POSITRON PRODUCTION IN A COULOMB FIELD WITH $Z > 137$

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The spontaneous production of positrons in a strong Coulomb field with $Z > Z_{cr}$ is considered, where Z_{cr} is the "critical" value of the nuclear charge (at $Z = Z_{cr}$ the energy of the $1S_{1/2}$ ground state passes below the boundary of the lower continuum $\epsilon = -mc^2$; the value Z_{cr} is always >137 and depends on the cut-off radius R and, to a lesser degree, on the electric charge distribution over the nuclear volume). It is shown that at threshold [i.e, for $(Z - Z_{cr}) \ll Z_{cr}$] the probability for spontaneous production of positrons vanishes exponentially. For the entire analysis it is very important that the electron state at $\epsilon = -mc^2$ remains localized within a distance of order $\hbar/mc$ from the nucleus (m is the electron mass); this is in sharp contrast to the usual nonrelativistic situation at $\epsilon \rightarrow +mc^2$. This property is characteristic for the relativistic Coulomb problem and can be understood already on the example of the Klein-Gordon equation. Spontaneous pair production in a short-range potential of square-well type is also considered.

1. INTRODUCTION

In a recent paper Gershteĭn and Zel'dovich[1] have posed themselves the interesting question of the spontaneous production of electron-positron pairs in the Coulomb field as the strength of the potential goes slowly (adiabatically) through a critical value.[1)] This situation arises when two bare nuclei with charge $Z > Z_{cr}/2$ approach each other up to a distance $r \lesssim \hbar/mc = 3.86 \times 10^{-11}$ cm, and it also obtains in a single nucleus with charge $Z > Z_{cr}$. Here Z_{cr} is the critical value of the nuclear charge for which the energy of the ground state of the electron drops below the boundary of the lower continuum $\epsilon = -mc^2$. The value of Z_{cr} is mainly determined by the charge radius R, but depends also on the character of the charge distribution over the nuclear volume; in all cases $Z_{cr} > 137$.

It should be noted that the relativistic Coulomb problem for $Z > 137$ has a number of specific features. Thus, the Dirac equation with a potential corresponding to a point charge Ze is not correct mathematically for $Z > 137$: here the "falling into the center" known from quantum mechanics occurs (for a state with angular momentum $j = 1/2$). In order to obtain physically reasonable results for $Z > 137$ one is therefore forced to take account of the finite dimensions of the nucleus. This approach to the problem is due to Pomeranchuk and Smorodinskiĭ,[2] who gave a (qualitatively) correct description of the phenomenon at $Z > 137$. However, their approximation in deriving an equation for Z_{cr} is too crude, so that the values for Z_{cr} quoted in[2] are too high.[4] A detailed calculation of Z_{cr} for different model charge distributions of the nucleus has been carried out in[5].

The study of the phenomena for $Z > 137$ is closely connected with the question of the polarization of the vacuum in a supercritical Coulomb field. In particular, the production of pairs for $Z > Z_{cr}$ gives rise to appearance of an imaginary part in the polarization operator. Although the parameter $\alpha = Ze^2/\hbar c = Z/137$ of perturbation theory exceeds unity in this case, the problem can be solved if the exact relativistic wave functions of the electron in a Coulomb field are used (the necessity of making such a calculation has been emphasized in[1]).

Let us now give a synopsis of the present paper. In Sec. 2 we consider the limiting case of a very small cut-off radius for the Coulomb field R, where $\ln[\hbar/(mcR)] = \eta^{-1} \gg 1$. In this case the small parameter η enters in the problem, so that the solution can be found in analytic form. In Sec. 3 these calculations are generalized to the realistic case of heavy nuclei, where the nuclear radius $R \ll \hbar/mc$, but η is not yet very small compared to unity.[2)] The formulas (28), (35), and (38) for the probability of spontaneous pair production w are obtained. The general conclusion of[1] about the production of pairs for $Z > Z_{cr}$ is confirmed, but the details of this process are not the same as assumed in[1]. In particular, the state of the electron at the boundary of the lower continuum $\epsilon = -mc^2$ remains localized, owing to the specifics of the relativistic Coulomb problem (cf.[4], and also Sec. 2). Therefore the probability w has a characteristic threshold behavior (28) for $Z \rightarrow Z_{cr}$ and the pair production process sets in slowly for $Z > Z_{cr}$.

2. THE LIMITING CASE $R \rightarrow 0$

When $Z > Z_{cr}$ a number of fundamental questions arise (the rearrangement of the vacuum because of pair production, the applicability of the single-particle approximation, etc.), which are answered most simply if one has a solution of the problem in analytical form. We therefore begin with the limiting case $R \rightarrow 0$ (more precisely, we assume that not only $R \ll 1$ but

1) Cf. also [2,3]. As communicated to the author by Ya. A. Smorodinskiĭ, the question of the pair production in the collision of two near-critical nuclei has been considered by I. Ya. Pomeranchuk already in 1945 (unpublished).

2) Thus the radius $R = 1.2 \times 10^{-12}$ cm corresponds to $\hbar/(mcR) = 32$ and $\eta = 0.3$; therefore the approximation $\eta \ll 1$ is very inaccurate. For example, the asymptotic formula (11) gives here $\alpha_{cr} - 1/2$ with an error of a factor of two.

also $|\ln R| \gg 1$). Then the small parameter η enters in the theory:

$$\eta = \left(\ln \frac{1}{R}\right)^{-1}. \tag{1}$$

For simplicity we begin with the case of a scalar particle. As long as $\alpha < 1/2$ the Coulomb problem for spin s = 0 is meaningful also for a point charge. The energy of the lowest 1S level is

$$\varepsilon = \left(\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha^2}\right)^{1/2}, \quad \alpha = \frac{Ze^2}{\hbar c} = \frac{Z}{137}. \tag{2}$$

For $\alpha > 1/2$ the quantity ϵ becomes complex, corresponding to the "falling into the center." Therefore the Coulomb potential must be cut off at small distances:

$$V(r) = \begin{cases} -\alpha/r & \text{for } r > R, \\ -\dfrac{\alpha}{R} f\left(\dfrac{r}{R}\right) & \text{for } r < R. \end{cases} \tag{3}$$

The form of the cut-off function $f(r/R)$ depends on the electric charge distribution over the nuclear volume.[3)] Solving the Klein-Gordon equation with the potential (3) we obtain for $l = 0$

$$\chi_0(r) = W_{p, ig}(2\lambda r) \quad (r > R), \tag{4}$$

where $\lambda = \sqrt{1-\epsilon^2}$, $p = \epsilon\alpha/\lambda$, $g = \sqrt{\alpha^2 - 1/4}$ and W is the Whittacker function [for arbitrary angular momentum l the solution also has the form (4) with $g = \sqrt{\alpha^2 - (l+1/2)^2}$]. For $1 > \epsilon > -1$ this wave function corresponds to a bound state and for $|\epsilon| > 1$ and $\lambda = -ik$ it has an asymptotic form of the type of a divergent wave:

$$\chi_0(r) \underset{r\to\infty}{\infty} \begin{cases} e^{-\lambda r} r^{\alpha\varepsilon/\lambda} & \text{for } |\varepsilon| < 1 \quad (5)\\ \exp\left\{i\left(kr + \dfrac{\alpha\varepsilon}{k}\ln r\right)\right\} & \text{for } |\varepsilon| > 1, \quad (5')\end{cases}$$

$$k = \sqrt{\varepsilon^2 - 1}.$$

In the internal region $r < R$, $\chi_0(r)$ is almost independent of ϵ,[5] so that the logarithmic derivative at the nuclear boundary $\gamma = [r\chi_0'/\chi_0]_{r=R}$ is determined by the quantity α and the form of the cut-off function $f(r/R)$. For example, for the simple cut-off $f(r/R) \equiv 1$ we have $\gamma = \alpha \cot \alpha$. The equation for the energy of the level ϵ takes the form

$$xW'_{p,ig}(x) = \gamma W_{p,ig}(x), \quad x = 2\lambda R. \tag{6}$$

Since $x \ll 1$, we use the asymptotic form

$$W_{p,ig}(x) \underset{(x\to 0)}{\approx} \sqrt{x}\left\{\frac{\Gamma(2ig)}{\Gamma(1/2 + ig - p)} x^{-ig} + \frac{\Gamma(-2ig)}{\Gamma(1/2 - ig - p)} x^{ig}\right\}, \tag{7}$$

with the help of which we can simplify Eq. (6): to

$$(2\lambda R)^{i\nu} = \frac{2\gamma - 1 + i\nu}{2\gamma - 1 - i\nu} \frac{\Gamma(1+i\nu)\Gamma((1-i\nu)/2 - p)}{\Gamma(1-i\nu)\Gamma((1+i\nu)/2 - p)} \tag{8}$$

(here $\nu = 2g = \sqrt{4\alpha^2 - 1}$). For $R \to 0$ the value of α is close to $1/2$ and $\nu \to 0$ (cf. Fig. 4 in[5]). Since for $\nu \to 0$

$$\frac{\Gamma(z+i\nu)}{\Gamma(z-i\nu)} = e^{2i\nu\psi(z)}[1 + O(\nu^2)], \tag{9}$$

where $\psi(z)$ is the logarithmic derivative of the Γ function, Eq. (6) is transformed to the final form

$$\ln 2\lambda R + \psi\left(\frac{\lambda - \varepsilon}{2\lambda}\right) - 2\psi(1) - \frac{2}{2\gamma - 1} = \frac{2\pi n}{\nu} \tag{10}$$

[the integer n appears when taking the logarithm of (8) since $\exp(2\pi i n) = 1$].

Let us determine the "critical" value α_{cr} for which the bound level passes beyond the boundary of the continuous spectrum $\epsilon = -1$. Since $\psi(z) = \ln z - 1/2 z + \ldots$ for $z \to \infty$, we find[4)]

$$\ln R - \frac{2}{2\gamma - 1} - 2\psi(1) = \frac{2\pi n}{\nu_{cr}}, \tag{10'}$$

and (for the ground state)

$$\nu_{cr} = 2\pi\eta + \ldots, \quad \alpha_{cr} = \frac{1}{2} + \pi^2\eta^2 + \ldots \tag{11}$$

The expansion parameter in (11) is η, and it is for this reason that we had to impose the condition $|\ln R| \gg 1$. We note that the main term in the asymptotic expression (11) contains only the cut-off radius R, whereas the specific form of the cut-off function $f(r/R)$ affects only the subsequent terms (via the parameter γ). From (10) and (10') we find an equation for the energy of the level:

$$f_0(\varepsilon) = \frac{1}{\nu} - \frac{1}{\nu_{cr}}, \tag{12}$$

where

$$f_0(\varepsilon) = -\frac{1}{2\pi}\left[\ln 2\lambda + \psi\left(\frac{\lambda - \varepsilon}{2\lambda}\right)\right] \quad (\lambda = \sqrt{1 - \varepsilon^2}) \tag{13}$$

The function $f_0(\epsilon)$ is real in the interval $2^{-1/2} > \epsilon > -1$, which corresponds to the existence of a bound state for $1/2 < \alpha < \alpha_{cr}$. For $\epsilon < -1$ it acquires an imaginary part

$$\operatorname{Im} f_0(\varepsilon) = \frac{1}{4}\left\{1 - \operatorname{th}\left(\frac{\pi}{2}\sqrt{\frac{\varepsilon^2}{\varepsilon^2 - 1}}\right)\right\}, \tag{13'}$$

which is exponentially small near $\epsilon = -1$. For $\epsilon \to -1$ we have the asymptotic expansion

$$f_0(\varepsilon) \approx \frac{1}{3\pi}(1+\varepsilon) + \frac{i}{2}\exp\left\{-\frac{\pi}{\sqrt{\varepsilon^2 - 1}}\right\}\theta(-\varepsilon - 1), \tag{14}$$

where $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$; the value of $f_0(\epsilon)$ is taken on the lower rim of the cut: $\operatorname{Im}\epsilon < 0$, $\lambda = -ik$.

From (12) and (14) we find the energy of the quasistationary level for $\alpha > \alpha_{cr}$:

$$\varepsilon = -\left(1 + \frac{k^2}{2}\right) - i\frac{3\pi}{2}e^{-\pi/k}, \quad k = \sqrt{6\pi\frac{\nu - \nu_{cr}}{\nu\nu_{cr}}}. \tag{15}$$

The imaginary part of ϵ determines the life time of the quasistationary state. Let us study its physical meaning.

The Klein-Gordon equation with the potential $V(r)$ is mathematically equivalent to a nonrelativistic Schrödinger equation with the effective energy E and the potential U:

$$E = 1/2(\varepsilon^2 - 1), \quad U = \varepsilon V - 1/2 V^2. \tag{16}$$

As in the nonrelativistic regime, the Coulomb interaction distorts the wave function for $r \to \infty$ in an essential way. For $V(r) = -\alpha/r$ the "tail" of the effective potential U has the form $U(r) \approx -\epsilon\alpha/r$, i.e., is repulsive for $\epsilon < 0$ [in contrast to $V(r)$ itself]. The region

[3)]If the charge density in the nucleus is written in the form $\rho(r) = \sigma(x)Ze/4\pi R^3$, where $x = r/R$ and $\int^1 \sigma(x)x^2\,dx = 1$ (normalization condition), then $f(x) = 1/x \int_0^x \sigma(y)y^2\,dy + \int_x^1 \sigma(y)y\,dy$. In particular, $f(1) = 1$.

[4)]Since $R \ll 1$, n can take only negative values. The ground state corresponds to the minimal ν_{crit} and hence, for it n = −1.

$\epsilon < -1$ corresponds in the nonrelativistic problem to states of the continuous spectrum $E > 0$, for which the penetrability of the Coulomb barrier (cf. Fig. 1) is equal to $(k = \sqrt{2E} = \sqrt{\epsilon^2 - 1})$

$$D \propto \exp\left\{-2\int_{r_1}^{r_2}\sqrt{-k^2+\frac{2|\varepsilon|\alpha}{r}-\frac{\alpha^2-{}^1/_4}{r^2}}\,dr\right\} = \exp\left\{-2\pi\left[\frac{\alpha|\varepsilon|}{k}-\sqrt{\alpha^2-\frac{1}{4}}\right]\right\}. \quad (17)$$

For $\epsilon = -1$ the penetrability is $D = 0$; therefore the state with $\epsilon = -1$ is localized. This can also be seen from the explicit formula for $\chi_0(r)$:

$$\chi_0(r) \underset{(\varepsilon=-1)}{=} r^{1/2}K_{i\nu}(\sqrt{8\alpha r}) \underset{r\to\infty}{\propto} r^{1/4}e^{-\sqrt{8\alpha r}}, \quad (18)$$

where $K_{i\nu}(z)$ is the MacDonald function. This state occurs for $\alpha = \alpha_{cr}$. With further increase of α it goes over into the quasistationary state (15) whose wave function goes over into the divergent wave (15′) for $r \to \infty$. From (15) we obtain the probability for pair production for $\alpha \to \alpha_{cr}$:

$$w_0 = -2\,\mathrm{Im}\,\varepsilon = 3\pi\exp\left\{-\sqrt{\frac{\pi}{6}\frac{\nu\nu_{cr}}{\nu-\nu_{cr}}}\right\}, \quad (19)$$

$$\nu = \sqrt{4\alpha^2-1}.$$

For spin $s = 1/2$ the calculations are analogous. We only quote the final results. The equation for the energy has the form (12) as before, where now $\nu = 2\sqrt{\alpha^2 - 1}$, and the function $f_0(\epsilon)$ is replaced by $f_{1/2}$:

$$f_{1/2}(\varepsilon) = -\frac{1}{2\pi}\left[\ln\lambda + \psi\left(-\frac{\varepsilon}{\lambda}\right) + \frac{1+\varepsilon}{1+\varepsilon+\lambda}\right]. \quad (20)$$

For $\epsilon < -1$ the function $f_{1/2}(\epsilon)$ acquires an imaginary part; however, this has a different sign from (13′):

$$\mathrm{Im}\,f_{1/2}(\varepsilon) = \frac{1}{4}\left\{1-\mathrm{cth}\left(\pi\sqrt{\frac{\varepsilon^2}{\varepsilon^2-1}}\right)\right\}. \quad (20')$$

This was to be expected, since we are not working in a second-quantized theory: as is known[6,7] the vacuum polarization matrix elements with an odd number of closed loops must be taken with different signs for bosons and fermions. In the formalism of second quantization (or in the equivalent Feynman method of transition amplitudes) the required sign appears automatically; cf. in this connection[8]. For $\alpha > \alpha_{cr}$ the energy of the quasistationary state is equal to

$$\varepsilon = -\left(1+\frac{\varkappa^2}{2}\right) - ie^{-2\pi/\varkappa}, \quad \varkappa = \sqrt{\frac{12\pi}{5}\frac{\nu-\nu_{cr}}{\nu\nu_{cr}}}. \quad (21)$$

From this we obtain for the probability of positron production in the Coulomb field for $\alpha > \alpha_{cr}$

$$w_{1/2} = \frac{6\pi}{5}\exp\left\{-\sqrt{\frac{5\pi}{3}\frac{\nu\nu_{cr}}{\nu-\nu_{cr}}}\right\}, \quad \nu = 2\sqrt{\alpha^2-1}. \quad (22)$$

With regard to formulas (19) and (22) we make the following remarks.

1. The probabilities w_0 and $w_{1/2}$ are here measured in the units $mc^2/\hbar$.

2. The pair production is a threshold effect and occurs only for $\alpha > \alpha_{cr}$.

3. For $\alpha \to \alpha_{cr}$ the probability w vanishes exponentially owing to the presence of a Coulomb barrier for positrons. The static field produces pairs only in that region where $|V(r)| > 2mc^2$, i.e., near the nucleus. In order that the positron escape to infinity it must penetrate through the Coulomb barrier (Fig. 1), whose penetrability is exponentially small for $\epsilon \to -1$.

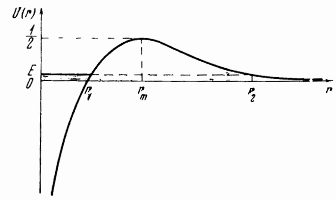

FIG. 1. Form of the effective potential (16) in the case where $v(r) = -\alpha/r$, and energy ϵ close to -1. The potential $U(r)$ takes its maximal value equal to $1/2$ for $r = r_m = \alpha$; r_1 and r_2 are the turning points; $r_1 = \alpha/2$, $r_2 = 2\alpha(\epsilon^2-1)^{-1}$. For $\epsilon \to -1$ the point r_2 is located far from the nucleus.

4. Formulas (19) and (22) have been obtained in the framework of the single-particle approximation. For their validity it is necessary, therefore, that the effect of pair production is small (otherwise an appreciable rearrangement of the vacuum would set in). This determines the region of applicability of the expressions obtained: it is necessary that the exponent be large compared to unity. In our case $\nu_{cr} \ll 1$, and already for $(\nu - \nu_{cr}) \sim \nu_{cr}^2$ the effect of pair production has the probability ~ 1, i.e., the dependence of w on α is very sharp.

5. The analytic properties of the function $f_s(\epsilon)$ are also of interest. Let, for example, $s = 1/2$. Going over to the variable $t = \sqrt{\epsilon^{-2} - 1}$ (here $\epsilon = -1$ corresponds to $t = 0$), we have from (20)

$$f_{1/2} = -\frac{1}{2\pi}\left\{\ln t + \psi\left(\frac{1}{t}\right) + \frac{1}{2}\left[1 + t - \sqrt{1+t^2} - \ln(1+t^2)\right]\right\}, \quad (23)$$

and

$$f_{1/2}(t) = \sum_{n=1}^{\infty} a_n t^{2n}, \quad (23')$$

$$a_n = \frac{1}{4\pi n}\left[B_{2n} + (-)^{n+1}\left(1 - \frac{\Gamma(n-{}^1/_2)}{\Gamma(n)\Gamma(-{}^1/_2)}\right)\right]$$

(here B_{2n} is a Bernoulli number).

Since $t^2 = (2z - z^2)/(1 - z)^2$, where $z = 1 + \epsilon$, and $t = t(z)$ is a function which is analytic inside the circle $|z| < 1$, we can rewrite (23) in the form of a series in powers of $z = 1 + \epsilon$:

$$f_{1/2}(\varepsilon) = \frac{5}{6\pi}\left[z + \frac{73}{100}z^2 + \frac{346}{525}z^3 + \dots\right]. \quad (23'')$$

As is known, for large n

$$B_{2n} \approx (-)^{n-1}\frac{(2n)!}{2^{2n-1}\pi^{2n}}. \quad (24)$$

It follows from (23′) that

$$|a_n| \underset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi n}}\left(\frac{n}{\pi e}\right)^{2n} \quad (25)$$

(here $e = 2.718\ldots$) and therefore the series (23″) does not converge, except asymptotically. This is reflected by the fact that the point $\epsilon = -1$ is a singular point of the function $f_s(\epsilon)$. We note, however, that this singularity is not connected with an infinity of the function itself or its derivative[5] but with the appearance of the imaginary part (20′) on the cut $-\infty < \epsilon$

[5] It is easy to see that $f_s(\epsilon = -1) = 0$ and that all derivatives $\partial^n f(\epsilon)/\partial\epsilon^n$ exist and are finite in the point $\epsilon = -1$.

< -1. Thus the point $\epsilon = -1$ is an essential singularity of $f_S(\epsilon)$.

6. The polarization of the vacuum in the field of a point charge with $V_0(r) = -\alpha/r$ without the usual assumption $\alpha \ll 1$ has been considered in[9], and it has been shown that the density of induced charge $\rho_P(r)$ and the polarization potential $V_P(r)$ can, in principle, be expressed in terms of the Coulomb Green's function in the form of contour integrals. However, because of the complicated form of the Coulomb propagator[6)] only the first two terms of the perturbation series could be determined so far:

$$V_P(r;\alpha) = \frac{\alpha_0}{2\pi}\sum_{n=0}^{\infty}\alpha^{2n+1}V_{2n+1}(r) \qquad (\alpha_0 = 1/137). \qquad (26)$$

The first term of the series $V_1(r)$ corresponds to the graph of Fig. 2 a; this is the so-called Uehling potential. Usually only this term is taken into account in estimates of the role of vacuum polarization.[19,20] Calculations show[9] that numerically $|V_3(r)| \ll |V_1(r)|$, so that even for $\alpha \sim 0.7$ the corrections to the level shifts in μ mesic nuclei due to $V_3(r)$ are negligibly small. For $\alpha \to 1$ one must, strictly speaking, sum over the entire infinite set of graphs (Fig. 2) since the point $\alpha = 1$ is a singularity of the series (26) in the case of a point charge. With account of the finite dimensions of the nucleus this singularity moves to the point $\alpha = \alpha_{cr} > 1$. What is the character of this singularity? In analogy to $f_S(\epsilon)$ one should suppose that α_{cr} is an essential singularity of the series (26) connected with the appearance of an imaginary part for $\alpha > \alpha_{cr}$ (which corresponds physically to pair production). The series (26) itself remains convergent at $\alpha = \alpha_{cr}$ and therefore the polarization potential $V_P(r;\alpha)$ gives, even for $\alpha = \alpha_{cr}$, in only a small correction of order $\alpha_0 = 1/137$ to the "bare" Coulomb potential $V_0(r) = -\alpha/r$ [this would evidently not be the case if the singularity of the series (26) were, say, a simple pole]. Thus, in calculating the wave functions of the electron one can neglect the distortion of the Coulomb potential owing to the polarization of the vacuum.

7. Gershteĭn and Zel'dovich[1] have proposed that a

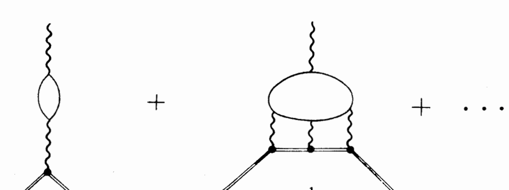

FIG. 2. Corrections to the Coulomb potential owing to vacuum polarization. The vertex Ze is marked by a black circle.

6)In the nonrelativistic case a closed expression is known for the Coulomb propagator in coordinate [10,11] as well as in momentum space [12] (and also for the N dimensional Kepler problem [13]). Unfortunately, in the relativistic propagator it is not possible to carry out the sum over intermediate states explicitly (certain approximate formulas for the Coulomb Green's function were obtained in [14]). The simplicity of the nonrelativistic case is explained by the high degree of symmetry of the Hamiltonian [group of the "hidden" symmetry 0(4)], of which little remains in the transition to the relativistic regime. [15-18]

delocalization of the vacuum polarization occurs for $\alpha \to \alpha_{cr}$, i.e., that the polarization charge extends over arbitrarily large distances from the nucleus. The principal argument in favor of this was that the wave function of the bound state $\chi_0(r) \sim e^{-\lambda r}$ and therefore the electron cloud would seem to become delocalized for $\epsilon \to -1$. Our analysis shows that this is not so: as seen from (18), $\chi_0(r) \sim \exp(-\sqrt{8\alpha r})$ for $r \to \infty$ and $\epsilon = -1$. The reason for this striking difference in the behavior of the wave function of the electron for $\epsilon = \pm 1$ is the dependence of the effective potential (16) on the sign of ϵ.

3. THRESHOLD BEHAVIOR OF THE PROBABILITY w

After having studied the qualitative nature of the phenomenon for $\alpha > \alpha_{cr}$ we now turn to a determination of the formulas which are valid in the region of realistic values of the nuclear radius $R \sim 10^{-12}$ cm. We obtain the threshold behavior of the probability w for the production of positrons for $\alpha \to \alpha_{cr}$ without assuming that $\alpha_{cr} \to 1$. Up to a factor we have

$$w = \exp\left(-\frac{2\pi\alpha|\varepsilon|}{k}\right), \quad k = \sqrt{\varepsilon^2 - 1}. \qquad (27)$$

Here we can set $\alpha \to \alpha_{cr}$, $|\epsilon| = 1$, and the momentum of the outgoing positron k must be expressed through $\Delta\alpha = \alpha - \alpha_{cr}$. As the coupling constant is increased from α_{cr} to $\alpha = \alpha_{cr} + \Delta\alpha$, the energy of the level is lowered by $\Delta\epsilon$: $\mathrm{Re}\,\epsilon = -1 + \Delta\epsilon$. Setting $\beta = -(\partial\epsilon/\partial\alpha)_{\alpha=\alpha_{cr}}$ we have

$$w \sim \exp\left\{-\sqrt{\frac{a}{\alpha - \alpha_{cr}}}\right\}, \quad a = 2\pi^2\frac{\alpha_{cr}^2}{\beta}. \qquad (28)$$

The coefficient a determines the width of the threshold region in which the pair production is still a small effect. The value of a depends on R and can be calculated for different cut-off models. To this end we note that the level shift $\Delta\epsilon$ can be found by perturbation theory (since the state of the electron for $\epsilon = -1$ remains localized). Together with (3) this gives

$$\beta = \frac{1}{R}\int_0^R (G^2 + F^2) f\left(\frac{r}{R}\right) dr + \int_R^\infty (G^2 + F^2)\frac{dr}{r}, \qquad (29)$$

where $G = rg(r)$, $F = rf(r)$, and g and f are radial functions for the upper and lower components of the Dirac bispinor normalized by

$$\int_0^\infty (g^2 + f^2) r^2\, dr = \int_0^\infty (G^2 + F^2)\, dr = 1. \qquad (29')$$

In our case $\epsilon = -1$ and therefore for $r > R$

$$G = cK_{i\nu}(\sqrt{8\alpha r}), \quad F = \frac{1}{\alpha}(rG' - G), \qquad (30)$$

where $\nu = 2\sqrt{\alpha^2 - 1}$ and $G(r)$ satisfies the equation

$$G'' + \frac{1}{r}G' + \left(\frac{\alpha^2 - 1}{r^2} - \frac{2\alpha}{r}\right)G = 0 \quad (r > R). \qquad (30')$$

For the calculation of the normalization constant c one can use (30) up to $r = 0$ since all integrals converge and the region $r < R$ makes a small contribution $\sim R \ll 1$ to the normalization integral. As a result

$$c = \left[\frac{12\alpha^3}{3 + 2\alpha^2}\frac{\operatorname{sh}\nu\pi}{\pi\nu}\right]^{1/2}. \qquad (30'')$$

Expressing F in (29) through G according to (30) we have

$$\int_R^\infty F^2(r)\frac{dr}{r} = \frac{1}{\alpha^2}\int_R^\infty \left(rG'^2 - 2GG' + \frac{G^2}{r}\right)dr. \tag{31}$$

Integrating the first term in (31) by parts and using (30′), we find after some calculation

$$\int_R^\infty (G^2+F^2)\frac{dr}{r} = 2\int_R^\infty \left(\frac{1}{r}-\frac{1}{\alpha}\right)G^2\,dr - \frac{1}{\alpha}[GF]_{r=R}. \tag{31′}$$

In the internal region $r < R$ we go over to the variable $x = r/R$ and take into account that G and F satisfy the equations (for $R \ll 1$)

$$G' = \frac{1}{x}G + \alpha f(x)F, \quad F' = -\alpha f(x)G - \frac{1}{x}F, \tag{32}$$

then, for $0 < x < 1$,

$$G'' - \frac{f'}{f}G' + \left[\alpha^2 f^2(x) + \frac{f'}{xf}\right]G = 0 \tag{32′}$$

[the dash in (32) and (33) denotes the derivative with respect to x). Therefore

$$\frac{1}{R}\int_0^R F^2(r)f\left(\frac{r}{R}\right)dr = \frac{1}{\alpha^2}\int_0^1 \frac{dx}{f(x)}\left(G'^2 - \frac{2}{x}GG' + \frac{1}{x^2}G^2\right). \tag{33}$$

With the help of (32′) we easily see that

$$\frac{1}{f}\left(G' - \frac{1}{x}G\right)^2 = \left[\frac{1}{f}\left(GG' - \frac{1}{x}G^2\right)\right]' + \alpha^2 fG^2. \tag{33′}$$

Since $f(1) = 1$ we find

$$\frac{1}{R}\int_0^R (G^2+F^2)f\left(\frac{r}{R}\right)dr = 2\int_0^1 G^2 f(x)\,dx + \frac{1}{\alpha}[GF]_{r=R}. \tag{33″}$$

Combining (31′) and (33″) we have[7)]

$$\frac{\beta}{2} = \frac{1}{R}\int_0^R G^2 f\left(\frac{r}{R}\right)dr + \int_R^\infty G^2\frac{dr}{r} - \frac{1}{\alpha}\int_R^\infty G^2\,dr. \tag{34}$$

Thus the function F is completely eliminated from the expression for the coefficient β. This has been possible because the Schrödinger equation simplifies significantly for $\epsilon = -1$. Using now expression (30) for $G(r)$, we have finally

$$a_{1/2} = \frac{\pi^2(3+2\alpha^2)}{24\alpha\Lambda(\alpha)}, \tag{35}$$

where everywhere $\alpha = \alpha_{cr}$ and

$$\Lambda(\alpha) = \frac{\operatorname{sh}\pi\nu}{\pi\nu}\left\{\int_z^\infty K_{i\nu}^2(x)\frac{dx}{x} + \rho K_{i\nu}^2(z)\right\} - \frac{1}{16\alpha^2}. \tag{36}$$

Here

$$\nu = 2\sqrt{\alpha^2-1}, \quad z = \sqrt{8\alpha R},$$

$$\rho = \frac{1}{2R}\int_0^R \left[\frac{G(r)}{G(R)}\right]^2 f\left(\frac{r}{R}\right)dr = \rho_{1/2}. \tag{36′}$$

Thus, for a rectangular cut-off $f(x) \equiv 1$

$$\rho = \frac{1}{4\sin^2\alpha}\left(1 - \frac{\sin 2\alpha}{2\alpha}\right). \tag{37}$$

For spin $s = 0$ the calculation is analogous to the one just presented (some details are given in Appendix B), and the final formula for the coefficient a has the form

$$a_0 = \pi^2\alpha / 12\Lambda(\alpha), \tag{38}$$

where $\Lambda(\alpha)$ agrees formally with (36), differing only in the value of the parameters $\nu = \sqrt{4\alpha^2 - 1}$ and ρ:

$$\rho = \rho_0 = \frac{1}{2R}\int_0^R \left[\frac{\chi_0(r)}{\chi_0(R)}\right]^2 f^2\left(\frac{r}{R}\right)dr \tag{38′}$$

[we note that for $f(x) \equiv 1$ the value of ρ_0 agrees with (37); in general $\rho_0 \neq \rho_{1/2}$].

For $R \ll 1$ that integral term dominates in (36) which represents the contribution from the outer region $r > R$ in (29). We note that

$$\int_z^\infty K_{i\nu}^2(x)\frac{dx}{x} = \begin{cases} \dfrac{\pi}{4z^2}e^{-2z}\left(1 - \dfrac{\nu^2 + 5/4}{z} + \ldots\right) & \text{as } z\to\infty, \quad (39) \\ \dfrac{\pi}{2\nu\operatorname{sh}\pi\nu}\ln\dfrac{1}{z} & \text{as } z \to 0. \quad (39′) \end{cases}$$

For $\alpha_{cr} = 1.25$ ($Z_{cr} = 170$) the ratio of the term $\rho K_{i\nu}(z)$ over the integral term in formula (36) amounts to about $1/6$. Therefore the coefficient a in (28) depends weakly on the specific form of the cut-off function.

In the limit $R \to 0$ we find, using the asymptotic form (39′) and $\nu_{cr} = 2\pi[\ln(1/R)]^{-1}$, that (35) and (38) agree with (19) and (22) of the previous section. In the region $R \sim 10^{-12}$ cm the asymptotic formulas (19) and (22) are inaccurate and a numerical calculation of the function $\Lambda(\alpha_{crit})$ is required. The results of such a calculation are shown in Fig. 3 for the simplest case of a rectangular cut-off $f(x) \equiv 1$.

If we extrapolate the dependence $R = r_0 A^{1/3}$ to the region $Z > 137$ assuming (as for heavy nuclei) $r_0 = 1.1$ F and $A = 2.5$ Z, then $Z_{cr} = 170$, where this value of Z_{cr} is weakly sensitive to the specific form of the nuclear charge distribution.[4] From Fig. 3 we find: $a_{1/2} = 3.75$, so that for $Z = Z_{cr} + 1$ the probability for the production of positrons $w \sim 10^{-10}\ mc^2/\hbar = 10^{11}\ \text{sec}^{-1}$. How strongly w depends on the excess of Z over Z_{cr} can be seen from the fact that the quantity w increases by three orders of magnitude as $Z = Z_{cr} + 1$ is increased to $Z = Z_{cr} + 2$.

4. PAIR PRODUCTION IN A SHORT-RANGE POTENTIAL

The case of a short-range potential differs significantly from the case just considered, since the Coulomb "tail" for $r \to \infty$ is not present in U(r). For simplicity we take the potential in the form of a square well, assuming a vector type interaction:

$$V(r) = -v\theta(r_0 - r) \tag{40}$$

(the depth v of the well is measured in the units mc^2). The motion of the levels in such a potential has been considered in[3]. For spin $s = 0$ and $l = 0$ the equation for the level energy ϵ coincides in form with the nonrelativistic equation:[3]

$$k\operatorname{ctg} kr_0 = -\lambda, \tag{41}$$

differing only in the values of the parameters k and λ:

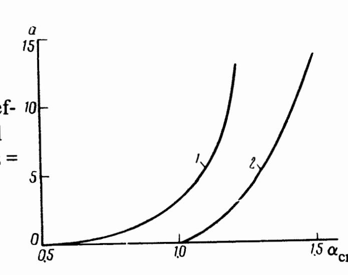

FIG. 3. Dependence of the coefficient a_s in (28) on α_{crit}. Curves 1 and 2 refer to the cases with spin s = 0 and s - 1/2.

7)In the last integral of (34) one can replace the lower limit by zero with an accuracy up to terms ~R.

$$k^2=(\varepsilon+v)^2-1, \quad \lambda=\sqrt{1-\varepsilon^2}. \tag{41'}$$

As the level appears, $\lambda = 0$. Denoting the corresponding depth of the well by w (for $\epsilon = \pm 1$) we have

$$v_n^{(\pm)}=\left\{1+\frac{\pi^2}{r_0^2}\left(n-\frac{1}{2}\right)^2\right\}^{1/2} \mp 1 \quad (n=1,2,\ldots). \tag{42}$$

The character of the dependence $\epsilon = \epsilon(v)$ is most easily studied in the limiting case of a narrow well $r_0 \ll 1$. Then we obtain from (41) for $n = 1$ (ground level)

$$v=\frac{\pi}{2r_0}+\frac{2}{\pi}\sqrt{1-\varepsilon^2}-\varepsilon. \tag{43}$$

From Fig. 4 we note the following somewhat unexpected result: the curve $\epsilon = \epsilon(v)$ folds back.[3] This can be interpreted in the following way: for $v = v_1^{(-)}$ a bound state for antiparticles appears in the well (lower branch of the curve). With further increase of v the levels for particles and antiparticles approach each other, touch for $v = v_{cr}$, and go off into the complex ϵ plane. The imaginary part of ϵ for $v > v_{cr}$ describes pair creation, where at threshold

$$w_0 \sim \sqrt{v-v_{cr}}. \tag{44}$$

It can be shown that in a narrow well of arbitrary form the function $\epsilon = \epsilon(v)$ for a scalar particle has qualitatively the same form as in Fig. 4 (cf. Appendix C). Thus, the results obtained do not depend on the sharp cut-off of $V(r)$ at $r = r_0$.

Let us now turn to the case of spin $s = 1/2$. The matching condition at $r = r_0$ for the Dirac equation is the continuity of the ratio F/G, which gives (for $\kappa = -1$, i.e., for $j = 1/2$)

$$k \operatorname{ctg} kr_0 = -\left[\lambda + v\frac{1+\lambda r_0}{(1+\varepsilon)r_0}\right], \tag{45}$$

where k and λ have the same values as above. The values of $v = v_n^{(+)}$ for which a level with $\epsilon = 1$ appears in the well are found from the equation

$$kr_0 \operatorname{ctg} kr_0 = -v/2 \quad (k=\sqrt{v(v+2)}), \tag{46}$$

and $v_n^{(-)}$ corresponding to the energy $\epsilon = -1$ is obtained from (45) in explicit form:

$$v_n^{(-)}=1+\sqrt{1+n^2(\pi/r_0)^2} \quad (n=1,2,\ldots). \tag{47}$$

For a narrow well $r_0 \ll 1$ we have approximately from (45)

$$v=\pi/r_0-(1+2\varepsilon) \quad (n=1), \tag{48}$$

i.e., the curve $\epsilon = \epsilon(v)$ falls off monotonically without a turning point. This difference between the spins $s = 0$ and $s = 1/2$ can be interpreted in the following way (cf. also Appendix C). The wave function for a scalar particle becomes, as in the nonrelativistic case, nonnormalizable for $\lambda = 0$: $\chi_0(r) \to c$ for $r \to \infty$ ($c \neq 0$). Calculating the level energy for a slightly deeper well, we find in first order of perturbation theory

$$\delta\varepsilon=\int_0^\infty \delta V(r)\chi_0^2(r)\,dr \Big/ \int_0^\infty \chi_0^2(r)\,dr = 0, \tag{49}$$

since the integral in the numerator converges [owing to $\delta V(r)$], while the denominator diverges. Hence $\delta\epsilon \sim (\delta V)^2$, so that curve 1 of Fig. 4 has a horizontal tangent at $\epsilon = \pm 1$. Furthermore, it follows from

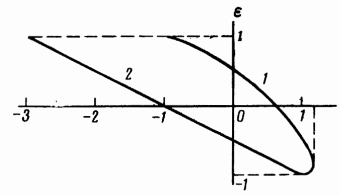

FIG. 4. Energy of the ground state ϵ as a function of the well depth for spin $s = 0$ (curve 1) and $s = 1/2$ (curve 2) for a narrow well. The abscissa is marked by the quantity $(v - \pi/2r_0)$ for $s = 0$ and by $(v - \pi/r_0)$ for $s = 1/2$.

(16) and (40) that the pairs of points (ϵ, v) and $(-\epsilon, v + 2\epsilon)$ correspond to the same effective potential $U(r)$ and to the same wave function. If the values (ϵ, v) satisfy (41), than $(-\epsilon, v + 2\epsilon)$ do as well; this leads immediately to the form of Fig. 4 for the qualitative behavior of curve $\epsilon = \epsilon(v)$ for spin $s = 0$.

On the other hand, for spin $s = 1/2$ the state with $\epsilon = -1$ is normalizable in an arbitrary short-range potential.[8)] Indeed, for $\kappa = -1$ and $V(r) \to 0$

$$F=\frac{1}{1+\varepsilon}\left(G'-\frac{1}{r}G\right). \tag{50}$$

For $\epsilon = -1$ we have $G(r) \to c$ for $r \to \infty$, and $F(r)$ would tend to infinity if $c \neq 0$. Therefore

$$|G(r)| < C_1 r^{-2}, \quad F(r) = C_2 r^{-1} \quad (r \to \infty). \tag{51}$$

which guarantees the convergence of the normalization integral (29′).

The normalizability of the states with $\epsilon = -1$ in a shortrange potential can also be explained in another way. Outside the range of the potential ($r \gg r_0$)

$$G = Ck_l(\lambda r), \quad F=-C\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\,k_{l'}(\lambda r), \tag{52}$$

where l and l' are the orbital angular momenta for the upper and lower components of the bispinor, and

$$k_l(z)=\sqrt{\frac{2z}{\pi}}K_{l+1/2}(z)=e^{-z}\sum_{n=0}^{l}\frac{(l+n)!}{(l-n)!\,n!\,(2z)^n}. \tag{52'}$$

Since the wave function remains finite for $\epsilon = -1$, it follows from this that the constant C in (52) must tend to zero in a definite way for $\epsilon \to -1$:

$$C \sim \lambda^L, \quad L = max(l, l'+1). \tag{53}$$

Therefore we have for the states with $\kappa < 0$ ($j = + 1/2$, $l' = l + 1$)

$$G(r)=0, \quad F(r)=cr^{-l'} \quad (r \gg r_0), \tag{54}$$

i.e., for $\epsilon = -1$ only the lower component of the bispinor corresponding to the largest orbital angular momentum l' survives in the asymptotic expression for $r \to \infty$. Since $l' \geq 1$ the normalization integral (29′) remains convergent for $\epsilon = -1$.

The situation is somewhat different for the states with $\kappa > 0$ ($j = l - 1/2$, $l' = l - 1$), for which we obtain instead of (54) ($\epsilon = -1$, $r \gg r_0$)

$$G(r)=cr^{-l}, \quad F(r)=-2cr^{-l'}. \tag{55}$$

Here both angular momenta l and l' are present in the asymptotic expression; therefore the state of the type $P_{1/2}$ is not normalized for $\epsilon = -1$. The other states ($j > 3/2$) remain normalized.

8) In the nonrelativistic case $\chi_l(r) \sim r^{-l}$ for $r \to \infty$ as soon as a level with orbital angular momentum l appears, i.e., $\chi_l(r)$ is normalized for $l \geq 1$. Correspondingly, for the Dirac equation the states with $j > 1/2$ are normalized for $\epsilon = -1$ as well as for $\epsilon = +1$.

For the lowest level $1S_{1/2}$ we have $\kappa = -1$ and (54) agrees with (51). Owing to the normalizability of the wave function for this state the level shift $\Delta\epsilon$ differs from zero already in first order perturbation theory [cf. (49)], i.e., $\delta\epsilon \sim \delta V$ and $k = \sqrt{\epsilon^2 - 1} = \sqrt{2|\delta V|}$. The behavior of $U_{eff}(r)$ for $r \gg r_0$ is determined by the centrifugal barrier, whose penetrability increases with the momentum k according to a power law: $D \sim k^{2l'+1}$ for $k \to 0$, and therefore $w \sim (V - V_{cr})^{l'+1/2}$ (these formulas refer to states with $\kappa < 0$).

In the concrete case of a square well the behavior of the probabilities for pair production w can easily be found directly from (45). According to what has been said, the critical value of the potential is obtained from (47) for $n = 1$:

$$v_{cr} = 1 + \sqrt{1 + (\pi / r_0)^2}. \tag{56}$$

For $v > v_{cr}$ the root of Eq. (45) becomes complex: $\epsilon \to \epsilon - i\gamma/2$. For the imaginary part we find[9)]

$$w_{1/2} = 2\gamma = A\,(v - v_{cr})^{3/2}, \tag{57}$$

where $(q = r_0/\pi)$

$$A = \frac{q}{2}\left(1 - \frac{q}{\sqrt{1+q^2}}\right)\left(1 - \frac{q}{2\sqrt{1+q^2}}\right)^{-3/2} = \begin{cases} r_0/2\pi & \text{for } r_0 \ll 1, \\ \pi/r_0\sqrt{2} & \text{for } r_0 \gg 1. \end{cases} \tag{58}$$

It is seen from this that the probability w does not vanish at threshold according to an exponential law in the case of short-range potential.

5. DISCUSSION OF RESULTS

1. As the charge Z goes through the critical value $Z_{cr}(R)$, spontaneous creation of positrons in the nuclear Coulomb field sets in. This process has the following characteristics. In the Coulomb field of a nucleus Z there is an unoccupied electron level for an electron lying in the lower continuum $\epsilon < -mc^2$. An electron of the Dirac sea makes a transition to this level, remaining localized near the nucleus (within distances of $r < \hbar/mc$), while the positron penetrates through the Coulomb barrier and leaves toward infinity as a real particle.[10)] Actually, two positrons appear owing to the twofold degeneracy of the $1S_{1/2}$ level with respect to spin, and the charge of the nucleus Z is lowered by two units for an outside observer.[1] Thereafter the level $1S_{1/2}$ remains completely filled, and the other levels lie still above $\epsilon = -mc^2$, so that by the Pauli principle, further production of positrons is forbidden [although perhaps $Z - 2 > Z_{cr}(R)$]. The entire process (production of two positrons and lowering of the nuclear charge) takes a time $\tau \lesssim 10^{-10}$ sec.

2. With further increase of the charge of the nucleus positron production starts again for such values of Z where the boundary $\epsilon = -mc^2$ is intersected by higher-lying levels. For the closest levels $2P_{1/2}$ and $2S_{1/2}$ this occurs at $\Delta Z = Z - Z_{cr}^{(0)} = 15$ and 50, respectively (cf. the table). Here $Z_{cr}^{(0)} \approx 170$ is the critical value of Z for the lowest level $1S_{1/2}$.

R, F	$1S_{1/2}$ I	$1S_{1/2}$ II	$2P_{1/2}$	$2S_{1/2}$
8	1.248	1.224	1.35	1.72
10	1.271	1.243	1.38	1.78
12	1.291	1.260	1.41	1.83

In the table we give the values of $\alpha_{cr} = Z_{cr}/137$ for the first three levels. The columns I and II refer to two cut-off models in (3): I) $f(x) \equiv 1$ and II) $f(x) = (3 - x^2)/2$, i.e., uniform charge distribution over the nuclear volume. The values of α_{cr} for the states $2P_{1/2}$ and $2S_{1/2}$ are calculated on the basis of model I.

3. The characteristic feature of the problem under consideration is the existence of a Coulomb barrier for the electron whose energy is close to the boundary of the lower continuum. The wave function $\chi_\epsilon(r)$ for energy ϵ has an asymptotic form of the type (5) for $r \to \infty$. The Coulomb interaction at large distances determines the coefficient $r^{\alpha\epsilon/\lambda}$ of the exponential in (5), which depends on the sign of ϵ. For $\epsilon \to +1$ the maximum of $\chi_\epsilon(r)$ goes off to large distances and the system becomes more and more dilute—delocalization sets in. This behavior of the states at the boundary of the continuous spectrum is well known from nonrelativistic quantum mechanics.[11)] For $\epsilon \to -1$ a completely different picture obtains: the coefficient $r^{\alpha\epsilon/\lambda}$ decreases more rapidly than any finite power of r and the electron remains localized near the nucleus. Accordingly, the behavior of the wave functions for $\epsilon = \pm 1$ is different:

$$\chi_\varepsilon(r) \underset{r\to\infty}{\propto} \begin{cases} \sin(\sqrt{8\alpha r} + \delta) & \text{for } \varepsilon = +1 \\ \exp(-\sqrt{8\alpha r}) & \text{for } \varepsilon = -1 \end{cases} \tag{59}$$

[the exponential decrease of $\chi_\epsilon(r)$ in the second case is nothing but the damping due to the Coulomb barrier].

By continuity, it is clear that the functions $\chi_\epsilon(r)$ and $\chi_{-\epsilon}(r)$ must differ strongly also in the continuous spectrum ($|\epsilon| > 1$), especially for $|\epsilon| \to 1$. In the quasiclassical approximation

$$\chi_\varepsilon(r) \propto \exp\left(i\int^r p\,dr\right), \quad p(r) = \sqrt{\varepsilon^2 - 1 + \frac{2\alpha\varepsilon}{r}}. \tag{60}$$

For $\epsilon > 1$ the quantity $p^2(r) > 0$ and $\chi_\epsilon(r)$ oscillates for all. If $\epsilon < -1$, we have a turning point $r_2 = 2\alpha|\epsilon|/k^2$, which is located far from the nucleus for small $k = \sqrt{\epsilon^2 - 1}$. In the classically forbidden region $r < r_2$ the wave function $\chi_\epsilon(r)$ is close to (18) and the oscillations described by the asymptotic form (5′) set in only for $r > r_2$ (these oscillations describe the positron wave going out to infinity). The probability for pair production $w \sim \exp(-2\,\mathrm{Im}\,S)$, where S is the classical action calculated along the trajectory beneath the barrier. The action S is independent of the spin, so that the probabilities w behave similarly for $s = 0$ and $s = 1/2$. For $\epsilon \to -1$ the exit point r_2 moves out to infinity and the probability w vanishes exponentially at threshold. Therefore the rearrangement

9)The power 3/2 in (57) corresponds to the fact that the orbital angular momentum $l' = 1$ remains at $r \gg r_0$ for the ground state.

10)In the language of second quantization this means creation of a pair at a distance $r < \hbar/mc$ from the nucleus, whereafter the electron of the pair settles in the K shell while the positron is repelled by the nucleus and escapes to infinity.

11)For example, for a hydrogen atom in a state with principal quantum number n we have $\chi_{nl}(r) \sim e^{-r/n} r^n$ for $r \to \infty$.

of the vacuum for $Z > Z_{cr}$ sets in continuously, not suddenly.

4. After the emission of the two positrons an atom with charge $Z - 2$ and a filled K shell remains. When an external positron is scattered on this atom, a narrow resonance will be observed:[12] the positron penetrates through the Coulomb barrier and annihilates with a K electron, as a result of which a nucleus with an unoccupied $1S_{1/2}$ level in the lower continuum remains; then a pair is created spontaneously. This process is analogous to the excitation of an atom by a photon with subsequent spontaneous emission of a photon of the same frequency (within the limits of the line width). The cross section for the scattering of the positrons must be described by a Breit-Wigner formula with a width equal to the probability for pair production w.

5. For potentials with a finite range there also exists a critical value of the well depth $V = V_{cr}$; when it is passed, spontaneous pair production sets in. However, in this case there is no Coulomb barrier for the positron, so that the probability w increases at the threshold $V \to V_{cr}$ much more rapidly than in the preceding case. Moreover, there is an appreciable spin dependence in the threshold behavior of w as well as in some qualitative features of the dependence $\epsilon = \epsilon(V)$. This is explained by the fact that here all processes take place at small distances (assuming $r_0 \lesssim \hbar/mc$, where r_0 is the radius of the potential well). Notwithstanding the absence of a Coulomb barrier, the states of the discrete spectrum remain normalized for $\epsilon \to -1$ (owing to the centrifugal barrier).

6. The problem considered is, at first glance, a problem of strong coupling, since the perturbation parameter $\alpha = Ze^2/\hbar c > 1$. Why is it possible to find an exact (in the known sense) solution for it? The explanation for this is that the interaction of the electron with a strong external field can be taken into account completely if the exact Coulomb wave functions are used, and the interaction of the electrons (real or virtual) between themselves can be neglected, since the corresponding coupling constant is $\alpha_0 = e^2/\hbar c = 1/137$.

7. At present it is not known if nuclei with $Z > 137$ exist in nature, so that the effects considered by us must so far be brought into connection with thought experiments.[13] However, the required situation can also be realized in another (apparently more realistic) way:[1] in collisions of two bare nuclei with charges Z_1 and Z_2 such that $Z_1, Z_2 < Z_{cr}(R)$, but $Z_1 + Z_2 > Z_{cr}(R)$. At the instant where these nuclei approach each other down to the distance R, the electric field for the electron is similar to the field of a nucleus with charge $Z_1 + Z_2$, and spontaneous pair production should also set in in this case. For $Z_1, Z_2 < 137$ each of the nuclei can be regarded as point-like, so that this problem reduces to the relativistic two-center problem. Unfortunately, the variables of this problem are no longer separable in elliptic coordinates $\xi = (r_1 + r_2)/R$, $\eta = (r_1 - r_2)/R$, φ (in contrast to the nonrelativistic case). This is due to the term V^2 in the effective potential (16). The non-separability of the variables leads to additional difficulties of numerical nature, but new qualitative features of the two-center problem do not, to all appearances, arise.

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APPENDIX A

In this Appendix we show how the limit $\epsilon \to -1$ in the wave function (4) is taken, when the argument of the Whittaker function $x = 2\lambda r \to 0$, its index $p = \epsilon\alpha/\lambda \to -\infty$, but their product remains finite. We start from the integral representation

$$W_{-k,\mu}(x) = \frac{e^{-x/2}x^{\mu+1/2}}{\Gamma(k+\mu+1/2)} \int_0^\infty e^{-xt} t^{\mu+k-1/2} (1+t)^{\mu-k-1/2}\, dt. \quad \text{(A.1)}$$

The integral in this formula is written as

$$I = \int_0^\infty e^{-\varphi(t)} [t(1+t)]^{\mu-1/2}\, dt, \quad \text{(A.2)}$$

where $\varphi(t) = xt + k \ln(1 + t^{-1})$. For $x \to 0$ and $k = -p \to \infty$ the function $\varphi(t)$ has a minimum for very large t of order $\sqrt{k/x}$, so that I is conveniently written in the form

$$I = \int_0^\infty e^{-(xt+kt^{-1})} t^{2\mu-1} \psi(t)\, dt, \quad \text{(A.3)}$$

where

$$\psi(t) = (1+t^{-1})^{\mu-1/2} \exp\{-k[\ln(1+t^{-1}) - t^{-1}]\}$$

$$= 1 + \frac{\mu - 1/2}{t} + \frac{k}{2t^2} + \ldots \quad (\text{as } t \to \infty). \quad \text{(A.4)}$$

Let us now use

$$\int_0^\infty e^{-(xt+kt^{-1})} t^{2\mu-1}\, dt = 2\left(\frac{k}{x}\right)^\mu K_{2\mu}(2\sqrt{kx}), \quad \text{(A.5)}$$

where $K_{2\mu}(z)$ is the MacDonald function. Writing $z = 2\sqrt{kx} = \sqrt{8\alpha r}$, we find

$$W_{-k,\mu}(x) = Ce^{-x/2} z \left\{ K_{2\mu}(z) + \left[\frac{1}{2} K_{2\mu-2}(z) + \frac{2\mu-1}{z} K_{2\mu-1}(z) \right] x + \ldots \right\}, \quad \text{(A.6)}$$

where $C = k^{\mu-1/2}/\Gamma(k + \mu + 1/2)_{(k \to \infty)}(k!)^{-1}$. The coefficient of the first power of x in the expansion (A.6) is equal to

$$\frac{1}{2}[K_{2\mu-2}(z) - K_{2\mu}(z)] + \frac{2\mu-1}{z} K_{2\mu-1}(z) \equiv 0 \quad \text{(A.7)}$$

owing to the recurrence relations between the K functions. As a result

$$\lim_{x\to 0,\ k\to\infty} W_{-k,\mu}(x) = CzK_{2\mu}(z)\{1 + O(x^2)\}. \quad \text{(A.8)}$$

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[12] This remark is due to Ya. B. Zel'dovich.

[13] On the other hand, the latest successes in the synthesis of superheavy nuclei allow one to hope to be able to reach the region of values of Z close to 137 (cf. [21,22]). Many theoretical calculations predict the existence of "islands" of unusally stable nuclei, in particular, near the magic numbers Z = 114 and Z = 126. [23]

Equation (6) now takes the form

$$z\frac{K_{i\nu}'(z)}{K_{i\nu}(z)} = 2\gamma - 1, \qquad \text{(A.9)}$$

where $\nu = 2g = \sqrt{4\alpha^2 - 1}$, $z = \sqrt{8\alpha}\,R$. The limit for spin $s = 1/2$ is taken in analogous fashion. The function $K_{i\nu}(z)$ entering in (A.9) is much simpler than the Whittaker function; it is defined by the integral

$$K_{i\nu}(z) = \int_0^\infty e^{-z\,\mathrm{ch}\,t} \cos \nu t \, dt, \qquad \text{(A.10)}$$

which converges rapidly and is convenient for numerical calculations. From this it is also seen that $K_{i\nu}(z)$ is real for real values of ν and z.

APPENDIX B

For spin $s = 0$ we start from the Klein-Gordon equation

$$\Delta\varphi + [(\varepsilon - V)^2 - 1]\varphi = 0. \qquad \text{(B.1)}$$

Setting here $\epsilon = -1 + \Delta\epsilon$, $V = V_{cr} + \delta V$, we find a perturbation formula for the level shift:

$$\Delta\varepsilon = \frac{\langle\varphi_0|(1 + V_{\text{кр}})\delta V|\varphi_0\rangle}{\langle\varphi_0|(1 + V_{\text{кр}})|\varphi_0\rangle}, \qquad \text{(B.2)}$$

where $\varphi_0(r)$ is the solution for $\epsilon = -1$, of the form $(r > R)$

$$\varphi_0(r) = \frac{\chi_0(r)}{r} = c_0 r^{-1/2} K_{i\nu}(\sqrt{8\alpha r}). \qquad \text{(B.3)}$$

For the normalization of the function $\chi_0(r)$ we extrapolate this expression down to $r = 0$; then

$$c_0 = \sqrt{24\frac{\mathrm{sh}\,\pi\nu}{\pi\nu}} \quad (\nu = \sqrt{4\alpha^2 - 1}). \qquad \text{(B.4)}$$

Then, for $\sigma > -2$,

$$\langle r^\sigma\rangle = \int_0^\infty \chi_0^2(r) r^\sigma dr = (8\alpha)^{-\sigma} \frac{\Gamma(\sigma+2)\Gamma(5/2)}{\Gamma(\sigma+5/2)} \left| \frac{\Gamma(\sigma+2+i\nu)}{\Gamma(2+i\nu)} \right|^2. \qquad \text{(B.5)}$$

In particular,

$$\langle r\rangle = \frac{4\alpha^2+3}{10\alpha}; \quad \langle r^{-1}\rangle = \frac{3}{\alpha}, \qquad \text{(B.6)}$$

so that $\langle \varphi_0 | 1 + V_{cr} | \varphi_0 \rangle = -2$ (we note that here α is everywhere understood to stand for $\alpha_{cr}(R)$, since we are considering the case $\epsilon = -1$).

The numerator of (B.2) contains the integral $\langle r^{-2}\rangle$, in which, in contrast to before, it is no longer possible to set the cut-off radius R equal to zero, since this would lead to a divergence (of logarithmic order). As a result we obtain

$$\beta = -\left(\frac{\partial\varepsilon}{\partial\alpha}\right)_{\varepsilon=-1} = \frac{\alpha}{2}\left\{\frac{1}{R^2}\int_0^R \chi_0^2(r) f^2\left(\frac{r}{R}\right) dr + \int_R^\infty \chi_0^2(r)\frac{dr}{r^2} - \frac{3}{\alpha^2}\right\}. \qquad \text{(B.7)}$$

Using (B.3) and (B.4), we have

$$\beta = 24\alpha\left\{\frac{\mathrm{sh}\,\pi\nu}{\pi\nu}\left[\int_z K_{i\nu}^2(x)\frac{dx}{x} + \rho K_{i\nu}^2(z)\right] - \frac{1}{16\alpha^2}\right\} \qquad \text{(B.8)}$$

where $z = \sqrt{8\alpha_{cr}}R$, and ρ is given by (38′). Substituting this in (28), we arrive at the desired expression (38) for the coefficient a_0.

APPENDIX C

The dependence of the energy of the ground state on v, $\epsilon = \epsilon(v)$, has been obtained for the special case of a rectangular well (cf. Sec. 4). Let us now show that also for an arbitrary short-range potential the curve $\epsilon = \epsilon(v)$ has qualitatively the same form. Consider a narrow well of arbitrary form:

$$V(r) = -vf(r/R), \quad R \ll 1, \qquad \text{(C.1)}$$

where $f(x)$ tends rapidly to zero for $x \gg 1$. For spin $s = 0$, introducing the variable $x = r/R$ in the Klein-Gordon equation, and expanding all quantities in R, we have

$$\chi(r) = \chi_0 + R\chi_1 + \ldots, \quad \alpha \equiv vR = \alpha_0 + R\alpha_1 + \ldots, \qquad \text{(C.2)}$$

where (for $l = 0$)

$$\chi_0'' + \alpha_0^2 f^2(x)\chi_0 = 0, \qquad \text{(C.3)}$$

$$\chi_1'' + \alpha_0^2 f^2\chi_1 = -2\alpha_0(\varepsilon + \alpha_1 f) f\chi_0. \qquad \text{(C.4)}$$

The boundary conditions for $\chi_0(x)$ are

$$\chi_0(0) = 0, \quad \chi_0(\infty) = 1. \qquad \text{(C.5)}$$

We note that α_0 is of order unity; it can be found from (C.3) and (C.5). Multiplying both sides of (C.4) by χ_0 and using (C.3), we find

$$[\chi_1'\chi_0 - \chi_1\chi_0']_0^\infty + 2\alpha_0\int_0^\infty (\varepsilon + \alpha_1 f) f\chi_0^2\, dx = 0. \qquad \text{(C.6)}$$

For $r \gg R$ the quantity $\chi(r) \sim e^{-\lambda r}$, so that there exists a region $R \ll r \ll 1$ in the case of $R \ll 1$ in which this exponential can be expanded in a series in terms of R; thus the boundary condition for χ_1 can be found:

$$\chi_1(0) = 0, \quad \chi_1(x) \approx -\lambda x \text{ for } x \gg 1 \quad (\lambda = \sqrt{1-\varepsilon^2}). \qquad \text{(C.7)}$$

Using (C.5) and (C.7) we find from (C.6) an equation for the energy of the level:

$$v = \frac{\alpha}{R} = \frac{\alpha_0}{R} + a_1\sqrt{1-\varepsilon^2} - a_2\varepsilon, \qquad \text{(C.8)}$$

where

$$a_1 = \left\{2\alpha_0\int_0^\infty f^2(x)\chi_0^2(x)dx\right\}^{-1}, \quad a_2 = \int_0^\infty f\chi_0^2\, dx \Big/ \int_0^\infty f^2\chi_0^2\, dx. \qquad \text{(C.9)}$$

Thus for $R \ll 1$ the concrete form of the potential affects only the values of the parameters α_0, a_1, and a_2. For an attractive potential $f(x) > 0$ and a_1, a_2 are positive. It follows from this that for $s = 0$ the curve $\epsilon = \epsilon(v)$ always has a turn-over point of the form shown in Fig. 4. The critical value of the potential corresponds to the vertex of this curve and is equal to

$$v_{cr} = \frac{\alpha_0}{R} + \sqrt{a_1^2 + a_2^2}. \qquad \text{(C.10)}$$

An analogous procedure applied to the Dirac equation leads, instead of (C.8), to

$$v = \frac{\beta_0}{R} + b_1 - b_2\varepsilon \qquad \text{(C.11)}$$

(for the states with $\kappa = -1$, $j = 1/2$). Here

$$b_1 = \int_0^\infty (G_0^2 - F_0^2)dx \Big/ \int_0^\infty f(x)(G_0^2 + F_0^2)dx,$$

$$b_2 = \left\{\int_0^\infty f(x)(G_0^2 + F_0^2)dx\right\}^{-1}, \qquad \text{(C.12)}$$

and $G_0(x)$, $F_0(x)$, and β_0 are solutions of the equations of zeroth order in R:

$$G' = \frac{1}{x}G + \beta f(x)F, \quad F' = -\beta f(x)G - \frac{1}{x}F \qquad \text{(C.13)}$$

with the boundary conditions

$$G(0) = F(0) = 0,$$
$$G(x) \approx 0, \quad F(x) \approx c/x \quad \text{for} \quad x \gg 1, \tag{C.14}$$

normalized according to (29). Here the dependence $\epsilon = \epsilon(v)$ is linear and there is no turn-over; the critical value of the potential corresponds to $\epsilon = -1$ and is equal to

$$v_{cr} = \frac{\beta_0}{R} + (b_1 + b_2). \tag{C.15}$$

For a square well $f(x) = \theta(1 - x)$ and we easily find

$$\beta_0 = 2a_0 = \pi, \quad a_1 = 2/\pi, \quad a_2 = -b_1 = 1, \quad b_2 = 2.$$

Here (C.8) and (C.11) go over into (43) and (48), respectively. Thus the picture obtained for the special example of the potential (40) is quite general.

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