpha_{01}(\mathbf{n}_c)|^2 - |\alpha_{-10}(\mathbf{n}_c)|^2 - |\alpha_{10}(\mathbf{n}_c)|^2 = 0,$$

is satisfied, and

$$\langle|\alpha_{01}(\mathbf{h})|^2\rangle = \langle|\alpha_{10}(\mathbf{h})|^2\rangle + (3A_2 - 1)\Delta/4.$$

Using the symmetry properties of T_{mmhk} and $\langle|\alpha_{nk}(\mathbf{h})|^2\rangle$ shown above, it is easy to establish from (29) that $r_{-1}(\mathbf{h}) = r_1(\mathbf{h})$:

$$r_1(\mathbf{h}) = B_1(1-LT_2)/3,\quad r_0(\mathbf{h}) = [B_0(1-LT_2)+L]/3,$$
$$L = {}^1/_4k^4\Delta(3A_2-1)[T_2B_1 + k^4\langle|\alpha_{10}(\mathbf{h})|^2\rangle(T_4-T_2)]^{-1}. \tag{37}$$

Substituting (37) in (11) and (12), we find the angular and polarization dependences of the quasi-equilibrium radiation in an aligned medium:

$$B_{\alpha\beta}^{(0)}(\mathbf{n}_1\mathbf{rh}) = {}^1/_3N_0R(\mathbf{rh})\{\delta_{\alpha\beta}B_1(1-LT_2) + [(B_0-B_1)(1-LT_2)+L]D_{\alpha0}^{(1)+}(\Omega_{h1})D_{0\beta}^{(1)}(\Omega_{h1})\},$$

$$\rho_{\alpha\beta}^{(0)}(\mathbf{n}_1\mathbf{rh}) = \frac{1}{8\pi}R(\mathbf{rh})\left\{\delta_{\alpha\beta}(1-LT_2) + \frac{LD_{\alpha0}^{(1)+}(\Omega_{h1})D_{0\beta}^{(1)}(\Omega_{h1})}{B_1\cos^2\beta_{h1} + B_0\sin^2\beta_{h1}}\right\}. \tag{38}$$

We note that for $L = 0$ the radiation inside the medium is unpolarized, while the radiation density possesses linear polarization $\sim (B_0 - B_1)$. This occurs as a consequence of the dependence of the cross section in an anisotropic medium on the polarization of the photon.

Substituting (37) in Eqs. (17), (18), (19), we obtain expressions for $\chi_{pq}(\mathbf{h}) = \delta_{pq}\chi_p(\mathbf{h})$, μ and $d_i(\mathbf{h})$ in an aligned medium:

$$\mu = \frac{1}{3}N_0(2C_1 + C_2)(1-LT_2) + \frac{1}{4}N_0L\int_0^1 dx(1-x^2)\cdot\times\frac{C_1(1+x^2)+C_2(1-x^2)}{B_1x^2+B_0(1-x^2)} \tag{39}$$

$$\chi_0(\mathbf{h}) = \frac{1}{16\pi N_0}\left\{\frac{1-LT_2}{B_1} + 3\int_0^1 dx\,x^2\left[\frac{1-LT_2}{B_1x^2+B_0(1-x^2)} + \frac{L(1-x^2)}{(B_1x^2+(1-x^2)B_0)^2}\right]\right\},$$

$$\chi_1(\mathbf{h}) = \chi_{-1}(\mathbf{h}) = \frac{1}{16\pi N_0}\left\{\frac{1-LT_2}{B_1} + \frac{3}{2}\int_0^1 dx(1-x^2)\cdot\times\left[\frac{1-LT_2}{B_1x^2+B_0(1-x^2)} + \frac{L(1-x^2)}{(B_1x^2+(1-x^2)B_0)^2}\right]\right\}.$$

The vector $d_i(\mathbf{h})$ in an aligned medium is equal to zero.

6. THE INTENSITY AND POLARIZATION OF RADIATION EMERGING FROM A PLANE LAYER OF GREAT OPTICAL THICKNESS

The equations obtained in the previous sections are easily generalized to the case of scattering from a plane layer of material. Let an unbounded beam of photons be incident on a plane layer of thickness L_0. We compute the density matrix of radiation emerging from the layer, $\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{h}0)$, by using the generalization of the method[11] previously suggested by the authors for the calculation of $\rho_{\alpha\beta}$ in the case of an isotropic medium. The essence of the method lies in the calculation of a series of successive approximations for $\rho_{\alpha\beta}$, and the rapid convergence of the series is connected with the successful choice of the initial approximation. Here we obtain the formula for the first approximation $\rho_{\alpha\beta}(\mathbf{n}_1\mathbf{h}0)$ in a somewhat different, physically clearer fashion, than in[11]. Inasmuch as the radiation has a diffuse character in the interior of the layer, we choose the solution of the diffusion equation (16) as the zeroth approximation for the matrix $\mathscr{B}_{ik}(\mathbf{n}_1\mathbf{h}z)$. Using Eqs. (11), (14) and (16), we get

$$\mathscr{B}_{ik}^{(0)}(\mathbf{n}_1\mathbf{h}z) = N_0u_{i\gamma}^{-1}(\mathbf{n}_1\mathbf{h})D_{\gamma n}^{(1)+}(\Omega_{h1})a_{nm}(\mathbf{h})D_{m\nu}^{(1)}(\Omega_{h1})\times[u^+(\mathbf{n}_1\mathbf{h})]_{\nu k}^{-1}(R_0+R_1z) \equiv N_0A_{ik}(\mathbf{n}_1\mathbf{h})(R_0+R_1z). \tag{40}$$

The coordinate z is measured in the depth of the layer from the surface z = 0 from which the radiation emerges. The connection between the constants R_0 and R_1 can be obtained from the boundary conditions.

It is easy to obtain the density matrix of the zeroth approximation, $\mathscr{R}_{ik}^{(0)}(\mathbf{n}_1\mathbf{h}z)$, by using the well-known formulas from the theory of radiation transfer:[5,6,11]

$$\mathscr{R}_{ik}(\mathbf{n}_1\mathbf{h}z)=|\sec\beta_{21}|\int_0^\infty dz'\exp[-|z-z'||\sec\beta_{21}|N_0 g_{ik}(\mathbf{n}_1\mathbf{h})]\mathscr{B}_{ik}(\mathbf{n}_1\mathbf{h}z')$$
$$\times[\theta(\cos\beta_{21})\theta(z'-z)+\theta(-\cos\beta_{21})\theta(z-z')]+O(e^{-N_0\sigma_0 L_0}). \quad (41)$$

Here the index 2 indicates the outward normal to the layer: $\mathbf{n}_2 = -\mathbf{n}_z$, and the function $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. By calculating $\mathscr{R}_{ik}^{(0)}(\mathbf{n}_1\mathbf{h}z)$ from Eq. (41), it is not difficult to find the expression for $\mathscr{B}_{ik}^{(1)}(\mathbf{n}_1\mathbf{h}z)$ and then the desired expression for $\mathscr{R}_{ik}^{(1)}(\mathbf{n}_1\mathbf{h}0)$ from (41):

$$\mathscr{R}_{ik}^{(1)}(\mathbf{n}_1\mathbf{h}0)=\Big\{\int_{\Omega+}d\mathbf{n}_x\langle s_{in}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\,s_{mk}^{+}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\rangle\frac{A_{nm}(\mathbf{n}_x\mathbf{h})}{g_{nm}(\mathbf{n}_x\mathbf{h})}\Big[R_0+\frac{R_1\cos\beta_{21}}{N_0 g_{ik}(\mathbf{n}_1\mathbf{h})}$$
$$+\frac{R_1\cos\beta_{2x}}{N_0 g_{nm}(\mathbf{n}_x\mathbf{h})}\Big]+\cos\beta_{21}\Big(R_0+\frac{R_1\cos\beta_{21}}{N_0 g_{ik}(\mathbf{n}_1\mathbf{h})}\Big)\int_{\Omega-}d\mathbf{n}_x\langle s_{in}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\,s_{mk}^{+}(\mathbf{n}_1\mathbf{n}_x\mathbf{h})\rangle$$
$$\times A_{nm}(\mathbf{n}_x\mathbf{h})\,[g_{nm}(\mathbf{n}_x\mathbf{h})\cos\beta_{21}-g_{ik}(\mathbf{n}_1\mathbf{h})\cos\beta_{2x}]^{-1}\Big\}\frac{1}{g_{ik}(\mathbf{n}_1\mathbf{h})}. \quad (42)$$

Here Ω^+ and Ω^- are solid angles with $\beta_{2x} \leq \pi/2$ and $\beta_{2x} > \pi/2$, respectively. By determining the connection between R_0 and R_1 from the boundary conditions, we can find the normalized intensity $J_0^{(1)}(\mathbf{n}_1\mathbf{h}0) \equiv I_0^{(1)}(\mathbf{n}_1\mathbf{h}0)/I^{(1)}$ from Eq. (42) ($\beta_{21} = \pi/2$; h; 0) (i.e., the angular distribution of the emerging radiation) and the degree of polarization $\xi_k(\mathbf{n}_1\mathbf{h}0) = I_k^{(1)}(\mathbf{n}_1\mathbf{h}0)/I_0^{(1)}(\mathbf{n}_1\mathbf{h}0)$. These quantities do not depend on the remaining unknown constants.

As an example, we compute the density matrix of the radiation emerging from a weakly gyrotropic medium by use of (42). Thus, for a plasma in the magnetic field H(Oe), the gyration vector $\mathbf{g} = 1.67 \times 10^{-16}\lambda^3 H$. We shall neglect terms of order η^2, where $\eta = 2g\,\mathrm{Re}\,\alpha/|\alpha|^2$. In the case of a weakly gyrotropic medium, the radiation inside the medium is, according to (27), unpolarized and isotropic. The angular distribution of the radiation emerging from the layer, according to[5,6,11], will be close to the angular distribution in the scattering in an isotropic medium which, as is well known,[5,6] is well approximated by the function $J_0^{(0)}(\mathbf{n}_1 0) = 1 + 2\cos\beta_{21}$. This leads directly to the relation $R_1 = 2R_0N_0\sigma$. In the linear approximation in η, we have

$$g_{\gamma\gamma}^{(s)}(\mathbf{n}_1\mathbf{g})=\sigma(1+\gamma\eta\cos\beta_{g1}), \qquad \sigma=(8\pi/3)k^4|\alpha|^2,$$

Computing the integrals (42) in the linear approximation in η, we get the following expressions for the unnormalized Stokes parameters $I^{(1)}(\mathbf{n}_1\mathbf{g}0)$ (we denote $\cos\beta_{21}$ by x):

$$I_0^{(1)}(\mathbf{n}_1\mathbf{g}0)={}^3/_4R_0[S_1(x)(1+x^2)+S_0(x)(1-x^2)],$$
$$S_1(x)=\frac{17}{12}+\frac{19}{12}x-x^3+\frac{1}{2}x(1+2x)(1+x^2)\ln\Big(1+\frac{1}{x}\Big), \quad (43)$$
$$S_0(x)=\frac{7}{6}+\frac{5}{6}x+2x^3+x(1+2x)(1-x^2)\ln\Big(1+\frac{1}{x}\Big).$$

For the Stokes parameter that determines the circular polarization, we have

$$I_2^{(1)}(\mathbf{n}_1\mathbf{g}0)=-\eta\,{}^3/_4R_0[L_\parallel(x)\cos\beta_{2g}+L_\perp(x)\sin\beta_{2g}\cos(\varphi_{2g}-\varphi_{21})],$$
$$L_\parallel(x)=x(1+x^2)P_1(x)+x(1-x^2)P_0(x)-x(1-x^2)[S_0(x)-S_1(x)]+A(x),$$
$$L_\perp(x)=\sqrt{1-x^2}\{(1+x^2)P_1(x)+(1-x^2)P_0(x)+x^2[S_0(x)-S_1(x)]+B(x)\}. \quad (44)$$

The functions of the cosine of the scattering angle entering in (44) are written in the form

$$P_1(x)=x\Big[\frac{48x^3+42x^2-7x-13}{12(1+x)}-\frac{1}{2}(8x^3+3x^2+4x+1)\ln\Big(1+\frac{1}{x}\Big)\Big] \quad (45)$$
$$P_0(x)=x\Big[-8x^2+x+\frac{1}{6}+(8x^3+3x^2-4x-1)\ln\Big(1+\frac{1}{x}\Big)\Big],$$
$$A(x)=-x+2x^2(1+2x)\Big[\frac{2x^2+x+1}{2(1+x)}-x^2\ln\Big(1+\frac{1}{x}\Big)\Big],$$
$$B(x)=-\frac{1}{2}-\frac{x}{2}(4x^2-1)+x(1-x^2)(1+2x)\ln\Big(1+\frac{1}{x}\Big).$$

The parameters $I_3^{(1)}$ and $I_1^{(1)}$, in the set of coordinates in which $\gamma_{21} = 0$, are respectively equal to

$$I_3^{(1)}(\mathbf{n}_1\mathbf{g}0)=(1-x^2)[Z_0(x;y)+yZ_1(x;y)]/u(y),$$
$$I_1^{(1)}(\mathbf{n}_1\mathbf{g}0)=-(1-x^2)[Z_1(x;y)-yZ_0(x;y)]/u(y), \quad (46)$$

where the following notation has been introduced:

$$y=\delta\cos\beta_{g1}, \qquad u(y)=1+y^2, \quad (47)$$
$$Z_0(x;y)=-\frac{1}{4}+\frac{x}{2}\Big\{-\frac{3}{2u}+\frac{6x^2}{u^3}(1-3y^2)+a(x;y)\varphi_1(x;y)-2xyb(x;y)\varphi_2(x;y)\Big\},$$
$$Z_1(x;y)=\frac{x}{2}\Big\{\frac{3y}{2u}-6yx^2(3-y^2)u^{-3}+b(x;y)\varphi_1(x;y)+2xya(x;y)\varphi_2(x;y)\Big\},$$
$$a(x;y)={}^1/_2\ln[(y^2+(1+x)^2)/x^2]+y\,\mathrm{arctg}\,(y/(1+x)),$$
$$b(x;y)=\mathrm{arctg}\,(y/(1+x))-{}^1/_2\ln[(y^2+(1+x)^2)/x^2],$$
$$\varphi_1(x;y)=[u^3+2xu^2-3x^2u(1-y^2)-6x^3(1-3y^2)]u^{-4},$$
$$\varphi_2(x;y)=[-u^2+3ux+3x^2(3-y^2)]u^{-4}.$$

The results of the calculation of the angular distribution and the normalized Stokes parameters from Eqs. (43)—(47) are given in Figs. 1—3.

We shall discuss these results briefly. We see that the intensity $I_0^{(1)}(\mathbf{n}_1\mathbf{g}0)$, with accuracy up to terms η^2, does not depend on the gyration vector $\mathbf{g}$. This is connected with the fact that the expansion parameter is

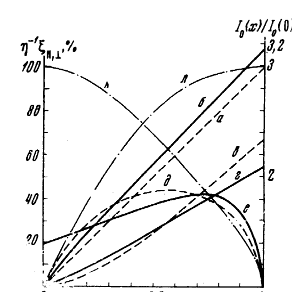

FIG. 1. Angular distribution and circular polarization of radiation emerging from a thick layer of weakly gyrotropic material: a, b—angular distribution $I_0(x)/I_0(0)$; c, d and e—functions $\xi_\parallel(x)\eta^{-1}$ and $\xi_\perp(x)\eta^{-1}$ in zeroth (dashed curve) and first (continuous curve) approximations; l and k—functions $\xi_\parallel(x)\eta^{-1}$ and $\xi_\perp(x)\eta^{-1}$ for single scattering of an unpolarized beam of photons.

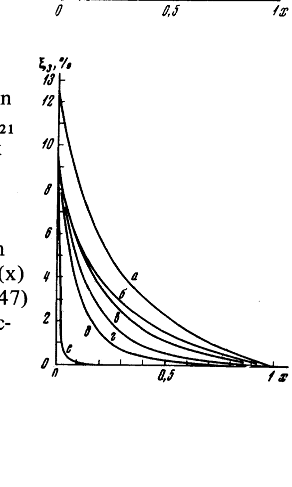

FIG. 2. Degree of linear polarization $\xi_3(x) = I_3(x)/I_0(x)$ (in a system with $\gamma_{21} = 0$) of radiation emerging from a thick layer of a weakly gyrotropic medium, when the gyration vector g is directed along the normal $\mathbf{n}_2$ to the layer: a—exact value of $\xi_3(x)$ for $\delta = 0$ (Rayleigh scattering); b, c, d, e, f—the function $\xi_3(x)$ calculated according to Eqs. (46) and (47) for the values of the parameter δ respectively equal to 0, 2, 5, 10, 100.

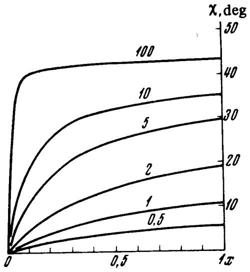

FIG. 3. Angle of rotation $\chi(\tan 2\chi = I_1(x)/I_3(x))$ of the plane of polarization of radiation emerging from the layer, relative to the plane of scattering $(n_2 n_1)$ in a reference frame with $\gamma_{21} = 0$, calculated from Eq. (47) for the case in which the gyration vector g is directed along the normal n_2 to the layer. The numbers above the curves indicate the value of the parameter δ on the figure, upper right χ, deg.

actually the parameter $\eta\alpha$, where $\alpha = 0, \pm 1$, and in the calculation of the sum of the diagonal elements of the density matrix $\rho^{(1)}_{\alpha\beta}$ the linear terms cancel one another. Figure 1 shows the normalized intensity $J_0^{(1)}(n_1 \mathbf{g} 0) = I_0^{(1)}(x)/I_0^{(1)}(0)$ and the initial angular distribution $J_0^{(0)}(x) = 1 + 2x$. We see that they are close to one another. This verifies the successful choice of the initial approximation. It is seen from Eq. (44) that the circular polarization $I_2^{(1)}(n_1 \mathbf{g} 0)$ is proportional to the small parameter η, depends linearly on the projection of the gyration vector on the normal to the surface layer $(L_{\parallel}(x)\cos\beta_{2g})$ and on the projection of g on the direction perpendicular to the plane of scattering $[L_{\perp}(x)\sin\beta_{2g}\cos(\varphi_{2g} - \varphi_{21})]$.

Figure 1 shows the values of $\xi_{\parallel}^{(1)} = L_{\parallel}(x)/I_0^{(1)}(x)$ and $\xi_{\perp}^{(1)} = L_{\perp}(x)/I_0^{(1)}(x)$ and a comparison of the circular polarization, which arises in single and multiple scattering. We note that the multiply scattered radiation has the opposite sign of polarization in comparison with the singly scattered case. This effect is explained by the fact that the components of the density matrix ρ_{-1-1} are attenuated less in the medium than the components of $\rho_{11}(\sigma_S^{(-1)}(n_1 g) \le \sigma_S^{(1)}(n_1 g))$, as a result of which $\rho_{-1-1} > \rho_{11}$ at the exit from the layer, although the opposite inequality holds in the radiation density $B_{\alpha\beta}$.

We now consider the linear polarization. It follows from (46) that the polarization is equal to zero in the linear approximation in η for x = 1, independent of the direction of **g**. The same also holds for single scattering if the density matrix is integrated over all the directions of the initial incidence n_0 of the unpolarized photons with *functions* depending only on the cosine of the angle between n_0 and the direction of the normal. Thus, even for linear polarization, the dependence of **g** for the forward direction arises only with account of terms that are quadratic in λ. Actually, $I_3^{(1)}$ and $I_1^{(1)}$ depend on the parameter $y = \delta\cos\beta_{g1}$, which is proportional to the difference in the phase velocities of the propagation of the components of the electromagnetic field E_{-1} and E_1 in an anisotropic medium. The parameter $\delta = 4\pi k g \sigma^{-1} \approx \lambda^3 \eta |\alpha|^{-1}$ depends on the wavelength λ and is always greater than η since in our case $\lambda^3/|\alpha| \sim \lambda^3/a_0^3 \gg 1$, where a_0 is the effective dimension of the scattering particle.

It follows from Fig. 2 that for $y \gg 1$, the polarization is of the order $y^{-1} \ll 1$ and for $y = 0$, $I_1^{(1)} = 0$ and $I_3^{(1)}$ is equal to the value of the polarization in Rayleigh scattering in an isotropic medium. Such a behavior of the polarization is explained by the interference of the waves, which have different propagation velocities in the medium. It is natural that there is no interference in the direction perpendicular to the field, when the velocities of the waves are the same, and the polarization is the same as in the case $\delta = 0$ (Rayleigh scattering).

Let us estimate the accuracy of the formulas developed here. It is seen from Fig. 1 that the error in the calculation of $I_0 \sim 5\%$. The accuracy of the formulas describing the linear polarization can be estimated by setting y = 0 and comparing them with the well-known numerical solutions for Rayleigh scattering. The error in this case does not exceed 25%. The maximum error is the calculation of circular polarization also reaches this value, which is established by comparing the successive approximations.

In conclusion, we put down some numerical estimates of the expansion parameters used here. Thus, in light scattering, with a wavelength $\lambda = 5 \times 10^{-5}$ cm in a plasma with a magnetic field of 42.5 Oe, we get $\eta = 1/4$ and $\delta = 1.3 \times 10^5$.

By means of the formulas obtained, it can easily be shown that the radiation flux from a distant object, which can be regarded as a point, will be polarized. Thus, in the scattering of electromagnetic radiation in a plasma in the dipole magnetic field $H_{eff} = m_0 r^{-3}$, the circular polarization of the radiation flux at large distances from the object reaches the value $0.4\,\eta_{eff}(n_1 \mathbf{m})$, where **m** is the direction of the dipole. The formulas obtained for the description of the polarization of the radiation from distant celestial objects.

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