CORRELATION FUNCTIONS IN THE THEORY OF PHASE TRANSITIONS: VIOLATION OF THE SCALING LAWS

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Static correlations at the critical point are investigated quantitatively by field theory methods. The correlation functions in the three-dimensional Ising, Heisenberg, and Bosegas models are found in the form of expansions in powers of a small parameter related to the critical index η. In the subcritical region of distances $r \gg r_0$, $\ln(r/r_0) \sim 1$, the scaling laws are found to be valid only within an accuracy of $\sim\eta$. In the critical region, $\ln(r/r_0) \gg 1$, there are two possible asymptotic behaviors, depending on the magnitude of η. For $\eta < \eta_1 \sim 10^{-3}$ and $\ln r \to \infty$, the corrections to the Ornstein-Zernike theory tend to zero and, in particular, $\eta_{\text{eff}} \to C \times 10^{-3} \ln^{-2} r$, where $C = 0.50$ (Ising), 0.64 (Bose) or 0.76 (Heisenberg). For $\eta > \eta_1$ and $\ln r \to \infty$, the effective indices tend to universal constants $\eta_2\ldots$, which depend only on the symmetry of the system; deviations from the scaling laws tend to zero as a small power of r_0/r.

1. INTRODUCTION

THE hypothesis of scale invariance in the critical region of temperature and momenta $(T \to T_c, k \to 0)$[1] has made it possible to describe phenomenologically a broad class of critical phenomena by means of two indices giving the behavior of the order parameter and of the correlation length. In the two-dimensional Ising model this hypothesis agrees with the exact solution; the critical region turns out to be broad and the asymptotic behavior is attained at momenta of the order of several reciprocal lattice constants: $k \lesssim r_0^{-1}$. In the two-dimensional model the indices and correlation functions differ markedly from those predicted by self-consistent field theory (Ornstein-Zernike correlation).

In a three-dimensional system, however, the situation is evidently different. Calculations using extrapolation of high-temperature expansions[2] show that the correlation of the spins at the critical point (the Green function $G(k)$) differs little from the Ornstein-Zernike function in its momentum-dependence

$$G(k)=\frac{z(\ln kr_0)}{k^2}, \quad T=T_c. \tag{1}$$

Here $z(\ln kr_0)$ is a slowly-varying function of the momentum k. The effective index, according to[2], is of order 10^{-2}

$$\eta_{\text{eff}}(\ln kr_0) = z'(\ln kr_0)/z(\ln kr_0) \sim 10^{-2}. \tag{2}$$

The thought naturally arises—is it not possible to use the small parameter η to construct a quantitative theory?

First of all one should understand which properties of the correlations are associated with a small value of the index η. For this it is convenient to go over from the correlation functions $Q_n = -\delta^n F/\delta h_1 \ldots \delta h_n$ (F is the free energy, and $h_i \equiv h(k_i)$ is a magnetic field in the language of the Ising model) to the vertex parts:

$$Q_n(k_1,\ldots,k_n) = G(k_1)\ldots G(k_n)\Gamma_n(k_i)\delta(\Sigma k_i),$$
$$Q_2(k_1,k_2) = G(k_1)\delta(k_1+k_2). \tag{3}$$

From the standpoint of phenomenological theory, the vertices $-\Gamma_n(k_i)$ are the Landau coefficients of the expansion of the free energy $\varphi_k = G(k)h_k$ in powers of the order parameter (or, in the Ising language, of the magnetic moment). The usual scaling forms in coordinate space for the correlation functions[1]

$$Q_n(r_1,\ldots,r_n) = r^{-n(d-2+\eta)/2}q_n(r_i/r) \tag{4}$$

correspond in momentum space to the following forms for the vertex parts:

$$\Gamma_n(k_i) = (z(k))^{-n/2}k^{d+n(1-d/2)}\gamma_n(k_i/k), \quad k_i \sim k \to 0; \tag{5}$$

$$z(k) = k^2G(k) \sim k^\eta. \tag{6}$$

In a three-dimensional system $(d = 3)$, the estimate (5) for $n = 4$ shows that the pair interaction is small, $\Gamma_4 \sim z^{-2}k \to 0$, i.e., the bare pair interaction V_0 is screened at small momenta $k \ll r_0^{-1}$ (r_0 is the interaction radius of the pair forces):

$$V_{\text{eff}} = \Gamma_4(k_i=0) = V_0 + (\text{contribution from diagrams}) = 0. \tag{7}$$

In the three-body interaction Γ_6 (in the Ising model language, the correlation of six spins), no screening occurs and Γ_6 depends slowly on the momenta:

$$\Gamma_6 \sim z^{-3}(k) - (\text{slow function of } k) \tag{8}$$

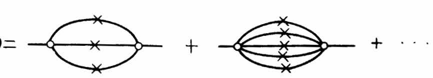
One must remember that, apart from the "main part" $\sim z^3 \sim k^{-3\eta}$, there is in Γ_6 a bare constant $\Gamma_6^{(0)} = g_0$, which will be unimportant only at exponentially small momenta $k \sim g_0^{1/3\eta}$. This means that the usual scaling laws can be true only in an exponentially narrow region and that in the subcritical region attainable in practice they are somehow modified and cease to be universal (g_0 is different for different systems). The fact that three-body forces play a distinct role for $\eta \ll 1$ was noted by Patashinskii[3].

As regards the remaining bare constants, they begin to be unimportant in the subcritical region $k \ll r_0^{-1}$, $\ln kr_0 \sim 1$, inasmuch as the vertex parts Γ_n for $n > 6$ are large: $\Gamma_{n>6} \sim k^{3-n/2} \gg \text{const}$.

The above properties of systems with small η make the problem more difficult rather than easier, but there is one more property which does lead to appreciable simplification, namely, that the smallness of the quantity η implies that the effective interaction is small:

$$z^{n/2}k^{n/2-3}\Gamma_n(k_i) = \gamma_n(k_i/k) \ll 1 \text{ for } \eta \ll 1. \quad (9)$$

This can be understood if we make use of the relativistic analogy of the papers of Polyakov[4] and the author[5] and write the "unitarity condition" for $\operatorname{Im} G(k^2 < 0)$:

$2\operatorname{Im} G(k^2<0) =$ 

$$= |G^2(k)| \sum_n \int |\Gamma^2_{n+1}(k, k_i)| z^n d\tau_n(k, k_i). \quad (10)$$

Here $d\tau_n(k, k_i) = k^{n-3} d\tau_n(1, k_i/k)$ is an element of the phase space of two-dimensional relativistic particles with zero mass. The details can be found in[5]; for our purposes, only the positiveness of each term in (10) is important.

The unitarity condition leads to a "sum rule" for the index η, since the imaginary part $\operatorname{Im} G(k^2)$ $= k^{-2}\operatorname{Im} z(\ln\sqrt{k^2})$ is proportional to the effective index: $z^{-1}\operatorname{Im} z(\ln\sqrt{|k_2|} + i\pi/2) = \frac{1}{2}\pi z'/z$. All powers of z and k in this sum rule are obviously cancelled if the scaling laws (1) and (5) are taken into account, and dimensionless vertices $\gamma_n(k_i/k)$ appear in the sum rule:

$$\pi z'/z = \pi\eta_{\text{eff}} = \sum_n \int |\gamma^2_{n+1}(k_i/k)| d\tau_n(1, k_i/k). \quad (11)$$

It is now clear that the smallness of the quantity η_{eff} means that each vertex γ_n is small, i.e., the effective interaction in the subcritical region is weak.

We emphasize that the smallness of η_{eff} has no connection with the magnitude of the bare pair constant V_0 or the other dimensional constants $\Gamma_n^{(0)}$. Only the dimensionless constant g_0 (three-body forces) must be small. Then all the vertices Γ_n will be given by diagrams containing g_0 and will, as we shall see, have the form (5) with small γ_n.

2. THREE-BODY FORCES IN THE SUBCRITICAL REGION

We thus arrive at the following model describing the correlations in the subcritical region of momenta ($kr_0 \ll 1$, $-\ln kr_0 \gtrsim 1$):

$$Z[h] = \int \delta\varphi(r) \exp\left[-\frac{1}{T}\int d^3r\left[\frac{1}{2}\varphi\nabla^2\varphi + \frac{1}{6!}g_0\varphi^6 - \varphi h\right]\right]. \quad (12)$$

Here Z is the partition function (a continuous integral), $\varphi(r)$ is the order parameter (magnetic moment in Ising model language), $h(r)$ is the source (magnetic field), T is the temperature, and $g_0 \ll 1$ is the coefficient of the three-body forces. The correlation functions Q_n are defined by variation with respect to the source $h(r)$:

$$Q_n(r_1, \ldots, r_n) = \delta^n \ln Z / \delta h(r_1) \ldots \delta h(r_n).$$

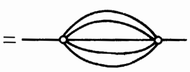
If one wishes, one can write (12) as the mean value $Z = \langle 0 | S | 0 \rangle$ of the S-matrix of a relativistic two-dimensional field $\hat{\varphi}$. For this one must go over to the pseudo-Euclidean metric $r = (it, \rho)$. The Laplace operator ∇^2 goes over to the d'Alembert operator $\Box = \nabla^2_\rho - \partial^2/\partial t^2$, and the factor i in the integral $\int d^3r = i\int dt d^2\rho$ ensures the unitarity of the S-matrix. In place of the continuous integral (12) one can write a T-product; a diagram technique for the correlation functions $Q_n(r_i) = \langle 0 | T\hat{\varphi}(r_i) \ldots \hat{\varphi}(r_n) | 0 \rangle$ is then constructed in the usual way using Wick's theorem.

We shall consider the first perturbation-theory corrections to the correlation functions. These corrections, as we shall see, are proportional to g_0^2, $g_0^2 \ln kr_0$, so that in the subcritical region ($-\ln kr_0 \sim 1$) perturbation theory is applicable at sufficiently small g_0. (We recall that ordinary perturbation theory in the pair interaction $V_0\varphi^4$ gives an expansion in the large parameter V_0/k[6]. But since we know that the pair interaction is screened ($V_{\text{eff}}(T = T_c) = 0$), it is not necessary to take these terms into account).

In the zeroth (Ornstein-Zernike) approximation, we shall normalize the Green function $Q_2 = G(k)$ to $1/r$ in coordinate space:

$$G_0(r) = 1/r, \quad G_0(k) = 4\pi/k^2. \quad (13)$$

(Changing the normalization of G(k) is equivalent to redefining the charge g and does not affect the dimensionless quantities in which we are interested). Then the correction of first order in g^2 is given by the diagram

$\delta G(k) =$ 

$$= G_0^2(k)\frac{g^2}{5!}\int (2\pi)^3\delta\left(k - \sum k_i\right)\prod_{i=1}^{5}\frac{d^3k_i}{(2\pi)^3}G_0(k_i). \quad (14)$$

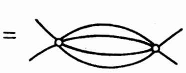
Integrals of this type are easily evaluated in coordinate space:

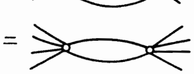
$$\delta G(k) = \left(\frac{4\pi}{k^2}\right)^2\frac{g^2}{5!}\int_{r_0}^{\infty}\frac{d^3r}{r^5}(e^{ikr} - 1) \to -\left(\frac{4\pi}{k^2}\right)^2\frac{g^2}{5!}\int_{r_0}^{1/k}\frac{d^3r}{r^5}\frac{(kr)^2}{2}. \quad (14')$$

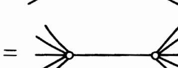
The subtraction at $k = 0$ in (14′) corresponds to redefining the transition point as a result of the contribution of the small distances $r \sim r_0$. To sum up, we have in the first approximation:

$$G(k) = \frac{4\pi}{k^2}\left(1 + \frac{\pi^2 g^2}{45}\ln kr_0\right). \quad (15)$$

In this same order (g^2) the vertex parts Γ_4, Γ_8 and Γ_{10} appear:

$\Gamma_4 =$  + (CRC) (16)

$\Gamma_8 =$  + (CRC) (17)

$\Gamma_{10} =$  + (CRC) (18)

(CRC = contributions from the remaining channels).

Evaluation of these diagrams in coordinate space gives

$$\Gamma_4 = g^2\frac{1}{4!}\int d^3r\frac{1}{r^4}[e^{i(k_1+k_2)r} - 1] + \text{(CRC)}$$

$$= -\frac{\pi^2 g^2}{24}(|k_1 + k_2| + |k_1 + k_3| + |k_1 + k_4|), \quad (16a)$$

$$\Gamma_8 = g^2\frac{1}{2!}\int d^3r\frac{1}{r^2}e^{i(k_1+k_2+k_3+k_4)r} + \text{(CRC)}$$

$$= \pi^2 g^2/|k_1 + k_2 + k_3 + k_4| + \text{(CRC)} \quad (17a)$$

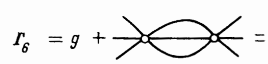
$$\Gamma_{10} = 4\pi g^2/(k_1 + k_2 + k_3 + k_4 + k_5)^2 + \text{(CRC)} \quad (18a)$$

The number of channels is equal to 3, 15 and 126 for Γ_4, Γ_8 and Γ_{10} respectively, and in the general case is given by the formula

$$\text{(number of channels for the transition } n_1 \to n_2) = \begin{cases} (n_1+n_2)!/n_1!n_2!, & n_1 \neq n_2, \\ (2n_1)!/2(n_1!)^2, & n_1 = n_2. \end{cases} \quad (19)$$

In (16a) we have made the subtraction at $k_i = 0$, in accordance with the initial assumption (7) ($\Gamma_4(k_i = 0) = 0$). This assumption means that the bare constant V_0 is cancelled by the contribution of the distances $r \sim r_0$ in the diagrams, and this must, of necessity, occur for $\eta < 1/2$. Various exactly soluble field-theory models[7] show that such a cancellation does not require a set of bare constants and occurs in a self-consistent manner, as in the two-dimensional Ising model. The reason for the screening is that Bose particles must necessarily repel each other ($V_0 < 0$), so that the diagrams are of alternating sign and cancel each other. An essential point is that no logarithmic terms appeared in Γ_4, Γ_8, and Γ_{10}, inasmuch as all the integrals converge in a power-law fashion. This also applies to the remaining vertices Γ_n, except for the unique dimensionless Γ_6.

The correction to Γ_6 is given by the diagram



$$= g + g^2 \frac{1}{3!} \int \frac{d^3r}{r^3} e^{i(k_1+k_2+k_3)r} + \text{(CRC)}$$

$$= g - g^2 \left[\frac{2\pi}{3} \ln |k_1 + k_2 + k_3| r_0 + \text{(CRC)} \right]. \quad (20)$$

We thus arrive at a logarithmic situation, characteristic of renormalizable field theories[8], and if the constant g is insufficiently small it is necessary to sum the principal logarithmic terms.

Knowing the order of magnitude of the effective subcritical indices $\eta \sim 10^{-2}$, $\alpha \sim 10^{-1}$, $\nu = 0.64$ found in[2] by extrapolation of high-temperature expansions, we can estimate the order of magnitude of g. Thus it is clear from (15) that the effective critical index η is, in the first approximation,

$$\eta_{\text{eff}}^{(1)} = {}^1/_{45}\pi^2 g^2. \quad (21)$$

Thus, $g^2 \sim 10^{-1}$.

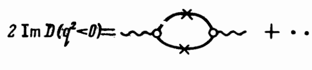
Another independent method of finding g is to estimate the correlation of the energy densities $D = \langle \epsilon\epsilon \rangle$, which determines the specific heat

$$C \sim \int D(r) d^3r \sim D(q \sim r_c^{-1}),$$

(r_c is the correlation length). In momentum space D(q) at $T = T_c$ behaves as[1,4,5]

$$D(q) = \text{const} \cdot q^{-\alpha/\nu}, \quad (22)$$

where $\alpha \sim 1/8$ is the specific heat index, $C \sim (T - T_c)^{-\alpha}$, and $\nu = 0.64$ is the correlation length index, $r_c \sim (T - T_c)^{-\nu}$. According to the unitarity condition



$$= \text{const} \cdot \int T^2(p, q) d^3p\, \delta(p^2) \delta((p-q)^2). \quad (23)$$

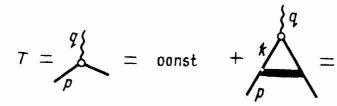
Consequently, the vertex T(p, q) behaves as

$$T(p, q) = p^{(1-\alpha/\nu)/2} t(q/p). \quad (24)$$

For $q \ll p$, the vertex T(p, q) is proportional to D(q)[5,9]:

$$T(p, q) \to \mathcal{T}(p, 0) D(q) = \text{const} \cdot p^{(1-\alpha/\nu)/2} (p/q)^{\alpha/\nu}. \quad (25)$$

We shall now find T(p, q) in our theory. The vertex T(p, q) satisfies the equation



$$= \text{const} + \int \frac{d^3k}{(2\pi)^3} \frac{T(k, q)(4\pi z)^2 \Gamma_4(p, -k, k+q, -p-q)}{2! k^2 (k+q)^2}. \quad (26)$$

Using for Γ_4 the first approximation (16a), putting $z = 1$ in the zeroth approximation (the results do not depend on the normalization of z) and going over to the correlation function

$$\langle \sigma_0 \varepsilon_q \sigma_r \rangle = F_q(r) = \int \frac{T(p, q)}{p^2 (p+q)^2} e^{ipr} \frac{d^3p}{(2\pi)^3}, \quad (27)$$

we obtain for $F_q(r)$ the equation

$$\nabla_r^2 (\nabla_r + iq)^2 F_q(r) = \delta(r) \left[\text{const} - \frac{\pi^2 g^2}{24} q F_q(0) \right] + \frac{2\pi^2 g^2}{3} r^{-4} F_q(r). \quad (28)$$

For $qr \ll 1$, according to (25) and (27),

$$F_q(r) \to \text{const} \cdot q^{-\alpha/\nu} r^f, \quad (29)$$

$$f \equiv 1/2 (1 - \alpha/\nu). \quad (30)$$

Substituting (29) into (28), for $qr \ll 1$ we obtain the relation

$$(f+1) f (f-1)(f-2) = {}^2/_3 \pi^2 g^2 = 30\eta + O(\eta^2). \quad (31)$$

In the latter equality we have made use of the formula (21), which is true in our approximation $\sim g^2$.

The experimental data $\alpha = 1/8$, $\nu = 0.64$ give $f \approx 0.4$ and then an estimate for η follows from (31):

$$\eta \approx {}^1/_{30} (f+1) f (f-1)(f-2) \approx 0.02. \quad (32)$$

We see that the various estimates of g do not contradict each other and that the relation (31) gives a reasonable value (32) for η.

It is interesting that the agreement with experiment was obtained as a result of the large numerical coefficient 30 in (31), which arose in a purely combinatorial manner: in (26) there are fewer factors of the type 1/n! than in the correction (14), (15) for η. Thus, even when $\eta = 2 \times 10^{-2}$, the interaction is great enough for the singularity in the specific heat to be weakened from $\alpha = 1/2$ (at $\eta = 0$) to $\alpha = 1/8 \to 0$. Here, of course, corrections of the next order in g^2 in (31), associated with corrections to Γ_4, can also turn out to be magnified, so that (31) is valid only in order of magnitude. One must also bear in mind that corrections $\sim g^4$ to η for $g^2 \sim 10^{-1}$ have the same order of magnitude as (21) and therefore in the subcritical region $\ln kr_0 \sim 1$ it is difficult to obtain exact relations. (Crudely speaking, the perturbation theory expansion parameter is not g^2 but $(\pi g)^2$.) However, in the critical region $-\ln kr_0 \gg 1$, as we shall see in the next section, the interaction will be weaker and perturbation theory in the effective charge g(ln k) will work better.

3. CORRELATIONS IN THE CRITICAL REGION ($|\ln kr_0| \gg 1$)

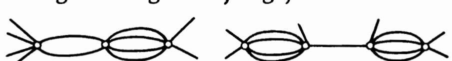
We shall now consider the region of momenta $kr_0 \ll 1$ so small that $\ln kr_0$ can be regarded as a large parameter. We shall use the logarithmic approximation, i.e., assume that $z = k^2 G(k)$ and $\Gamma_6(k_i)$ are slow (logarithmic) functions of the momenta. This approximation is justified by the small experimental value of η_{eff} and in addition is found to be self-consistent—from our equations it will be seen that η is small: $\eta \mid_{\ln k \to -\infty} \sim 10^{-4}$ (i.e., even smaller than in the subcritical region).

In the logarithmic approximation the unitarity conditions for Γ_6 lead, as for (11), to a differential equation of the form

$$\Gamma_6'(l) = \Gamma_6(l) F(\Gamma_6(l) z^3(l)), \quad l = \ln(1/kr_0). \tag{33}$$

For a better understanding of the meaning of this equation and the properties of the function $F(g)$, we shall derive it by a more usual method, namely, by direct summing of diagrams. The diagrams for Γ_6 can be divided into three classes:

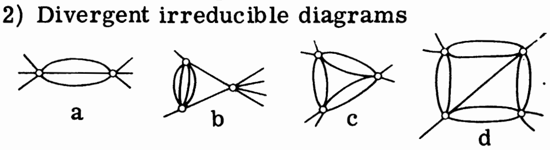
1) Convergent diagrams, e.g.,



a b (34)

In these diagrams the important internal momenta are those of the order of the external k_i; they therefore depend on the ratios k_i/k_j, so that the logarithmic approximation is not applicable to them.

2) Divergent irreducible diagrams



a b c d . . . (35)

In these diagrams the large internal momenta $k_i' \gg k_i$, $k_i' \ll r_0^{-1}$ are the important ones, so that the external momenta do not appear in the propagators and only determine the lower limit of the logarithmic integral $\int_k^{1/r_0} dk'/k'$. Therefore, for $k_i \sim k$ the divergent diagrams depend on one variable $l = \ln(1/kr_0)$. For example, diagrams (35)a, b and c respectively give the contributions

$$\frac{2\pi}{3} \int_k^{1/r_0} \frac{dk'}{k'} \Gamma_6^2(k') z^3(k') = \frac{2\pi}{3} \int_0^l dl_1 \Gamma_6^2(l_1) z^3(l_1), \tag{36}$$

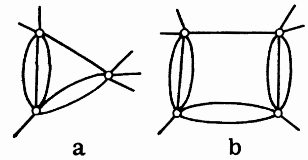
$$-\frac{\pi^2}{3} \int_0^l dl_1 \Gamma_6^3(l_1) z^6(l_1), \tag{37}$$

$$\frac{\pi^4}{2} \int_0 dl_1 \Gamma_6^3(l_1) z^6(l_1). \tag{38}$$

These expressions must then be multiplied by the number of diagrams of one type which differ only in the external momentum configurations (for $k_i \sim k \to 0$, the contributions of diagrams of one type are, with logarithmic accuracy ($\ln k_i = \ln k$), the same). The number of diagrams of the type (35)a, b and c is

$$N(a) = 6!/2 \cdot 3!3! = 10, \quad N(б) = 6!/2 \cdot 1!1!4! = 15,$$
$$N(в) = 6!/6 \cdot 2!2!2! = 15.$$

3) Divergent reducible diagrams, e.g.,



a b (39)

These diagrams contain several logarithmic integrations, corresponding to corrections to the internal vertices Γ_6.

It is obvious that in the critical region $l = \ln(1/kr_0) \to \infty$, the divergent diagrams are the most important ones. They can be grouped in such a way that within them there are exact vertex parts Γ_6 in place of the bare g_0, and exact Green functions $G(k') = z(k')/k'^2$.

For the sum of the divergent diagrams $\Gamma_6(l)$ we can write

$$\Gamma_6(l) = g_0 + \int_0^l \Gamma_6(l_1) K(l_1) dl_1. \tag{40}$$

We shall show that the kernel $K(l_1)$ possesses the property of renormalizability

$$K(l_1) = F(\Gamma_6(l_1) z^3(l_1)). \tag{41}$$

For the irreducible diagrams this property is obvious, since these contain only one logarithmic integration. The integrals over all the internal momenta $k_2', \ldots k_n'$ apart from one, k_1', converge, so that the factors $\Gamma_6(k_i')$ and $z(k_i')$ can be taken out for $k_i' = k_1'$. After this there remains a logarithmic integration over k_1' of a function of the form $[\Gamma_6(k_1')]^{m+1} [z(k_1')]^{3m}$. For the reducible diagrams we must remember that the logarithmic part of the correction to the internal vertices $\Gamma_6(k_i')$ is already accounted for in the irreducible diagrams, and so we must make the corresponding subtraction in the internal integrals for $\Gamma_6(k_i')$, after which the integrals over all the momenta $k_2', \ldots k_n'$ will converge and we shall again obtain an integral over k_1' of the form

$$\int_k^{1/r_0} \frac{dk'}{k'} \Gamma_6(k') [\Gamma_6(k') z^3(k')]^m.$$

For example, diagram (39)a contains two logarithmic integrations[1)]

$$\frac{1}{3!} \int \frac{d^3k}{k^3} z^3 \Gamma_6(k) \delta\Gamma_6(k) = \frac{2\pi}{3} \int_0 dl_1 \Gamma_6(l_1) \delta\Gamma_6(l_1) z^3(l_1), \tag{42}$$

$$\delta\Gamma_6 = \frac{1}{3!} \int \Gamma_6^2 z^3 e^{ikr} \frac{d^3r}{r^3} = \frac{2\pi}{3} \int_{r_0} \Gamma_6^2 z^3 \frac{dr}{kr^2} \sin kr$$
$$= \frac{2\pi}{3} \int_0^l \Gamma_6^2 z^3 \, dl_1 + \frac{2\pi}{3} \Gamma_6^2(l) z^3(l) \left[\ln kr_0 + \int_{r_0}^\infty \frac{dr}{kr^2} \sin kr \right]$$
$$= \frac{2\pi}{3} \int_0^l \Gamma_6^2(l_1) z^3(l_1) \, dl_1 + \frac{2\pi}{3} \Gamma_6^2(l) z^3(l) (1 - C). \tag{43}$$

Here C is Euler's constant. In (43) we include the

[1)]It is essential to note that in separating out the convergent part from the logarithmic integral for $\delta\Gamma_6$ the answer depends on the ratios of the momenta and on the method of cutoff. Therefore we can only obtain an estimate of the contribution of the reducible diagrams in this way. However, this estimate shows that the reducible diagrams make a relatively small contribution to the function F, e.g., the coefficient $(60(1-c) \times (2\pi/3)^2 \sim 1.2 \times 10^2)$ of $\Gamma_6^2 z^6$ in (44) is almost an order of magnitude smaller than the contribution of the irreducible diagram (38) - $15\pi^4/2 \sim 7 \times 10^2$.

first term in the divergent part $\Gamma_6(l)$; it is, therefore, already accounted for in the "second-order" diagram in (35). There remains the second term; it gives a "third-order" contribution

$$60\left(\frac{2\pi}{3}\right)^2(1-C)\int_0^l \Gamma_6^3(l_1)z^6(l_1)\,dl_1. \tag{44}$$

The number of diagrams of the type (39)a is N(a) = 6!/1!2!3! = 60. Similarly, we can verify that the reducible diagrams of higher orders also make a contribution of the form (40), (41). We shall now investigate Eqs. (11) and (33) as $l \to \infty$. First of all, it is obvious that the right-hand side of Eq. (11) depends, like F in (33), only on the renormalized charge $g(l) = z^3(l)\Gamma_6(l)$:

$$z'(l)\,/\,z(l) = -\eta(g(l)). \tag{45}$$

Then, eliminating z and Γ_6 from the equations, we obtain our equation for the renormalized charge[2)]:

$$g'(l) = g[F(g) - 3\eta(g)] \equiv \Phi(g). \tag{46}$$

As $l \to \infty$, as is known from the theory of differential equations, $g(l) \to g(\infty)$, where $g(\infty)$ is a stable root of the function $\Phi(g)$. Also the effective index

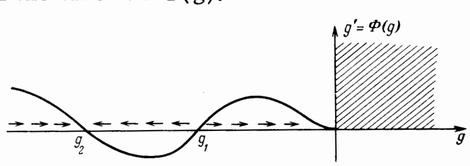
$$\eta_{\text{eff}} = -z'/z = \eta(g(l))$$

tends to a constant $\eta(g(\infty))$, so that the scaling law

$$G = \text{const}\cdot k^{\eta-2}, \quad \Gamma_6 \to g(\infty)z^{-3} \to \text{const}\cdot k^{-3\eta}.$$

is asymptotically true.

In order to estimate the index η_∞ and elucidate how the scaling laws are modified in the pre-asymptotic region $\ln(1/kr_0) \gg 1$, we shall consider the expected graph of the function $\Phi(g)$:



First of all, only one sign of g, corresponding to repulsion, has physical meaning; in our normalization it is negative. (Attraction in a Bose-system would lead to a first-order phase transition.) The behavior of $\Phi(g)$ as $g \to -0$ is determined by perturbation theory:

$$\Phi(g) \to {}^{20}/_3\pi g^2 + (\sim 10^3)g^3 + (\sim 10^5)g^4 + \ldots. \tag{47}$$

Thus, the zero root g = 0 is doubly degenerate and stable.

The first root g_1 of the equation $\Phi(g) = 0$, as can be seen from the figure, is unstable: $\Phi'(g_1) > 0$. The second root g_2 will be stable. The order of magnitude of $g_{1,2}$ can be estimated by the Padé method or as the ratio of the Taylor coefficients of the function $g^2/\Phi(g)$. The first terms of (47) give a very small value, which justifies the logarithmic approximation

$$g_{1,2} \sim 10^{-2}, \quad \eta_\infty \sim {}^1/_{45}\pi^2 g_2^2 \sim 10^{-4}.$$

We can now analyze the qualitative behavior of the renormalized charge $g(l)$ and of the effective index $\eta(g(l))$ on increase of $l = \ln(1/kr_0)$.

2) Equations (33), (45) and (46) are equations of the Lie renormalization group.

1. If the bare value g_0 is in the interval $g_1 < g_0 < 0$, $g(l)$ increases monotonically and tends to "−0" in accordance with the law:

$$g(l) \to -\frac{3}{20\pi l} = \frac{3}{20\pi \ln kr_0}. \tag{48}$$

The effective index η_{eff} tends to zero in accordance with the law

$$\eta_{\text{eff}} \to \pi^2 g^2/45 \to 1/2000(\ln kr_0)^2. \tag{49}$$

In this case, perturbation theory in the effective charge $g(l)$ gives an expansion in inverse powers of $\ln kr_0$ and is applicable even when $\ln kr_0 \sim 1$.

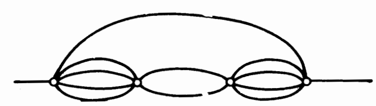
2. If the bare constant $g_0 < g_1$, then $g(l)$ tends to g_2 in accordance with the law:

$$g(l) \to g_2 + \text{const}\cdot\exp\{\Phi'(g_2)l\} = g_2 + \text{const}\cdot(kr_0)^{-\Phi'(g_2)}.$$

The effective indices η, α, etc., also tend to their asymptotic values in accordance with the same law:

$$\eta_{\text{eff}} \to \eta(g_2) + \text{const}\cdot(kr_0)^{-\Phi'(g_2)}. \tag{50}$$

If we take the estimate $g \sim 10^{-2}$ which follows from the first terms of the expansion of $\Phi(g)$, perturbation theory will be applicable for $\eta(g)$. For example, the diagram


(51)

contributes to $\eta(g)$

$$\delta\eta = \frac{8\pi^2}{3}\left(\frac{\pi^2 g^2}{24}\right)^2 \sim 10^{-6} - 10^{-7},$$

which is two orders of magnitude smaller than the main contribution (21):

$$\eta \sim \pi^2 g^2/45 \sim 10^{-4}.$$

The reason for this is that there are fewer diagrams in the Green function than in the vertex Γ_6, so that, although the higher-order corrections to Γ_6 (i.e., to $\Phi(g)$) are not small for $g \sim 10^{-2}$, the corrections to the Green function (i.e., to $\eta(g)$) will be small.

We note that for $g \sim 10^{-2}$, i.e., $\eta \sim 10^{-5}$, the specific heat index will already be close to the classical value $\alpha = \nu = 1/2$, so that, in this case, in the critical region $\ln k \gg 1$ the index $\alpha(\ln k)$ must begin to increase.

Thus, we see that two solutions are possible, depending on the magnitude of the bare constant g_0: at sufficiently small $g_0 < 0$, the effective interaction tends logarithmically to zero and the indices slowly approach the free values (i.e., from the Landau theory); with stronger repulsion, the interaction $g(l)$ and the indices $\eta(l)$, $\alpha(l)$ and $\nu(l)$ tend to universal constants, as was suggested earlier[5]. The asymptotic index η is very small in both cases and therefore the logarithmic approximation used is justified. In this one can assume for practical calculations, that $\eta = 0$, $G = 4\pi/k^2$ and calculate only the function F(g) in (46). The small values of η_∞ and g_∞ are associated with the rapid growth of the Taylor coefficients in the function F(g), i.e., with the increase in the number of high-order diagrams. We can hope, nevertheless, that it will be possible to determine the position of the zero of the function F(g) sufficiently accurately by the Padé method, in a manner analogous to the determination of the transition point T_c from high-temperature expansions[10].

4. GENERALIZATION TO OTHER SYSTEMS

First we shall explain why our methods are not applicable to the two-dimensional Ising model, i.e., why the index η cannot be small in a two-dimensional system. The point is that as $\eta \to 0$ the spin correlation in two-dimensional space increases indefinitely:

$$G(r) = \int d^2k \cdot k^{\eta-2} e^{ikr} \sim \frac{r^{-\eta}}{\eta} \to \infty. \tag{52}$$

Therefore, if we choose $G_0 = 1/k^2$ as the zeroth approximation we obtain enormous corrections in the next approximation, proportional to ln R where R is the lattice dimension. This means that in the two-dimensional system η must be fairly large and the logarithmic approximation is inapplicable. Also, it is clear why in the two-dimensional Ising model the scaling laws have a wide region of applicability ($kr_0 \gtrsim 1$) – self-consistent field theory has no region of applicability at all in two-dimensional space. Thus, the two-dimensional Ising model is not a completely felicitous analogy for the three-dimensional problem.

The theory considered above was concerned with the three-dimensional Ising model, i.e., with systems with a scalar real order parameter (critical points of liquids, etc.). The generalization to the Bose-liquid (the λ-point of helium) and the Heisenberg model (the Curie point of a ferromagnet) creates no difficulties.

A Bose-liquid close to the λ-point is described by a complex classical field ψ or, equivalently, by a two-dimensional vector $\varphi_\alpha = (\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)$ (the lattice of plane dipoles of the paper by Vaks and Larkin[11]). The gauge invariance of the complex field $\psi \to e^{i\alpha}\psi$ implies invariance with respect to rotation of the vector φ in the two-dimensional space. This can be seen from the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\psi^*(\nabla^2+\tau)\psi + \frac{1}{4!}V_0(\psi^*\psi)^2 + \frac{1}{6!}g_0(\psi^*\psi)^3 + \dots \\ &= \frac{1}{2}\varphi_\alpha(\nabla^2+\tau)\varphi_\alpha + \frac{1}{4!}V_0(\varphi_\alpha{}^2)^2 + \frac{1}{6!}g_0(\varphi_\alpha{}^2)^3 + \dots . \end{aligned} \tag{53}$$

The partition function for $T \to T_c$ can be expressed as a continuous integral

$$e^{-F/T} = Z = \int \delta\varphi_\alpha \exp\left\{-\frac{1}{T}\int d^3r[H(\varphi) - \varphi_\alpha h_\alpha]\right\}, \tag{54}$$

and the Green functions are defined as functional derivatives with respect to the source $h = h_\alpha$:

$$Q_n^{\alpha_1\dots\alpha_n}(r_1,\dots,r_n) = \delta^n F/\delta h_{\alpha_1}(r_1)\dots\delta h_{\alpha_n}(r_n). \tag{55}$$

The Heisenberg model near the Curie point is described by a classical vector field φ_α (three-component). (For the proof we can make use of the diagram technique[12] and, as $T \to T_c$ and $k_i \to 0$, retain the zeroth terms in the sums over the frequencies.) The effective Hamiltonian of the classical field in the Heisenberg model has the form (53), where the constants $V_0, g_0, \dots$ depend on the spin S, and the vector φ_α is three-dimensional.

Thus the whole difference between the Bose-liquid and the Heisenberg ferromagnetic on the one hand, and the Ising model on the other reduces, close to the Curie point, to the fact that the corresponding classical field φ, i.e., the order parameter, has an internal degree of freedom, analogous to isotopic spin. The vertex parts $\Gamma_n(k_i)$ will be tensors of rank n in the two- or three-dimensional isotopic space, symmetric with respect to interchanges of any pair of variables $k_i\alpha_i \longleftrightarrow k_j\alpha_j$. (Here α_i = 1, 2 or 1, 2, 3 are the tensor indices.) This was noted by Larkin and Khmel'nitskiĭ[13] with four-dimensional models as examples.

After this the construction of the theory is analogous to that in Secs. 2 and 3. In the logarithmic approximation the Green function is equal to

$$G^{\alpha\beta}(k) = 4\pi k^{-2} z(\ln kr_0)\delta_{\alpha\beta}. \tag{56}$$

The three-body interaction has the form (with logarithmic exactness)

$$z^3\Gamma_6^{\alpha_1\dots\alpha_6}(l) = 1/_{15}g(l)[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\delta_{\alpha_5\alpha_6} + (14\text{ interchange})]. \tag{57}$$

In second order in g the pair interaction is (diagram (16))

$$\begin{aligned} z^2\Gamma_4(k_i,\alpha_i) = &-\frac{\pi^2 g^2}{24}\frac{(a+4)}{75}\{4(k_{12}+k_{13}+k_{23})(\delta_{12}\delta_{34}+\delta_{13}\delta_{24}+\delta_{23}\delta_{14}) \\ &+(a+2)[k_{12}\delta_{12}\delta_{34}+k_{13}\delta_{13}\delta_{24}+k_{23}\delta_{23}\delta_{14}]\} + O(g^3). \end{aligned} \tag{58}$$

Here a is the dimensionality of the isotopic space: a = 1 (Ising), a = 2 (Bose), or a = 3 (Heisenberg). The abbreviated notation

$$\delta_{ij} \equiv \delta_{\alpha_i\alpha_j}, \quad k_{ij} = |k_i + k_j|.$$

is used. The function g(l) is determined by Eq. (46), where $\Phi(g)$ depends on a in the following way:

$$\Phi(g) \approx g^2\frac{4\pi}{15}(3a+22) + g^3\frac{\pi^4}{75}(2720+620a+34a^2+a^3) + \dots \tag{59}$$

The effective index η is

$$\eta(g) = \frac{\pi^2\sigma^2}{45}\frac{(a+2)(a+4)}{15} + O(g^4). \tag{60}$$

It is interesting that the relation (31), which, in second order in g, relates the indices $f = \frac{1}{2}(1 - \alpha/\nu)$ and η, is conserved. This is explained by the fact that in the diagrams for the vertex $T_{\alpha\beta} = T\delta_{\alpha\beta}$ and the Green function, the sums over the tensor indices coincide. Inasmuch as, in all systems in the subcritical region, $\alpha/2\nu \lesssim 1/10$, i.e., $f \approx 0.5$, this means that, to an accuracy within terms $\sim g^4 \sim \eta^2$, the subcritical index η has the order of magnitude (32), irrespective of the symmetry of the system.

In the asymptotic region $\ln(1/kr_0) \gg 1$, the critical indices and functions will depend only on the symmetry of the system, i.e., on the quantity a; if the asymptotic values $g(\infty)$ and $\eta(\infty)$ are determined by the higher terms in g in the equations $\Phi(g_\infty) = 0$, $\eta_\infty = \eta(g_\infty)$, they will be small and will depend weakly on a. The contributions of the higher terms in g are determined chiefly by the large number of the diagrams and not by their magnitude. This can already be seen in third order (59). In the case of weak coupling, when g ln k $\to 0$ asymptotically, the effective index η will tend to zero in accordance with the law

$$\eta_{\text{eff}} \to \frac{(a+2)(a+4)}{48(3a+22)^2}\ln^{-2}k. \tag{61}$$

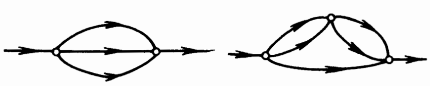
This formula will be obtained if we find $g(l)$ from (46) and (59) and substitute it in (60).

A separate analysis is required for the phase transition in uniaxial ferroelectrics[13], where the dipole-

dipole interaction weakens the fluctuations of the dipole moment P_z and, in place of the Ornstein-Zernike law, the phenomenological theory at $T = T_c$ reduces to the correlation[14,13]

$$\langle P_z(k)P_z(-k)\rangle = (k_\perp^2 + k_z^2 + \alpha^2 k_z^2 / k_\perp^2 + k_z^2)^{-1} = {}^1/_2(i\alpha k_z + k_z^2 + k_\perp^2)^{-1} + {}^1/_2(-i\alpha k_z + k_z^2 + k_\perp^2)^{-1}. \quad (62)$$

For sufficiently large momenta $k_\perp$, $k_z \gg \alpha$, the problem reduces to the three-dimensional Ising model and our theory can be applied to it; in the critical region $k_\perp$, $k_z \ll \alpha$, however, the dipole-dipole term $\pm i\alpha k_z$ substantially changes the situation. If we introduce a "time" $t = iz$ and an "energy" $\epsilon = ik_z$, the theory becomes equivalent to quantum field theory in the two-dimensional space (x, y) (k_x, k_y), but is now nonrelativistic—the spectrum of the particles has the form $\epsilon(k) = \pm k^2/\alpha$. The corrections to the phenomenological theory (62) will correspond to decays of one particle into several others:



a b (63)

A field theory formally analogous to this was considered in[15] (Sec. 5). The corrections of the higher approximations contain powers $V_0^m \ln kr_0)^n$, where V_0 is the amplitude of the "decay" $(1 \to 3)$. In this respect the situation is the same as in the four-dimensional Ising model. (This was noted by Larkin and Khmel'nitskiĭ[13].) For sufficiently small $V_0 < 0$, $|V_0| \ll 1$, summing the leading terms of the form $(V_0 \ln k)^n$ leads to zero charge $V_{eff} \to \mathrm{const}/\ln k$[13]. In the general case, the behavior of the effective charge $V_{eff}(\ln k)$ is given by an equation of the type (46), where the function $\Phi(V)$ will have the qualitative form shown in the figure. Then, if the bare constant $V_0 < 0$ is greater in absolute magnitude than the critical value V_1, the renormalized charge $V_{eff}(\ln k)$ will tend to a constant $V(-\infty) = V_2$, as in our theory. For the same reasons as those given above (the increase of the Taylor coefficients of the function $\Phi(V)$), the renormalized charge $V_{eff}(-\infty) = V_2$, the index η will be numerically small, and the logarithmic approximation will be applicable in the whole region $k \ll \alpha$. For calculations in the logarithmic approximation, one can, as in[13], use the analogy with the four-dimensional Ising model (the quantity $\alpha k_z/k$ plays the part of the fourth momentum component in (62)).

Then, in second order in V (diagram a in (63)), we obtain the following value of the effective index η:

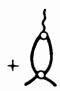
$$\eta_{\text{eff}} = {}^1/_3\pi^4 V^2 \quad (64)$$

(we use the normalization $G_0(k) = (2\pi)^2 k^{-2}$, $G_0(r) = r^{-2}$ in the four-dimensional Ising model).

The calculation of the index α/ν for the specific heat $C \sim r_c^{\alpha/\nu}$ is analogous to that in Sec. 3 of our paper. The calculations can be simplified if we assume that $\alpha \ll \nu$ and $\eta \ll 1$, in agreement with experiment. Then the vertex $T(p, q) \sim p^{-\alpha/2\nu} t(q/p)$ depends slowly on p and q for $p \sim q$, and the logarithmic approximation can be applied to it (the behavior of the vertex is found from the unitarity condition, as with (23)):

$$\operatorname{Im} D(q) \sim \int T^2 d^4p\, \delta(p^2)\delta((p-q)^2) \sim T^2 \quad (p \sim q),$$
$$\operatorname{Im} D(q) \sim \operatorname{Re} D(q) \sim q^{-\alpha/\nu}.$$

In the logarithmic approximation the equation for $T(p, q)$ has the form

$T(p, q) = 1 +$ 

$$= 1 + \frac{1}{2}\int_p^{1/r_0} V(\ln k)\frac{d^4k}{(2\pi)^4}\frac{(2\pi)^4}{k^4}T(k, q),$$

$$T(\ln pr_0) = 1 + \pi^2 \int_{\ln pr_0}^{0} d\ln(kr_0)\, V(\ln kr_0)\, T(\ln kr_0). \quad (65)$$

Hence the effective index $\alpha/2\nu$ is equal to

$$\frac{\alpha}{2\nu} = -\frac{d\ln T(p\sim q)}{d\ln p} = \pi^2 V. \quad (66)$$

Since $\pi^2 V$ is small and negative (repulsion), the index α will also be small and negative, i.e., the specific heat tends to a finite limit in accordance with the law $C \to C_0 + (T - T_c)^{-\alpha}$. Eliminating V from (64) and (66), we find a connection between the effective indices η, α and ν, accurate up to terms of order $V^2 \sim \eta$:

$$\alpha = -2\nu\sqrt{3\eta} + O(\eta). \quad (67)$$

The experimental data on ferroelectrics are insufficiently accurate to test this relation; we can, however, make use of the results of high-temperature expansions[16] for the four-dimensional Ising model:

$$\alpha = -0.12 \pm 0.03, \quad \nu = 0.536 \pm 0.003,$$
$$\gamma = (2-\eta)\nu = 1.065 \pm 0.003.$$

These values give $\eta = 0.014 \pm 0.018$, and the relation (67) is fulfilled within the large error limits of the calculations of Moore[16].

It is interesting that the negative value of α allows us to reject the solution with zero charge; in this solution, according to[13], the specific heat tends logarithmically to infinity: $C \sim (\ln(T - T_c))^{1/3}$, and the effective index $\alpha = -d\ln C/d\ln(T - T_c) = -{}^1/_3\ln(T - T_c)$ is positive.

From the standpoint of our theory it is natural that, in the four-dimensional Ising model, the charge V does not tend to zero; the bare interaction V_0 in the Ising model is not small. In real ferroelectrics, the bare interaction V_0 can be small (see[13]) and then the charge becomes zero. In the four-dimensional Heisenberg model or in the four-dimensional Bose-gas (plane dipoles in a four-dimensional lattice) the relation (67) will look like:

$$\alpha = -2\nu\sqrt{(a+2)\eta} + O(\eta), \quad (68)$$

where $a = 3$ (Heisenberg), $a = 2$ (Bose), or $a = 1$ (Ising). Systems which reduce to these four-dimensional models are as yet unknown, but it would be interesting to test (68) by means of high-temperature expansions.

5. DISCUSSION OF THE RESULTS

Our theory is based on the experimental fact that the fourth term $V_0\varphi^4$ in the Landau expansion of the free-energy density in powers of the order parameter $\varphi(r)$

$$F(r) = \tfrac{1}{2}A\varphi\Delta\varphi + V_0\varphi^4/4! + g_0\varphi^6/6! + \ldots \quad (69)$$

vanishes at the transition point

$$V_0(T_c) = 0. \quad (70)$$

Here the correlation of the order parameters at the transition point

$$G(r) = \langle\varphi(0)\varphi(r)\rangle = \int \delta\varphi\, \varphi(0)\varphi(r) \exp\left[-\int F\, d^3r/T_c\right] \quad (71)$$

differs from the Ornstein-Zernike correlation

$$G_0(r) = \int \delta\varphi\, \varphi(0)\varphi(r) \exp\left(-\frac{A}{2T_c}\int \varphi\Delta\varphi\, d^3r\right) = \frac{\text{const}}{r} \quad (72)$$

only because of the term $g\varphi^6$ in the free energy (69). The corrections to the Ornstein-Zernike law are calculated by means of diagram technique ((69) plays the role of the Lagrangian) and for $g_0 \ll 1$ these corrections are small:

$$rG(r) = \text{const}\cdot(1 - \eta \ln r), \quad \eta = \tfrac{1}{45}\pi^2 g_0^2 + O(g_0^4). \quad (73)$$

Therefore there is a broad subcritical region $r \gg r_0$, $\ln(r/r_0) \sim 1$ in which the Ornstein-Zernike law is valid. The density correlation $\langle \varphi^2(r_1)\varphi^2(r_2)\rangle$ in this region varies more strongly:

$$D(r) = \langle\varphi^2(0)\varphi^2(r)\rangle = \text{const}\cdot r^{-3+\alpha/\nu}. \quad (74)$$

The index α/ν is connected with the index η by the relation (33), (34) (accurate up to terms $\sim\eta^2$). Because of the large numerical coefficient 30 in (31), the index $\alpha/\nu \sim 0.2$ differs markedly from the classical value $\alpha/\nu = 1$. The relation (31) is not very sensitive to the magnitude of α/ν and to find α/ν from the magnitude of η and explain the small experimental value $\alpha/\nu \approx 1/5$ is difficult in practice.

The effective indices $\eta_{\text{eff}} = -d \ln rG/d \ln r$ and $(\alpha/\nu)_{\text{eff}} = d \ln r^3 D/d \ln r$ in the subcritical region $\ln(r/r_0) \gtrsim 1$ are not universal (g_0 depends not only on the symmetry but also on the parameters of the system) and depend logarithmically on r. In the asymptotic region $\ln(r/r_0) \gg 1$, there will be either strong or weak coupling, depending on the magnitude of the constant g_0. In the case of weak coupling the effective interaction $g \ln r$ tends logarithmically to zero, while the effective indices tend to their classical values in accordance with the law

$$\eta_{\text{eff}} \to c\cdot 10^{-3}(\ln r)^{-2} \to 0, \quad (\alpha/\nu)_{\text{eff}} \to 1 - 30\eta_{\text{eff}} \to 1, \quad (75)$$

where $c = 0.5$ (Ising), 0.64 (Bose), or 0.76 (Heisenberg).

In the case of strong coupling the effective interaction $g \ln r$ tends to a universal constant $g(\infty)$ in accordance with the power law (50); the indices behave analogously and the asymptotic values η_∞, $(\alpha/\nu)_\infty$ should be not very different from the classical values. (From the standpoint of the bootstrap equations of[5], this means that the renormalized vertex $\mathcal{T}_c(p^2)$ depends weakly on the momentum, i.e., $\mathcal{T}_c \approx$ const, so that the index $\eta = 2[1 - \mathcal{T}_c(0)/\mathcal{T}_c(-1)]$ is numerically small. The constancy of the vertex $\mathcal{T}(p^2)$ is connected with the fact that the amplitudes Γ_4, $\Gamma_6, \ldots, \Gamma_n$ are small as $\eta \to 0$ and therefore in the unitarity condition for $\mathcal{T}_c(p^2)$ the imaginary part $\text{Im}\,\mathcal{T}_c \sim \Sigma\int \mathcal{T}_n\Gamma_n d\tau_n$ will be small, so that $\mathcal{T}_c(p^2)$ is determined by the constant real part.)

The values of the indices known from experiments and high temperature expansions are, apparently, evidence in favor of strong coupling.

Concerning estimates of the index η from high-temperature expansions, we should, however, make the following remark. There are grounds for hoping that these expansions can give good estimates of the behavior of thermodynamic quantities (specific heat, susceptibility, spontaneous moment, etc.) which have one singularity $T = T_c$. But high-temperature expansions for the correlation functions, especially in the region $T - T_c \ll k^{1/\nu}$, are inapplicable for the following reason. As was remarked in[4,5], the correlation functions have complex temperature singularities at

$$T = T_c \pm \text{const}\cdot e^{i\pi/2\nu}(k/n)^{1/\nu}, \quad n = 1, 2, 3, \ldots. \quad (76)$$

For $\nu > 0.5$ these singularities are on the physical sheet of the T-plane and are concentrated towards $T = T_c$. Therefore, high-temperature expansions are valid up to the first singularity, i.e., for $|T - T_c| \rangle$ const $\times\, k^{1/\nu}$, whereas in the region $|T - T_c| \sim k^{1/\nu}$ the correlation functions change their structure (e.g., they have minima and maxima[9]) and the expansions have no meaning.

Therefore, the statement of Ferer, Moore and Wortis[2] about the violation of the scaling laws in the region $|T - T_c| \ll k^{1/\nu}$ and the estimate $\eta = 0.041$ seem doubtful. More reliable are the statements[2,16] on the violation of the relation $\alpha = 2 - d\nu$, inasmuch as the correlation length $r_c \sim (T - T_c)^{-\nu}$ can be found exactly from the asymptotic behavior $rG(r) \to e^{-r/r_c}$ in the region of "applicability" $|T - T_c| \gg r^{1/\nu}$ of the high-temperature expansions.

The reasons for the violation of the relation $\alpha_{\text{eff}} = 2 - 3\nu_{\text{eff}}$ can be seen within the framework of our approach; this relation should be fulfilled only in the asymptotic region $\ln r$, $\ln(T - T_c) \gg 1$, while in the subcritical region the interaction leads to corrections $\sim\eta$.

To summarize, one can say that the phenomenological theory of Landau is a more successful analogy for critical phenomena than the two-dimensional Ising model, provided that we take into account that the free energy expansion starts at the sixth power of the order parameter, and with a small coefficient. After this, the corrections to the phenomenological theory from fluctuations will be small and it is not difficult to find them.

In the next paper this idea will be used to give an account of critical phenomena above and below T_c in an external field.

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