DESTRUCTION OF LONG-RANGE ORDER IN ONE-DIMENSIONAL AND TWO-DIMENSIONAL SYSTEMS HAVING A CONTINUOUS SYMMETRY GROUP I. CLASSICAL SYSTEMS

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The low-temperature state of two-dimensional classical systems, which in the three-dimensional case have an ordered phase with a spontaneous violation of a continuous symmetry (magnetic substances, crystals), is considered. It is shown that for arbitrary dimension the long-range correlations are determined by an expression for the energy of the long wavelength fluctuations, which is quadratic with respect to the gradients. The distinctive feature of the one- and two-dimensional cases is that the fluctuation deflections grow with distance and at sufficiently large distances may reach a finite value, which leads to the necessity to take account of the effects associated with these. Thus, for a lattice of plane classical spins (Sec. 1) the contribution from configurations, where the spin vector on a path between sufficiently distant points is turned through an angle containing several complete revolutions, becomes essential.

The following new results are contained in this article: A complete description of the low-temperature state is obtained for a lattice of plane classical spins (Sec. 1) and two-dimensional crystals (Sec. 3), i.e., all of the n-point distribution functions are found, and the method is generalized to an arbitrary lattice system with a commutative continuous group; the two-point distribution function and the transformation law for the n-point functions associated with the homogeneous dilatation of all distances are found for a classical Heisenberg ferromagnetic substance (Sec. 2); also expressions are found for the free energy of magnetic substances (Secs. 1 and 2) in a weak external field, from which the necessity of a phase transition in these two-dimensional systems follows.

IT is known that in two-dimensional systems there cannot be a spontaneous violation of continuous symmetry even if this occurs in the analogous three-dimensional system. The following are examples of such systems: crystals (violation of translational invariance), isotropic magnetic substances (violation of rotational symmetry), and systems possessing superfluidity or superconductivity (violation of gauge invariance). The impossibility of long-range order in such two-dimensional systems follows from a simple calculation that goes back to Peierls[1] and Landau,[2] from which it is clear that in a two-dimensional ordered state (if it were to exist) the fluctuations of the ordering parameter would increase without any limit with increasing size of the system (see also[3]). Recently the vanishing of long-range order upon going to the thermodynamic limit has been rigorously proved for all such two-dimensional systems on the basis of Bogolyubov's inequality (see[4]). On the other hand, since the ground state of the system is ordered, the ordering must be preserved over all larger distances as $T \to 0$, and a question arises concerning the description of the state of the system at sufficiently low temperatures when the destruction of long-range order occurs over macroscopic distances. Such a description is also obtained in the present article for classical systems. Although real systems at low temperatures are essentially quantum systems, a consideration of classical systems is justified for the following reasons: In them it is easier to clarify the fundamental side of the situation, which should not change substantially upon taking account of quantum effects, since the question is the behavior over macroscopic distances. It is proposed to examine quantum systems, and also other systems for which the extension of the obtained results does not appear to be completely trivial (we mention systems having a long-range interaction and systems with randomly distributed magnetic moments) in a subsequent article.

1. LATTICE SYSTEMS WITH A COMMUTATIVE SYMMETRY GROUP

Let us consider this question, using as an example a model consisting of a simple lattice, at each site of which there is a plane classical spin—a vector $\mathbf{s}$ of fixed (unit) length which can rotate only in the one plane. Let us denote the positions of the lattice sites by $\mathbf{r} = \Sigma n_i \mathbf{a}_i$, where the n_i are integers and the $\mathbf{a}_i$ are the basis vectors of the lattice; their length (the lattice constant) is denoted by a. One can specify the state of each spin $\mathbf{s_r}$ by a single quantity—the angle $\varphi_\mathbf{r}$, namely,

$$s_r = \{\cos\varphi_r;\ \sin\varphi_r\}. \tag{1}$$

The configuration of the lattice is given by the set of angles $\varphi_\mathbf{r}$ for all sites $\mathbf{r}$. In this connection the Hamiltonian is expressed in the following way ($J > 0$):

$$H = -\frac{J}{2}\sum_{|r-r'|=a}\sum (s_r s_{r'}) = -\frac{J}{2}\sum_{|r-r'|=a}\sum \cos(\varphi_r - \varphi_{r'}). \tag{2}$$

The minimum of (2) is reached when all of the angles $\varphi_\mathbf{r}$ are the same, but as $T \to 0$ the probability of finite deflections of neighboring spins tends to zero, so that it is natural to use the quadratic expansion of (2) near the minimum:

$$H - E_0 \approx \frac{J}{2}\sum_{|r-r'|=a}\sum \frac{1}{2}(\varphi_r - \varphi_{r'})^2 = -\frac{J}{2}\sum_r\sum_{r'} \varphi_r \Delta_{rr'}\varphi_{r'}, \tag{3}$$

where E_0 denotes the energy of the ground state, and $\Delta_{\mathbf{r}\mathbf{r}'}$ is a matrix operator representing the difference analog of the Laplacian operator and is defined by the equation ($\psi_{\mathbf{r}}$ is an arbitrary function)

$$\sum_{\mathbf{r}'} \Delta_{\mathbf{r}\mathbf{r}'}\psi_{\mathbf{r}'} = \sum_{(i)} (\psi_{\mathbf{r}+\mathbf{a}_i} + \psi_{\mathbf{r}-\mathbf{a}_i} - 2\psi_{\mathbf{r}}). \tag{4}$$

It is easy to construct a low-temperature expansion for the statistical sum, in which the major term corresponds to (3), and the subsequent terms represent corrections to (3) (of the order of $(\varphi_{\mathbf{r}} - \varphi_{\mathbf{r}'})^4$ and higher). From this expansion it is clear that (3) can be used in order to calculate the free energy at low temperatures. However, in other problems there may be doubt concerning the applicability of (3). In fact (3) leads to a Gaussian distribution for $\varphi_{\mathbf{r}}$ with a correlation matrix $\langle \varphi_{\mathbf{r}}\varphi_{\mathbf{r}'}\rangle = J/TG_{\mathbf{r}\mathbf{r}'}$, where $G_{\mathbf{r}\mathbf{r}'} = -(\Delta^{-1})_{\mathbf{r}\mathbf{r}'}$ denotes the Green's function for the operator (4). In the two-dimensional case $G_{\mathbf{r}\mathbf{r}'}$ increases logarithmically with distance whereas $\langle \varphi_{\mathbf{r}}\varphi_{\mathbf{r}'}\rangle$ obviously cannot increase without limit. If one calculates the free energy in the presence of an external field, by having added the energy of the spins in this field to (3), expanded up to terms of order $\varphi_{\mathbf{r}}^2$, then one finds[1)]

$$\frac{F-E_0}{N} = \frac{T}{2}\ln\frac{2\pi}{T} - h + \frac{T}{2}\int (d\mathbf{k})\ln(J\Delta(\mathbf{k}) + h), \tag{5}$$

which corresponds to an average moment per site given by

$$\langle m\rangle = -\frac{1}{N}\frac{\partial F}{\partial h} = 1 - \frac{T}{2}\int \frac{(d\mathbf{k})}{J\Delta(\mathbf{k}) + h}. \tag{6}$$

Here $\Delta(\mathbf{k})$ denotes the Fourier representation of the operator (4):

$$\Delta(\mathbf{k}) = 4\sum_{(i)} \sin^2\left(\frac{\mathbf{k}\mathbf{a}_i}{2a}\right). \tag{7}$$

It is clear that in the two-dimensional case expression (6) for h = 0 (i.e., the spontaneous moment) diverges like ln R where R denotes the size of the system ($k_{min} \sim 1/R$).

In order to clarify just what the situation is here, it is useful to consider the one-dimensional case where the analogous divergences are even stronger ($\sim$ R) and, on the other hand, the problem can be solved exactly. Namely, for a chain consisting of N + 1 spins, in which the angles φ_0 and φ_N of the boundary spins are fixed, the density of the probability distribution for l isolated spins at the sites $\mathbf{r}_1 = n_1\mathbf{a}$, $\mathbf{r}_2 = n_2\mathbf{a}, \ldots, \mathbf{r}_l = n_l\mathbf{a}$ has the form

$$P(\varphi_1, n_1, \ldots; \varphi_l, n_l) \sim P_{n_1}(\varphi_0 - \varphi_1)P_{n_2-n_1}(\varphi_1 - \varphi_2)\ldots P_{N-n_l}(\varphi_l - \varphi_N), \tag{8}$$

where $P_n(\varphi - \varphi')$ is a function of the angle $(\varphi - \varphi')$ whose Fourier components are equal to $(c_m/c_0)^n$ while the c_m are the Fourier coefficients of the function $\exp\{-JT^{-1}\cos(\varphi - \varphi')\}$. As T → 0 the limit $P_n(\varphi - \varphi')$ is proportional to $\delta(\varphi - \varphi')$. However if one considers the asymptotic behavior over distances of the order of J/T, i.e., the limit when T → 0, n → ∞, but nT/J tends to a finite limit which is equal to t, then $(c_m/c_0)^n \to \exp\{-(1/2)m^2t\}$ and

$$P_n(\varphi - \varphi') \approx \vartheta\left(\varphi - \varphi' \middle| \frac{nT}{J}\right) \quad \text{for } \frac{T}{J} \ll 1,\ n \gg 1,\ \frac{nT}{J} = O(1), \tag{9}$$

where the customary notation is used for the function

$$\vartheta(\varphi|t) = 1 + 2\sum_{m=1}^{\infty} e^{-\frac{1}{2}m^2t}\cos m\varphi. \tag{10}$$

This function is a solution of the equation

$$\frac{\partial\vartheta}{\partial t} = \frac{1}{2}\frac{\partial^2\vartheta}{\partial\varphi^2}$$

under the following conditions: $\theta(\varphi \mid 0) = \delta(\varphi)$; $\theta(\varphi + 2\pi \mid) = \theta(\varphi \mid)$, i.e., it is the transition probability for a random walk (diffusion) around the circumference. It may be obtained from the corresponding function for diffusion along a straight line $-\infty < \varphi < +\infty$ if the sources are located at the points $\varphi_n = 2\pi n$, n = 0, ±1, ±2, ...

$$\vartheta(\varphi|t) = \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{+\infty} \exp\left\{-\frac{1}{2}(\varphi - 2\pi n)^2\right\}. \tag{11}$$

For small values of t one can keep only the term with n = 0, and for $t \gg 1$ the function (10) rapidly approaches an equilibrium distribution along the circumference.

From the derivation it is evident that the asymptotic form (9) is determined by the second derivatives of H at the minimum point, that is, by expression (3). Its form is associated with taking account of the following properties: (a) the probability of finite deflections for nearest neighbors is small, and in the limit T → 0 the angle $\varphi_{\mathbf{r}}$ changes continuously; (b) the angular deviation of spins which are sufficiently far apart may have a finite value (because the dispersion $\langle(\varphi_{\mathbf{r}} - \varphi_{\mathbf{r}'})^2\rangle$ increases linearly with the distance). On the strength of (a), for any arbitrarily distant spins one can determine the total change of angle, which is equal to the angle between the spins plus 2πn, where n denotes the number of rotations of the spin through 2π associated with a continuous transition from one site to the other; in virtue of (b), the contribution from the configurations with several revolutions is essential for the correlations at large distances. It is easy to see that a Gaussian distribution corresponding to (3) holds for the total angle, and (9) is obtained from it by reduction to the interval of periodicity (i.e., taking into consideration the fact that total angles which differ by 2πn correspond to one and the same mutual alignment of the spins). For other dimensions the differences will be associated only with (b): in the two-dimensional case the dispersion of the angular deflection increases logarithmically with distance and finite deflections accumulate slowly; in the three-dimensional case the dispersion remains bounded and finite deflections are generally not important.

Thus, in the two-dimensional case of interest to us one can use the Gaussian distribution corresponding to (3) (below this will be denoted by GDC (3)) provided it is applied to the total angles $\varphi_{\mathbf{r}}$ with intervals of variation $-\infty < \varphi_{\mathbf{r}} < \infty$.[2)] It is most convenient of all to accomplish the reduction of a distribution by having

[1)] In this and similar formulas, the integration over k is carried out over a cell of the reciprocal lattice, $|k_i| \leq \pi$; (dk) denotes $\Pi_i(dk_i/2\pi)$.

[2)] We define the difference of the total angles at distant points as the total change of the angle along any arbitrary path connecting these points. The fact that the result is independent of the path follows from the fact that going around a closed contour gives zero: for a small contour this follows from a), and one can construct a circuit around a large contour out of circuits around small contours.

noted that such a reduction is carried out automatically if the averages of periodic functions of the angles $\varphi_{\mathbf{r}}$ are calculated according to the GDC (3). It is sufficient for an arbitrary set of points $\mathbf{r}_k$ ($k = 1, 2, \ldots, n$) to evaluate the average of the Fourier exponential functions:

$$\left\langle \exp\left\{ i \sum_k m_k \varphi_{\mathbf{r}_k} \right\} \right\rangle = \exp\left\{ -\frac{T}{2J} \sum_k \sum_{k'} m_k m_{k'} G_{\mathbf{r}_k \mathbf{r}_{k'}} \right\}. \quad (12)$$

Here the m_k are arbitrary integers, and $G_{\mathbf{r}\mathbf{r}'}$ is the matrix which is the inverse of the matrix $\Delta_{\mathbf{r}\mathbf{r}'}$, i.e., the Green's function of the operator (4). Actually, for averages over the total angles Eq. (12) is valid for arbitrary m_k (the characteristic function for the GDC (3) stands on the right, see[5]), but according to what has been said above, in that case when the m_k are integers and hence the averaged function is periodic, its average with respect to the GDC (3) agrees with the Gibbs average (for $T/J \ll 1$).

Certain asymptotic forms of the two-dimensional $G_{\mathbf{r}\mathbf{r}'}$ are needed for a consideration of the consequences of (12). Namely, when the size of the lattice $R \gg a$, the points $\mathbf{r}$ and $\mathbf{r}'$ are far away from the boundaries, and $|\mathbf{r} - \mathbf{r}'| \ll R$, the quantity $G_{\mathbf{r}\mathbf{r}'}$ can be represented in the form

$$G_{\mathbf{r}\mathbf{r}'} \approx g\left(\frac{\mathbf{r}-\mathbf{r}'}{a}\right) - \frac{1}{2\pi} \ln r_0 + \frac{1}{2\pi} \ln(AR). \quad (13)$$

Here $g(\mathbf{r}/a)$ is the Green's function of an infinite lattice such that $g(0) = 0$:

$$g\left(\frac{\mathbf{r}}{a}\right) = \int \frac{(d\mathbf{k})}{\Delta(\mathbf{k})} (e^{i\mathbf{k}\mathbf{r}/a} - 1); \quad (14)$$

$r_0 \approx 0.2\,a$ is the length entering into the asymptotic behavior of $g(\mathbf{r}/a)$ for $|\mathbf{r}| \gg a$:[3)]

$$g\left(\frac{\mathbf{r}}{a}\right) = -\frac{1}{2\pi} \ln \frac{|\mathbf{r}|}{r_0} + (\text{term} \to 0 \text{ as } |\mathbf{r}| \to \infty); \quad (15)$$

r_0 is determined from the equation ($\gamma = 0.5772\ldots$ is Euler's constant):

$$\ln \frac{r_0}{a} = -\gamma - \ln \frac{\pi}{2} - \lim_{\delta \to 0} \left\{ \int_{|\mathbf{k}| > \delta} \frac{(d\mathbf{k})}{\Delta(\mathbf{k})} + \ln \delta \right\} = -\gamma - \frac{3}{2} \ln 2; \quad (16)$$

finally, the last term in (13), strictly speaking, must be the "regular part" of the Green's function of the usual (differential) Laplacian operator Green's function minus $(1/2\pi) \ln |\mathbf{r} - \mathbf{r}'|^{-1}$); since it has the form $(1/2\pi) \ln(AR)$ where $A = A(\mathbf{r}/R, \mathbf{r}'/R) \approx A(0, 0)$ (for $|\mathbf{r}|, |\mathbf{r}'|, |\mathbf{r} - \mathbf{r}'| \ll R$), we obtain (13).

Now, using Eqs. (12)–(16), let us consider certain properties of a two-dimensional lattice. For convenience of notation we introduce the dimensionless parameter

$$\alpha = T/2\pi J. \quad (17)$$

a) Influence of the boundaries. The density of the distribution[4)] of the angle $\varphi_{\mathbf{r}} = \varphi$ is given by

$$P(\varphi, \mathbf{r}) = \sum_{m=-\infty}^{+\infty} e^{-(T/2J) G_{\mathbf{r}\mathbf{r}} m^2} e^{im\varphi} = \vartheta\left(\varphi \middle| \frac{T}{J} G_{\mathbf{r}\mathbf{r}}\right). \quad (18)$$

For $(T/J) G_{\mathbf{r}\mathbf{r}} \approx \alpha \ln(AR/r_0) \gg 1$ only the term with $m = 0$ remains, i.e., the angle $\varphi_{\mathbf{r}}$ is uniformly distributed, and there is no spontaneous violation of symmetry.

b) Two-point distribution. Formula (12) gives the Fourier coefficients for the density of the distribution of the angles $\varphi = \varphi_{\mathbf{r}}$ and $\varphi' = \varphi_{\mathbf{r}'}$; the last two terms of (13) give a factor

$$\exp\left\{ -\frac{1}{2} \alpha (m + m')^2 \ln \frac{AR}{r_0} \right\},$$

in these coefficients so that for $\alpha \ln(AR/r_0) \gg 1$ only the terms with $m + m' = 0$ remain, and we obtain

$$P(\varphi, \mathbf{r}; \varphi', \mathbf{r}') = \sum_{m=-\infty}^{+\infty} \exp\left\{ \frac{T}{J} m^2 g\left(\frac{\mathbf{r}-\mathbf{r}'}{a}\right) \right\} e^{im(\varphi - \varphi')}$$

$$= \vartheta\left(\varphi - \varphi' \middle| -2\frac{T}{J} g\left(\frac{\mathbf{r}-\mathbf{r}'}{a}\right)\right). \quad (19)$$

By virtue of Eq. (15) over distances $|\mathbf{r} - \mathbf{r}'| \gg a$ this gives

$$P(\varphi, \mathbf{r}; \varphi', \mathbf{r}') = \vartheta\left(\varphi - \varphi' \middle| 2\alpha \ln \frac{|\mathbf{r}-\mathbf{r}'|}{r_0}\right)$$

$$= \sum_{m=-\infty}^{+\infty} e^{im(\varphi - \varphi')} \left| \frac{\mathbf{r}-\mathbf{r}'}{r_0} \right|^{-\alpha m^2}. \quad (20)$$

c) Many-point distributions. In analogy to (20) we obtain the distribution for the angles φ_k describing the spins at the points $\mathbf{r}_k$ ($k = 1, 2, \ldots, n$) when the points $\mathbf{r}_k$ are far away from the boundary and $|\mathbf{r}_k - \mathbf{r}_{k'}| \gg a$:

$$P(\varphi_1, \mathbf{r}_1; \ldots; \varphi_n, \mathbf{r}_n) = \sum_{m_1 + \ldots + m_n = 0} \ldots \sum \exp\left\{ i \sum_k m_k \varphi_k \right\}$$

$$\times \prod_{k \neq k'} \prod \left| \frac{\mathbf{r}_k - \mathbf{r}_{k'}}{r_0} \right|^{-1/2 \alpha m_k m_{k'}}. \quad (21)$$

d) Homogeneous transformation of the distributions. In connection with a homogeneous dilation of all distances: $\mathbf{r}_k \to \mathbf{r}_k' = \lambda \mathbf{r}_k$, each term in (21) is multiplied by $|\lambda|^{\alpha s}$, where the fact that $\Sigma m_k = 0$, $s = -(1/2)\Sigma m_k^2 = -(1/2)(m_1^2 + \ldots + m_n^2)$ in all terms has been taken into account. Thus

$$\left\langle \exp\left\{ i \sum_k m_k \varphi_{\mathbf{r}'_k} \right\} \right\rangle = |\lambda|^{-1/2 \alpha \sum_k m_k^2} \left\langle \exp\left\{ i \sum_k m_k \varphi_{\mathbf{r}_k} \right\} \right\rangle, \quad (22)$$

where $\mathbf{r}_k' = \lambda \mathbf{r}_k$. Having taken an infinitesimal transformation $\lambda = 1 + \delta\lambda$, we obtain the equivalent relations:

$$\left(\sum_k \mathbf{r}_k \frac{\partial}{\partial \mathbf{r}_k} \right) P(\varphi_1 \mathbf{r}_1, \ldots, \varphi_n \mathbf{r}_n) = \frac{1}{2} \alpha \left(\sum_k \frac{\partial^2}{\partial \varphi_k^2} \right) P(\varphi_1 \mathbf{r}_1, \ldots, \varphi_n \mathbf{r}_n). \quad (23)$$

e) Correlation of the spins. For $|\mathbf{r} - \mathbf{r}'| \gg a$ we have:

$$\langle \mathbf{s}_r \mathbf{s}_{r'} \rangle = \langle \cos(\varphi_r - \varphi_{r'}) \rangle = \left| \frac{\mathbf{r}-\mathbf{r}'}{r_0} \right|^{-\alpha}. \quad (24)$$

f) Weak external field. Let us consider the change of the free energy in the presence of a weak external field, where (5) is not applicable (it is impossible to expand $\cos \varphi_{\mathbf{r}}$ because only the differences between neighboring or close angles are small, but not the angles themselves). In a weak field it is necessary to take the expression

$$\Delta F = -T \ln \left\langle \exp\left\{ \frac{h}{T} \sum_r \cos \varphi_r \right\} \right\rangle. \quad (25)$$

[3)] See [6]. The connection between the notations is as follows: the "potential kernel" $a(\mathbf{x})$ from [6] is equal to $-4g(\mathbf{x})$.

[4)] Referred to $d\varphi/2\pi$, but in Eqs. (19)–(21) referred to the product $\Pi\, (d\varphi_k/2\pi)$.
k

for the free energy. A formal expansion in powers of h would give $\Delta F/N = (1/2)\chi h^2$, where $\chi = \int (d\mathbf{r})\langle \mathbf{s}_0 \cdot \mathbf{s}_\mathbf{r}\rangle$, but in virtue of Eq. (24) this integral diverges. Therefore F has a singularity for small values of h. In order to clarify its form, let us consider the case when the field is enclosed in a finite region of dimension R, where R is large but still finite. Let us denote by $\Delta F(R, h)$ the change of the free energy in the presence of such a field. A general term of the series for $\exp\{-T^{-1}\Delta F(R, h)\}$ is proportional to the integral

$$\int\limits_{|\mathbf{r}_k|\leqslant R} \ldots \int (d\mathbf{r}_1)\ldots(d\mathbf{r}_n)\langle(\cos\varphi_{\mathbf{r}_1})^{p_1}\ldots(\cos\varphi_{\mathbf{r}_n})^{p_n}\rangle (h/T)^n.$$

First let us consider the terms where $p_1 = p_2 = \ldots = p_n = 1$. Having carried out the substitution $\mathbf{r}_k = R\mathbf{r}_k'$ and taking (22) into consideration we see that the general term $\sim R^{2n+\alpha s}$, where $s = -(1/2)\Sigma m_k^2 = -(1/2)n$ (because all $m_k = \pm 1$). For the sum of the terms of the indicated form we obtain the series

$$\sum_{n=0}^{\infty} C_n(\alpha) R^{(2-\alpha/2)n}\left(\frac{h}{T}\right)^n = f(\alpha, x), \qquad x = R^{2-\alpha/2}\frac{h}{T}. \tag{26}$$

The remaining terms (where just one $p_k \geq 2$) have a higher power of h at the same power of R; thus the contributions of the terms with $p_1 = 2, p_2 = \ldots = p_n = 1$ have the form $R^{-2+\alpha}f_1(\alpha, x)$ and $R^{-2-\alpha}f_2(\alpha, x)$. A representation of the following form is obtained for $\Delta F(R, h)$:

$$\frac{1}{T}\Delta F(R, h) = f_0\left(\alpha, \frac{h}{T}R^{2-\alpha/2}\right) + R^{-2+\alpha} f_1\left(\alpha, \frac{h}{T}R^{2-\alpha/2}\right) + \ldots \tag{27}$$

Now let us take into consideration that as $R \to \infty$ the limit

$$\lim_{R\to\infty}\frac{\Delta F(R, h)}{R^2} = \frac{\pi}{a^2}\frac{\Delta F(h)}{N}, \tag{28}$$

should exist, which is proportional to the specific free energy. From here and from (27) it follows, as one can easily see, that the functions entering into (27) must be power-law functions for $x \gg 1$:

$$f_0(\alpha, x) = C_0(\alpha)x^{b_0}, \qquad f_1(\alpha, x) = C_1(\alpha)x^{b_1}, \ldots,$$

where $b_0 = 4/(4-\alpha)$, $b_1 = 1 + 4/(4-\alpha), \ldots$. Comparing with the expansion (5) (applicable for $\alpha \ln(h/T) \ll 1$, whereas the expansions obtained below are valid for $\alpha \ll 1$, $h/T \ll 1$), we find that $C_0(\alpha) = -1 + o(\alpha)$. Thus, the principal terms of the expansions as $h \to 0$ will be

$$\frac{\Delta F}{N} = -T\left(\frac{h}{T}\right)^{4/(4-\alpha)} + \ldots \tag{29}$$

and for the specific moment

$$\langle m\rangle = -\frac{1}{N}\frac{\partial F}{\partial h} = \frac{4}{4-\alpha}\left(\frac{h}{T}\right)^{\alpha/(4-\alpha)} + \ldots \tag{30}$$

g) Existence of a phase transition. Since the usual dependences $\Delta F \sim h^2$ and $\langle m\rangle \sim h$ hold at high temperatures, a comparison with (29) and (30) shows that a break in the analyticity must occur between high and low temperatures, i.e., a certain phase transition. Such a conclusion was reached in articles[7] on the basis of a numerical analysis of the high-temperature expansions.

2. LATTICE SYSTEMS THAT ARE INVARIANT UNDER THE ROTATION GROUP

Let us again consider the question for the case of a model, taking the model to be a classical Heisenberg ferromagnetic substance. In this model a three-dimensional unit vector $\mathbf{n}_\mathbf{r}$ is located at each site $\mathbf{r}$. In the system of spherical coordinates θ and ψ its components and the volume element are given by

$$\mathbf{n}_\mathbf{r} = \{\sin\vartheta_\mathbf{r}\cos\psi_\mathbf{r}; \sin\vartheta_\mathbf{r}\sin\psi_\mathbf{r}; \cos\vartheta_\mathbf{r}\}, \quad (d\mathbf{n}_\mathbf{r}) = \frac{1}{4\pi}\sin\vartheta_\mathbf{r}\, d\vartheta_\mathbf{r}\, d\psi_\mathbf{r}. \tag{31}$$

The Hamiltonian of the system has the following form ($J > 0$, $\gamma_{\mathbf{r}\mathbf{r}'}$ denotes the angle between $\mathbf{n}_\mathbf{r}$ and $\mathbf{n}_{\mathbf{r}'}$):

$$H = -\tfrac{1}{2}J\sum_{|\mathbf{r}-\mathbf{r}'|=a}\sum (\mathbf{n}_\mathbf{r}\mathbf{n}_{\mathbf{r}'}) = -\tfrac{1}{2}J\sum_{|\mathbf{r}-\mathbf{r}'|=a}\sum \cos\gamma_{\mathbf{r}\mathbf{r}'}. \tag{32}$$

For $T = 0$ all the $\mathbf{n}_\mathbf{r}$ are directed in the same direction, but as $T \to 0$ for neighboring spins $\gamma_{\mathbf{r}\mathbf{r}'} \ll 1$, and one can write (still using $\sin\theta_\mathbf{r}\sin\theta_{\mathbf{r}'} \approx \sin^2\theta_\mathbf{r} \approx \sin^2\theta_{\mathbf{r}'}$):

$$H - E_0 \approx \frac{1}{2}\cdot J\sum_{|\mathbf{r}-\mathbf{r}'|=a}\sum \frac{1}{2}(\gamma_{\mathbf{r}\mathbf{r}'})^2 \approx$$

$$\approx \frac{1}{2}J\sum_{|\mathbf{r}-\mathbf{r}'|=a}\sum \frac{1}{2}\{(\vartheta_\mathbf{r}-\vartheta_{\mathbf{r}'})^2 + (\psi_\mathbf{r}-\psi_{\mathbf{r}'})^2\sin^2\vartheta_\mathbf{r}\}. \tag{33}$$

The Gibbs distribution corresponding to (33) is not Gaussian and in the general case does not reduce to such a distribution (this is associated with the noncommutative nature of the rotation group. see Sec. 3). One can explicitly evaluate the n-point distributions only in the one-dimensional case where they, in analogy to (8), are expressed in terms of the two-point distribution which has the form

$$P(\mathbf{n}, \mathbf{r}; \mathbf{n}', \mathbf{r}') = \Omega\left(\mathbf{n}, \mathbf{n}'\left|\frac{J}{T}\frac{|\mathbf{r}-\mathbf{r}'|}{a}\right.\right). \tag{34}$$

Here $\Omega(\mathbf{n}, \mathbf{n}'|t)$ denotes the transition probability for rotational Brownian motion (see[8]); it is given by a series in terms of the spherical harmonics

$$\Omega(\mathbf{n}, \mathbf{n}'|t) = \sum_{l=0}^{\infty}(2l+1)P_l(\mathbf{n}\mathbf{n}')e^{-\frac{1}{2}l(l+1)t}$$

$$= \sum_{l=0}^{\infty}\sum_{|m|\leqslant l} Y_l^m(\mathbf{n})Y_l^{*m}(\mathbf{n}')e^{-\frac{1}{2}l(l+1)t} \tag{35}$$

and is a fundamental solution of the equation $\partial\Omega/\partial t = -(1/2)\hat{\mathcal{L}}^2\Omega$, where

$$-\hat{\mathcal{L}}^2 = (\sin\vartheta)^{-1}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + (\sin\vartheta)^{-2}\frac{\partial^2}{\partial\psi^2}. \tag{36}$$

A comparison of (34) with (9) and (20) leads to the natural supposition that in the two-dimensional case for $|\mathbf{r} - \mathbf{r}'| \gg a$ we will have

$$P(\mathbf{n}, \mathbf{r}; \mathbf{n}', \mathbf{r}') = \Omega\left(\mathbf{n}, \mathbf{n}'\left|2a\ln\frac{|\mathbf{r}-\mathbf{r}'|}{r_0}\right.\right). \tag{37}$$

And in fact (37) turns out to be a consequence of relations (42) which are proved below. One can arrive at these relations by considering the structure of the n-point distributions over distances much larger than the lattice constant a. The form of the corresponding asymptotes should primarily be determined by the energy of the long wavelength fluctuations; this can be verified directly for the system of Section 1. In fact (21) follows from (15), and the form of (15) is determined by only the quadratic term in the long wave-

length expansion of $\Delta(\mathbf{k})$, which corresponds to the expression $(1/2)\int(d\mathbf{r})(\nabla\varphi)^2$ for the energy; only the coefficient of proportionality between $\mathbf{r}_0$ and a (see Eq. (16)) depends on the short wavelength behavior of $\Delta(\mathbf{k})$. In the case under consideration one can also confirm that the overwhelming contribution to the statistical sum comes from the configurations in which $\theta_{\mathbf{r}}$ and $\psi_{\mathbf{r}}$ vary slowly and smoothly; for such configurations one can regard $\theta_{\mathbf{r}}$ and $\psi_{\mathbf{r}}$ as the values of continuous and differentiable functions at the lattice points; the energy (33) for such configurations is transformed into the form

$$H - E_0 \approx \frac{1}{2} J \iint (d\mathbf{r}) \{(\nabla\vartheta)^2 + (\nabla\psi)^2 \sin^2\vartheta\}. \tag{38}$$

In order to evaluate the n-point functions of the distribution $P(\mathbf{n}_i, \mathbf{r}_i)$ it is necessary to fix the values of the spins at the points $\mathbf{r}_i$ to be equal to $\mathbf{n}_i$ and to integrate over the remaining spins; in this connection expression (38) will not be applicable near each fixed spin. Therefore we surround each point $\mathbf{r}_i$ by a circle of radius ρ such that $\rho \gg a$, but $\rho \ll \min|\mathbf{r}_i - \mathbf{r}_k|$. Outside of the circles (38) is applicable, and inside the circles another approximation is valid: Here the dominant contribution comes from the configurations in which $\theta_{\mathbf{r}}$ and $\psi_{\mathbf{r}}$ differ very little from the values which correspond to the spin fixed at the center of the circle. For such configurations the Gibbs distribution with an energy (33) reduces to a Gaussian distribution. In fact, let all the $\mathbf{n}_{\mathbf{r}}$ be close to one and the same value $\bar{\mathbf{n}}$, i.e., $\mathbf{n}_{\mathbf{r}} = \bar{\mathbf{n}} + \boldsymbol{\epsilon}_{\mathbf{r}}$, where $|\boldsymbol{\epsilon}_{\mathbf{r}}| \ll 1$ and $(\mathbf{n}\cdot\boldsymbol{\epsilon}_{\mathbf{r}}) = 0$. Then, having chosen $\bar{\mathbf{n}}$ as the axis of the spherical coordinate system, we will have $|\theta_{\mathbf{r}}|, |\psi_{\mathbf{r}}| \ll 1$, $\sin\theta_{\mathbf{r}} \approx \theta_{\mathbf{r}}$, and the expression for the statistical weight takes the form

$$dZ \sim \exp\Big\{-\frac{J}{2T}\sum_{|\mathbf{r}-\mathbf{r}'|=a}\sum \frac{1}{2}[(\vartheta_{\mathbf{r}}-\vartheta_{\mathbf{r}'})^2 + \vartheta_{\mathbf{r}}^2(\psi_{\mathbf{r}}-\psi_{\mathbf{r}'})^2]\Big\}\prod_{(\mathbf{r})}(\vartheta_{\mathbf{r}}\,d\vartheta_{\mathbf{r}}\,d\psi_{\mathbf{r}}). \tag{39}$$

If $\theta_{\mathbf{r}}$ and $\psi_{\mathbf{r}}$ are regarded as polar coordinates of a two-dimensional vector $\boldsymbol{\epsilon}_{\mathbf{r}} = \{\epsilon_{\mathbf{r}}^{(1)}, \epsilon_{\mathbf{r}}^{(2)}\} = \{\theta_{\mathbf{r}}\cos\psi_{\mathbf{r}}, \theta_{\mathbf{r}}\sin\psi_{\mathbf{r}}\}$, then with reference to $\boldsymbol{\epsilon}_{\mathbf{r}}$ the distribution (39) is Gaussian:

$$dZ \sim \exp\Big\{-\frac{J}{2T}\sum_{|\mathbf{r}-\mathbf{r}'|=a}\sum \frac{1}{2}[(\epsilon_{\mathbf{r}}^{(1)}-\epsilon_{\mathbf{r}'}^{(1)})^2 + (\epsilon_{\mathbf{r}}^{(2)}-\epsilon_{\mathbf{r}'}^{(2)})^2]\Big\}\prod_{(\mathbf{r})}(d\epsilon_{\mathbf{r}}^{(1)}\,d\epsilon_{\mathbf{r}}^{(2)}) \tag{40}$$

Expression (40) is applicable in the three-dimensional case for a system of arbitrary dimensions, and in the two-dimensional case—so long as $\langle|\boldsymbol{\epsilon}_{\mathbf{r}}|^2\rangle \ll 1$, which gives the condition $(T/J)\ln\rho/a \ll 1$ for the radius of the circle. It is essential that for $T/J \ll 1$ one can select radii ρ which satisfy this condition and also such that $\rho \gg a$; if we take two such radii, ρ and ρ', then both (38) as well as (40) will be applicable inside the ring $\rho' \leq |\mathbf{r} - \mathbf{r}_i| \leq \rho$; the statistical sum of such a ring gives a Gaussian functional integral (which is a completely definite mathematical concept, see[9]):

$$Z_{\rho\rho'} \sim \int\ldots\int \exp\Big\{-\frac{J}{2T}\iint\limits_{\rho'\leq|\mathbf{r}-\mathbf{r}_i|\leq\rho}(d\mathbf{r})\,[(\nabla\varepsilon^{(1)})^2 + (\nabla\varepsilon^{(2)})^2]\Big\}\prod_{\mathbf{r}}(d\varepsilon_{\mathbf{r}}^{(1)}d\varepsilon_{\mathbf{r}}^{(2)}). \tag{41}$$

From what has been said it is clear how distant correlations are determined by relation (38). In fact, let us discard the spins falling inside the circles, and let us consider the (normalized) statistical sum of the obtained "hole" system provided that all spins of the boundary of the i-th circle take one and the same value $\mathbf{n}_i$. Let us denote this quantity by $P_\rho(\mathbf{n}_i, \mathbf{r}_i)$; it possesses the following properties, of which (a) and (b) are proved in the Appendix: (a) the asymptotic behavior, which is of interest to us, of the n-point functions distribution $P(\mathbf{n}_i, \mathbf{r}_i)$ are obtained from $P_\rho(\mathbf{n}_i, \mathbf{r}_i)$ by the substitution $\rho = \mathbf{r}_0$ (where $\mathbf{r}_0$ is given by Eq. (16)); (b) from (41) one obtains the following equations (A.11) for the dependence of $P_\rho(\mathbf{n}_i, \mathbf{r}_i)$ on ρ:

$$\rho\frac{\partial}{\partial\rho}P_\rho(\mathbf{n}_i, \mathbf{r}_i) = \frac{1}{2}\alpha\Big(\sum_i \hat{\mathscr{L}}_i^2\Big)P_\rho(\mathbf{n}_i, \mathbf{r}_i);$$

(c) the $P_\rho(\mathbf{n}_i, \mathbf{r}_i)$ only depend on the ratios $\mathbf{r}_i/\rho$. In fact, (38) is applicable outside of the circles, and as is directly evident the energy (38) does not vary under the transformation $\mathbf{r} \to \lambda\mathbf{r}$, $\theta(\mathbf{r}), \psi(\mathbf{r}) \to \theta(\lambda\mathbf{r})$, $\psi(\lambda\mathbf{r})$ (the factor $|\lambda|^{-2}$ coming from the square of the gradient is cancelled by the factor $|\lambda|^2$ coming from the volume element). By virtue of (c) one can replace $\rho\partial/\partial\rho$ in Eqs. (A.11) by $-\Sigma\mathbf{r}_i\partial/\partial\mathbf{r}_i$; then by using (a) we obtain the desired relations for the n-point distribution functions:

$$\Big(\sum_i \mathbf{r}_i\frac{\partial}{\partial\mathbf{r}_i}\Big)P(\mathbf{n}_1, \mathbf{r}_1; \ldots; \mathbf{n}_n, \mathbf{r}_n) = -\frac{1}{2}\alpha\Big(\sum_i \hat{\mathscr{L}}_i^2\Big)P(\mathbf{n}_1, \mathbf{r}_1; \ldots; \mathbf{n}_n, \mathbf{r}_n) \tag{42}$$

($\hat{\mathscr{L}}_i^2$ denotes the operator (36) which operates on the variables θ_i and ψ_i). For $n = 2$ from here we obtain (37) (taking into consideration that the two-point functions must depend only on $(\mathbf{n}\cdot\mathbf{n}')$ and $|\mathbf{r} - \mathbf{r}'|$), and for arbitrary n we obtain formulas analogous to (22):

$$\Big\langle\prod_{(i)} Y_{l_i}^{m_i}(\mathbf{n}_{\mathbf{r}_i'})\Big\rangle = |\lambda|^{-\frac{1}{2}\alpha\sum_i l_i(l_i+1)}\Big\langle\prod_{(i)} Y_{l_i}^{m_i}(\mathbf{n}_{\mathbf{r}_i})\Big\rangle,\quad \mathbf{r}_i' = \lambda\mathbf{r}_i. \tag{43}$$

From (43), in the same way as in Sec. 1, expansions are obtained for the free energy and for the moment in a weak external field. Their dominant terms differ from (29) and (30) only by the replacement of α by 2α, and these expansions lead to a similar conclusion, namely, that a phase transition must occur in a classical two-dimensional Heisenberg ferromagnetic. This conclusion was also obtained earlier on the basis of machine calculations (see[7]).

3. GENERAL LATTICE SYSTEMS: TWO-DIMENSIONAL CRYSTALS

The titles of Secs. 1 and 2 emphasize that although the investigation was carried out for specific models, it is actually applicable to an arbitrary lattice system. In fact, let the state of each site be described by the variables $\xi = \{\xi^i\}$ and the interaction (having a finite radius) is invariant under the group of transformations $\xi \to \xi' = U\xi$. It is easy to see that the expression for the energy density of the long wavelength fluctuations should have the following form: If $ds^2 = g_{ik}d\xi^i d\xi^k$ is a distance invariant under group transformations in the space ξ, then the energy density is proportional to $g_{ik}(\nabla\xi^i)\cdot(\nabla\xi^k)$. For commutative groups (and only for them) instead of ξ^i one can choose the variables ψ^i such that ds^2 will have a Euclidean form in them, and the energy density will be proportional to the sum of the squares of the gradients ψ^i. Applying the discussions of Sec. 1 to each ψ^i, we arrive at a Gaussian distribution over the space, associated with a universal covering group; then it is necessary to reduce to a dis-

tribution on the original space. If the group is noncommutative, g_{ik} depends on ξ (see (38)), but one can obtain formulas analogous to (42) and (43) for arbitrary groups.

Let us still consider the case of two-dimensional crystals. Let the two-dimensional system in its ground state be a crystal with a lattice constant a (for simplicity we assume the lattice to be square). At low temperatures in the vicinity of each atom its neighbors are arranged near the lattice sites so that one can attribute each atom to a definite site. Moving along a continuous path, one can realize such a correspondence even for distant atoms; here in general different paths might lead to different correspondences for one and the same atom—this will happen in that case when there are dislocations located somewhere between these paths. However since each dislocation involves a finite energy, at sufficiently low temperatures it is possible to ignore the dislocations (compare with the similar considerations in Sec. 1). We arrive at the result that at sufficiently low temperatures one can refer each atom to a definite lattice site so that the coordinates of the atom attributed to the site $\mathbf{r}$ are represented in the form[5)]

$$\mathbf{q}_\mathbf{r} = \mathbf{r} + \mathbf{u}_\mathbf{r}. \tag{44}$$

In terms of the variables $\mathbf{u_r}$ the long wavelength part of the energy is given by (see[10]):

$$H - E_0 \approx \iint \Big[\frac{J'}{2a^2} \Big(\Big(\frac{\partial u_1}{\partial x_1} \Big)^2 + \Big(\frac{\partial u_2}{\partial x_2} \Big)^2 \Big) + \frac{J''}{a^2} \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \frac{J'''}{2a^2} \Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \Big)^2 \Big] (d\mathbf{r}) \tag{45}$$

(J', J'', and J''' have the dimensions of energy). A Gaussian distribution exists for the quantities $\mathbf{u_r}$ so that in analogy to (12) we obtain

$$\Big\langle \exp\Big\{ i \sum_\mathbf{r} \mathbf{k}_\mathbf{r} \mathbf{u}_\mathbf{r} \Big\} \Big\rangle = \exp\Big\{ -\frac{1}{2T} \sum_\mathbf{r} \sum_{\mathbf{r}'} \mathbf{k}_\mathbf{r} g\Big(\frac{\mathbf{r} - \mathbf{r}'}{a} \Big) \mathbf{k}_{\mathbf{r}'} \Big\}, \tag{46}$$

where $g(\mathbf{r}/a) = g_{jj'}(\mathbf{r}/a)$ ($j, j' = 1, 2$) is the Green's function which is defined in analogy with (14) only instead of $\Delta^{-1}(\mathbf{k})$ it is necessary to take the matrix which is the inverse of the matrix $\Delta_{jj'}(\mathbf{k})$; the long wavelength expansion of this last matrix (correct to terms of order k^2) is determined by expression (45). The asymptotic expression for $|\mathbf{r}| \gg a$, which is analogous to (15), also has a logarithmic character but in the general case it still depends on the orientation of $|\mathbf{r}|$ relative to the crystalline axes; for systems with random isotropy ($J' = J'' = J$; $J''' = -J$)

$$g_{jj'}\Big(\frac{\mathbf{r}}{a} \Big) \approx \delta_{jj'} \frac{2a^2}{2\pi J} \ln \frac{|\mathbf{r}|}{r_0};$$

we do not write down the general expressions.

In the case of a crystal, not the distributions for $\mathbf{u_r}$ but the ordinary distribution functions $f_n(\mathbf{q}_1, \ldots, \mathbf{q}_n)$ are of interest, where the latter are defined as the probabilities of finding n arbitrary particles near the points $\mathbf{q}_1, \ldots, \mathbf{q}_n$, relative to $N^{-n}(d\mathbf{q}_1) \ldots (d\mathbf{q}_n)$. One can represent the Fourier transforms of these functions as the averages of products of the quantity $\rho(\mathbf{p}) = N^{-1} \Sigma \exp\{-i\mathbf{p} \cdot \mathbf{q}_l\}$ (the summation goes over all of the particles); according to (44) one can also write $\rho(\mathbf{p}) = N^{-1} \Sigma \exp\{-i\mathbf{p} \cdot \mathbf{r} - i\mathbf{p} \cdot \mathbf{u_r}\}$ (summation over the lattice sites), and from (46) we then obtain the following result for the distribution functions

$$f_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) = a^{-2} \int \ldots \int (d\mathbf{p}_1) \ldots (d\mathbf{p}_n) \delta(\mathbf{p}_1 + \ldots + \mathbf{p}_n) \times \frac{1}{N} \sum_{\mathbf{r}_1} \ldots \sum_{\mathbf{r}_n} \exp\Big\{ i \sum_{l=1}^{n} \mathbf{p}_l (\mathbf{q}_l - \mathbf{r}_l) - \frac{1}{2T} \sum_{l} \sum_{l' \neq l} \mathbf{p}_l g\Big(\frac{\mathbf{r}_l - \mathbf{r}_{l'}}{a} \Big) \mathbf{p}_{l'} \Big\} \tag{47}$$

(in formula (47) one can simultaneously omit the factor $1/N$ and the summation over one of the $\mathbf{r}_l$ since actually the expression being summed only depends on the differences $\mathbf{r}_l - \mathbf{r}_{l'}$). Thus, in the isotropic case we have

$$a^2 f_2(\mathbf{q}, \mathbf{q}') = \iint \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \sum_\mathbf{r} \exp\Big\{ i\mathbf{p}(\mathbf{q} - \mathbf{q}' - \mathbf{r}) - \frac{T}{J} p^2 g\Big(\frac{\mathbf{r}}{a} \Big) \Big\}, \tag{48}$$

where $g(\mathbf{r}/a)$ is the function given in Eq. (14). For $|\mathbf{q} - \mathbf{q}'| \gg a$ the contributions from sites occurring for points $\mathbf{q} - \mathbf{q}'$ at a distance $\lesssim a\alpha \ln(|\mathbf{q} - \mathbf{q}'|/a)$ are essential; one can use (15) for all of these sites and, by integrating over $\mathbf{p}$ we obtain

$$a^2 f_2(\mathbf{q}, \mathbf{q}') = \sum_\mathbf{r} \Big(4\pi a^2 \alpha \ln \frac{|\mathbf{r}|}{r_0} \Big)^{-1} \exp\Big\{ -\frac{(\mathbf{q} - \mathbf{q}' - \mathbf{r})^2}{a^2 \alpha \ln(|\mathbf{r}|/r_0)} \Big\} \approx F\Big(\frac{\mathbf{q} - \mathbf{q}'}{a} \Big| 2\alpha \ln \frac{|\mathbf{q} - \mathbf{q}'|}{r_0} \Big). \tag{49}$$

The last equation is obtained by replacing $\alpha \ln(|\mathbf{r}|/r_0)$ by $\alpha \ln(|\mathbf{q} - \mathbf{q}'|/r_0)$, which is valid for $\alpha = T/2\pi J \ll 1$. The solution of the two-dimensional diffusion equation $\partial F/\partial t = (1/2)\nabla^2 F$ is denoted by $F(x, t)$ with the following initial condition: at $t = 0$ the particles are concentrated at the lattice sites (i.e., $F(\mathbf{x}|0) = \sum_\mathbf{r} \delta(\mathbf{x} - \mathbf{r})$). As $t \to \infty$ they spread out in the plane and $F(x, t) \to a^{-2}$, which corresponds to the expected behavior of $f_2(\mathbf{q}, \mathbf{q}')$ in the absence of long-range order.

In conclusion I express my gratitude to G. V. Ryazanov who pointed out an error in the initial proof of relations (42).

APPENDIX

PROPERTIES OF THE FUNCTIONS $P_\rho(\mathbf{n}_i, \mathbf{r}_i)$ USED IN THE PROOF OF (42)

Let us surround the point $\mathbf{r}_0 = 0$ by a circle of radius ρ, as described in Sec. 2, we neglect the spins falling inside the circle, and we consider the (normalized) partition function of the resulting "perforated" system for a fixed configuration on the boundary of the circle. In virtue of the macroscopic nature of ρ, one can specify the boundary configuration by a continuous function $\mathbf{n}(\varphi)$ of the polar angle φ corresponding to points on the boundary of the circle. The introduced statistical sum will then be a functional of $\mathbf{n}(\varphi)$, which we denote by $P_\rho(\mathbf{n}(\cdot))$. It is obvious that for a sufficiently small ρ the creation of configurations with large deflections along the boundary requires a very large energy, so that $P_\rho(\mathbf{n}(\cdot))$ will have an appreciable value only for configurations in which all spins $\mathbf{n}(\varphi)$ are close to each other and to a certain average

[5)] The displacements $\mathbf{u_r}$ are of the order of $a\sqrt{\ln N}$ where N is the number of particles; the differences $\mathbf{u_r} - \mathbf{u_{r'}}$ should be small (in comparison with (a)) for neighboring sites.

value $\bar{n}$ so that $n(\varphi) = \bar{n} + \epsilon(\varphi)$ where $\epsilon(\varphi)$ is small. It is most convenient to eliminate the ambiguity in the choice of n by imposing the condition

$$\varepsilon_0 = \int \varepsilon(\varphi)\, d\varphi/2\pi = 0. \tag{A.1}$$

If $\epsilon(\varphi)$ is expanded in a Fourier series, then $P_\rho(n(\cdot))$ will be a function $P_\rho(\bar{n}, \epsilon_m)$ of $\bar{n}$ and of the Fourier coefficients $\epsilon_m (m \neq 0)$. Now let $\rho' < \rho$ be the radius of a smaller circle. It is obvious that $P_{\rho'}(\bar{n}', \epsilon'_m)$ and $P_\rho(n, \epsilon_m)$ are connected by the following relation:

$$P_{\rho'}(\bar{n}', \varepsilon_m') = \int Z_{\rho'\rho}(\bar{n}', \varepsilon_m' | \bar{n}, \varepsilon_m) P_\rho(\bar{n}, \varepsilon_m) \prod_{m\neq 0} (d\varepsilon_m)\,(d\bar{n}), \tag{A.2}$$

where the kernel $Z_{\rho'\rho}$ is given by the functional integral (41) with respect to the functions $\epsilon(r, \varphi)$ defined in the ring $\rho' \leq r \leq \rho$ and satisfying the following conditions on the boundaries of the ring:

$$\varepsilon(\rho, \varphi) = \sum_{m\neq 0} \varepsilon_m e^{im\varphi}, \quad \varepsilon(\rho', \varphi) = (\bar{n}' - \bar{n}) + \sum_{m\neq 0} \varepsilon_m' e^{im\varphi} \tag{A.3}$$

(the form of the conditions on the inner boundary follows from Eq. (A.1)). By virtue of its Gaussian nature we have (correct to within a normalization factor)

$$Z_{\rho'\rho}(n', \varepsilon_m' | n, \varepsilon_m) \sim \exp\left\{-\frac{1}{T} E_{min}(n', \varepsilon_m'; n, \varepsilon_m)\right\}, \tag{A.4}$$

where $E_{min}(\ldots)$ denotes the minimum of the energy in the exponential of Eq. (41) for the boundary conditions (A.3). From the solution of Laplace's equation we obtain

$$\frac{1}{T} E_{min}(\ldots) = \frac{(\bar{n}' - \bar{n})^2}{2\alpha\Delta t} + \frac{1}{\alpha} \sum_{m=1}^{\infty} m \left\{ \frac{\mathrm{ch}(m\Delta t)}{\mathrm{sh}(m\Delta t)} (|\varepsilon_m'|^2 + |\varepsilon_m|^2) - \frac{1}{\mathrm{sh}(m\Delta t)} (\varepsilon_m' \dot{\varepsilon}_m^* + \dot{\varepsilon}_m' \varepsilon_m) \right\}, \tag{A.5}$$

where the following notation is used: $t = \ln(1/\rho)$, $t' = \ln(1/\rho')$, $\Delta t = t' - t = \ln(\rho'/\rho)$, and $\alpha = T/2\pi J$.

Expression (A.4) is valid as long as $\langle(\bar{n}' - \bar{n})^2\rangle = \alpha\Delta t \ll 1$; since $\alpha \ll 1$ then one can choose Δt such that $\alpha\Delta t \ll 1$, but $e^{\Delta t} \gg 1$. Here $\tanh(m\Delta t) \approx 1$, $1/\sinh(m\Delta t) \approx 0$, and from Eqs. (A.4) and (A.5) we obtain

$$Z_{\rho'\rho}(\bar{n}', \varepsilon_m' | n, \varepsilon_m) \sim \exp\left\{-\frac{(\bar{n}' - \bar{n})^2}{2\alpha\Delta t}\right\} \times \prod_{m=1}^{\infty} \exp\left\{-\frac{m}{\alpha}|\varepsilon_m'|^2\right\} \prod_{m=1}^{\infty} \exp\left\{-\frac{m}{\alpha}|\varepsilon_m|^2\right\}. \tag{A.6}$$

From Eqs. (A.2) and (A.6) it now follows that $P_\rho(\bar{n}, \epsilon_m)$ should have the form

$$P_\rho(\bar{n}, \varepsilon_m) = P_\rho(\bar{n}) \exp\left\{-\frac{1}{\alpha} \sum_{m=1}^{\infty} m |\varepsilon_m|^2\right\}, \tag{A.7}$$

where the variation of the function $P_\rho(n)$ with a change of ρ is given by the following relation:

$$P_{t+\Delta t}(\bar{n}') \sim \int \exp\left\{\frac{(\bar{n}' - \bar{n})^2}{2\alpha\Delta t}\right\} P_t(\bar{n})\,(d\bar{n}), \qquad t = \ln\frac{1}{\rho}. \tag{A.8}$$

Formula (A.8) indicates that $P_\rho(\bar{n})$ satisfies the Fokker-Planck equation for rotational Brownian motion (see[8]) in which $t = -\ln\rho$ plays the role of the time, namely

$$\frac{\partial}{\partial(-\ln\rho)} P_\rho(\bar{n}) = -\frac{1}{2} \alpha \hat{\mathscr{L}}^2 P_\rho(\bar{n}), \tag{A.9}$$

where $\hat{\mathscr{L}}^2$ is the operator (36). We note that the function $P_\rho(\bar{n})$ is, obviously, the value of the initial functional $P_\rho(n(\cdot))$ for the configuration $n(\varphi) \equiv \bar{n}$ (i.e., when all the spins on the boundary are identical and equal to $\bar{n}$).

Now let us consider the distribution of the probability $P(n)$ for values of the spin located at the center of the circle. In analogy to Eq. (A.2) we have

$$P(n) = \int Z_{0\rho}(n | \bar{n}, \varepsilon_m) P_\rho(\bar{n}, \varepsilon_m) \prod_{m\neq 0} (d\varepsilon_m)\,(d\bar{n}), \tag{A.10}$$

where $Z_{0\rho}$ is determined by the minimum of the energy in the exponential of (40) under the following conditions: $\epsilon_r = n - \bar{n}$ at the center of the circle and (A.3) applies on its boundary. This minimum is expressed in terms of the value G_{00} of the Green's function at the center of the circle. Using (15) and (A.9) it is not difficult to show that $P(\bar{n})$ is obtained from the analytic expression for $P_\rho(\bar{n})$ by substituting $\rho = r_0$, where r_0 is given by Eq. (16).

Hence follows the validity of assertion (a) which was used in Sec. 2 in order to prove (42). In fact, let us take n points r_i distributed over macroscopic distances, let us enclose each of these points by a circle of radius ρ_i, and having rejected the inner parts of the circles, let us consider the functional $P^i_\rho(n_i(\cdot), r_i)$—the probability of the configurations $n_i(\varphi)$ on the boundaries of the circles. Applying the arguments stated above to each circle, we see that $P_{\rho_i}(n_i(\cdot), r_i)$ is represented in the form of the product of certain function $P_\rho(n_i, r_i)$ times a product of Gaussian exponentials from formula (A.7), referring to each point r_i. When all $\rho_i = \rho$, the function $P_{\rho_i}(n_i, r_i)$ coincides with the function $P_\rho(n_i, r_i)$ introduced in Sec. 2 and from the fact that the assertions proved above are valid for the dependence of each ρ_i, the validity of a) follows from Sec. 2 and the validity of b) follows from Eqs. (A.10) for each pair $\rho_i, \bar{n}_i$, that is, the equation

$$-\rho \frac{\partial}{\partial\rho} P_\rho(n_i, r_i) = -\frac{1}{2} \alpha \left(\sum_i \hat{\mathscr{L}}_i^2\right) P_\rho(n_i, r_i). \tag{A.11}$$

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