QUANTUM PROCESSES IN A CONSTANT ELECTRIC FIELD

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The differential probability of emission of a photon by an electron situated in a constant electric field is obtained, together with the differential probabilities of the remaining processes represented by the three-point diagram under consideration.

1. INTRODUCTION

A study of pair-generating quantum processes in an external field is of general theoretical interest. The simplest process of this type is pair production by the field itself. As early as in 1934, Landau and Lifshitz considered the process of pair production by a classical current of two charged particles[1]. The Bethe-Heitler and Breit-Wheeler processes can also be regarded as generation of a pair by an external field (Coulomb plus field of a plane wave in the former case and fields of two plane waves in the latter)[2,3]. In this case the external field is taken into account by perturbation theory. In a number of cases, part of the pair-generating external field can be taken into account accurately[4-6].

The simplest external field that can be accounted for accurately is a constant electric field. The probability of pair production by such a field has a surprisingly simple form[7,8]. It is also clear that the results obtained for it are useful also for different estimates of more realistic cases (see, for example, estimates of pair production probability in the field of a focused laser beam[9]).

In this paper we consider the emission of a photon by an electron, pair production by a photon, emission of a photon by a pair, and the inverse processes.

2. EMISSION OF A PHOTON BY AN ELECTRON

We use units in which $\hbar = c = 1$, $e^2/4\pi = \alpha = 1/137$, and the same representation for the γ matrices as in[10]. The electron charge is denoted by $-e$. The external field is described by a vector potential

$$A_1 = A_2 = A_0 = 0, \quad A_3 = -Et. \tag{1}$$

The orthonormal solutions of the Dirac equation with such a potential and the Green's function were obtained in[8]. The positive ${}^{+}\psi$ (${}_{+}\psi$) and negative ${}^{-}\psi$ (${}_{-}\psi$) frequency solutions as $t \to -\infty$) are of the form

$$\begin{aligned} {}^{-}{}_{+}\psi_r &= \exp\{-{}^1/_8\pi\lambda + i\mathbf{px}\}(2VeE\lambda)^{-1/2}\cdot \\ &\times[u_r D_{i\lambda/2}(\mp(1-i)\xi) \mp {}^1/_2 u_r'\sqrt{eE}\lambda(1-i)D_{i\lambda/2-1}(\mp(1-i)\xi)], \\ {}^{+}{}_{-}\psi_r &= \exp\{-{}^1/_8\pi\lambda + i\mathbf{px}\}(4VeE)^{-1/2}[u_r D_{-i\lambda/2-1}(\mp(1+i)\xi) \\ &\pm u_r'\sqrt{eE}(1-i)D_{-i\lambda/2}(\mp(1+i)\xi)], \end{aligned} \tag{2}$$

where

$$\xi = \frac{p_3 - eEt}{\sqrt{eE}}, \quad \lambda = \frac{p_\perp^2 + m^2}{eE}, \quad p_\perp^2 = p_1^2 + p_2^2,$$

$$u_1 = \begin{bmatrix} p_1 - ip_2 \\ m \\ p_1 - ip_2 \\ -m \end{bmatrix}, \quad u_1' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} m \\ -(p_1 + ip_2) \\ m \\ p_1 + ip_2 \end{bmatrix}, \quad u_2' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Here $D_\nu(z)$ is the parabolic-cylinder function, V the normalization volume, and $r = 1, 2$ the indices of the spin states.

The Green's function is

$$\begin{aligned} G(x', x) &= \sum_n {}^{+}\psi_n(x')\,{}_{+}\bar{\psi}_n(x) N_n \quad \text{for } t' > t, \\ G(x', x) &= -\sum_n {}_{-}\psi_n(x')\,{}^{-}\bar{\psi}_n(x) N_n \quad \text{for } t' < t, \end{aligned} \tag{3}$$

where

$$N_n = -{}^1/_2\Gamma(i\lambda/2)\sqrt{\lambda/\pi}e^{\pi\lambda/4}, \quad |N_n|^2 = (1 - e^{-\pi\lambda})^{-1}, \quad n = r, \mathbf{p}.$$

In the case of a pair-generating field, a distinction is made between relative and absolute probability[11]. It suffices for us to find the relative probabilities. In the case of necessity one can change over to absolute probabilities (see[8 12]). The amplitude of the relative probability for the emission of a photon by an electron is (see[11,13])

$$\begin{aligned} M_{n'n} &= \int_{\substack{t''\to\infty \\ t'\to-\infty}} {}^{+}\bar{\psi}_{n'}(x'')\beta G(x'', x) e\hat{A}'(x)\cdot \\ &\cdot G(x, x')\beta\,{}_{+}\psi_n(x')\, d^3x'' d^4x d^3x' \end{aligned} \tag{4}$$

$$= N_{n'}N_n \int {}_{+}\bar{\psi}_{n'}(x) e\hat{A}'(x)\,{}^{+}\psi_n(x) d^4x, \tag{4'}$$

where

$$\hat{A}'(x) = \gamma_\mu e_\mu' e^{-ikx}/\sqrt{2Vk_0}.$$

The integrals with respect to t contained in (4′) are calculated in the Appendix. For simplicity we are not interested in the polarization of the photon and we sum the square of the matrix element over the polarization e'. An elementary calculation yields the following differential probability of emission of a photon in the entire volume and during the entire time:

$$dW_{n'n} = \sum_{e'} |M_{n'n}|^2 \frac{Vd^3k}{(2\pi)^3} = Q|\quad|_{r'r}^2 \frac{d^3k}{(2\pi)^3},$$

$$Q = \frac{e^{-\pi(\lambda+\lambda')/4}}{E^2 k_0 \lambda'(1 - e^{-\pi\lambda})(1 - e^{-\pi\lambda'})}, \quad \mathbf{p} = \mathbf{p}' + \mathbf{k}. \tag{5}$$

Here $r', r = 1, 2$ indicate the spin states of the final and initial electrons, and

$$\begin{aligned} |\quad|_{11}^2 &= p_\perp'^2|I_{00}|^2 + {}^1/_4 p_\perp^2\lambda'^2|I_{11}|^2 \\ &-\lambda' \operatorname{Re}\{[(p_1' + ip_2')(p_1 - ip_2) + m^2]I_{01}I_{10}^*\}, \end{aligned}$$

$$| \;|_{22}^2 = p_\perp'^2|I_{00}|^2 + {}^1\!/_4 p_\perp^2\lambda'^2|I_{11}|^2 - \lambda' \operatorname{Re}\{[(p_1' - ip_2')(p_1 + ip_2) + m^2]I_{01}I_{10}^*\},$$
$$| \;|_{12}^2 = | \;|_{21}^2 = m^2(|I_{00}|^2 + {}^1\!/_4\lambda'^2|I_{11}|^2 - \lambda' \operatorname{Im} I_{00}I_{11}^*). \tag{6}$$

The functions $I_{j'j} \equiv I_{j'j}(\rho)$, $\rho = k_\perp/\sqrt{eE}$ are defined in the Appendix (formula (A.10). Since there are certain relations between the confluent hypergeometric functions Ψ in terms of which the $I_{j'j}$ are expressed[14], we can express $| \;|^2_{r'r}$ in terms of either $\Psi(i\lambda/2, 1 + i\nu; \zeta)$ and its derivative with respect to ζ, or else in terms of $I_{00}(\rho)$ and $I'_{00}(\rho) = dI_{00}(\rho)/d\rho$. Here $\zeta = -i\rho^2/2$, $\nu = (\lambda - \lambda')/2$. In the former case we have

$$\frac{1}{2}\sum_{r',r=1}^{2} | \;|^2_{r'r} = \pi e^{3\pi\nu/2}\Big\{(p_\perp^2 + p_\perp'^2)|\Psi|^2 + [2(p_1p_1' + p_2p_2' + m^2) + eE\rho^2]\frac{\rho^2}{\lambda}|\Psi'|^2 - [p_\perp^2\rho^2 + 2\nu(p_1p_1' + p_2p_2' + m^2)]\frac{2}{\lambda}\operatorname{Re}\Psi'\Psi^*\Big\}, \tag{7}$$

$$| \;|_{12}^2 = | \;|_{21}^2 = m^2\pi e^{3\pi\nu/2}(\rho^4/\lambda^2)|\Psi'|^2,$$
$$\Psi = \Psi(i\lambda/2, 1 + i\nu; \zeta). \tag{7'}$$

For brevity we do not present the expressions for $| \;|^2_{11}$ and $| \;|^2_{22}$ separately. In the latter case we obtain

$$\frac{1}{2}\sum_{r',r=1}^{2} | \;|^2_{r'r} = \frac{1}{\lambda}\Big\{\Big[eE\lambda\lambda' + \frac{eE}{4}(\rho^2 - \lambda - \lambda')^2 + m^2(\rho^2 - \lambda - \lambda') + \frac{1}{2\rho^2}(p_1p_1' + p_2p_2' + m^2)(\rho^4 - 4\nu^2)\Big]|I_{00}|^2 + [2(p_1p_1' + p_2p_2' + m^2) + eE\rho^2]|I_{00}'|^2 + [2(p_1p_1' + p_2p_2' + 2m^2) + eE(\rho^2 - \lambda - \lambda']\rho \operatorname{Im} I_{00}I_{00}'^*\Big\}, \tag{8}$$

$$| \;|_{12}^2 = | \;|_{21}^2 = (m^2/\lambda^2)\{{}^1\!/_4(\rho^2 + 2\nu)^2|I_{00}|^2 + \rho^2|I_{00}'|^2 + \rho(\rho^2 + 2\nu)\operatorname{Im} I_{00}I_{00}'^*\}. \tag{8'}$$

We now proceed to discuss formula (5). It can be integrated with respect to **k** only between such limits that leave the radiation probability much smaller than unity, since the attenuation of the initial state of the electron has not been taken into account. The integration of the radiation probability over all the **k** in a field of unlimited dimensions gives a divergent expression. In our case the integral with respect to k_3 diverges logarithmically:

$$\int_{-k_\parallel}^{k_\parallel} \frac{dk_3}{\sqrt{k_\perp^2 + k_3^2}} = 2\operatorname{Arsh}\frac{k_\parallel}{k_\perp}. \tag{9}$$

It must be recognized, however, that for a quantum transition with emission of a photon **k** an important role is played by a definite interval of the time t. This is the interval that includes the times that make the main contribution to the integrals in (A.1). A detailed examination of the question of the formation of radiation in the classical limit is given in[15]. It suffices here to confine ourselves to qualitative considerations. It is natural to assume that the points $\xi' = 0$ and $\xi = 0$ should lie inside the formation interval. It follows therefore that the formation time is $t \propto p_3'/eE$ and the width of the formation interval is $\Delta t \propto |\xi' - \xi|/\sqrt{eE} = |k_3|/eE$. Thus, integration with respect to k_3 (or p_3') is, roughly speaking, equivalent to integration with respect to the time:

$$\operatorname{Ar sh}\frac{k_\parallel}{k_\perp} \approx \ln 2\frac{k_\parallel}{k_\perp} = \ln\frac{eEt}{m} + \text{const} = \frac{eE}{m}s + \text{const}, \tag{10}$$

where s is the proper time of the initial particle. Symbolically

$$\frac{dk_3}{k_0} = \frac{eE}{m}ds. \tag{11}$$

This relation makes it possible to determine the probability of radiation per unit proper time.

Since the radiation-formation interval is proportional to k_3, for sufficiently large k_3 it is necessary to take into account the attenuation of the initial state of the electron owing to the emission of softer quanta. Incidentally, it should be borne in mind that by a suitable choice of the reference system it is possible, without changing the field, to reduce any fixed longitudinal momentum to zero.

We note further that in accordance with formula (5) radiation is possible also with the transition of the electron into a state having a higher transverse energy, i.e., with $\nu = (\lambda - \lambda')/2 < 0$. From (A.2) we see, however, that

$$I_{00}(\rho) \equiv I_{00}(\rho, \lambda', \lambda) = e^{\pi\nu}I_{00}(\rho, \lambda, \lambda'). \tag{12}$$

In conjunction with (8) this means that such inverse transitions are suppressed by the factor $\exp(-2\pi|\nu|)$ compared with the direct transitions ($\nu > 0$).

Let us consider two limiting cases of formula (5). In the classical limit

$$\Psi\Big(\frac{i\lambda}{2}, 1 + i\nu; \zeta\Big) = i\sqrt{\frac{\lambda}{\pi}}\exp\Big\{\theta - \frac{\pi\nu}{4} - i\nu\ln\frac{\rho}{\sqrt{2}}\Big\}K_{i\nu}(z),$$
$$\Psi'\Big(\frac{i\lambda}{2}, 1 + i\nu; \zeta\Big) = -\sqrt{\frac{\lambda}{\pi}}\exp\Big\{\theta - \frac{\pi\nu}{4} - i\nu\ln\frac{\rho}{\sqrt{2}}\Big\} \times \frac{\lambda}{z}\Big[K_{i\nu}'(z) - \frac{i\nu}{z}K_{i\nu}(z)\Big], \tag{13}$$
$$\theta = \frac{\pi}{8}(\lambda + \lambda') - \frac{i\pi}{4} + \frac{i\lambda'}{4}\Big(1 - \ln\frac{\lambda'}{2}\Big) + \frac{i\lambda}{4}\Big(1 - \ln\frac{\lambda}{2}\Big), \quad z = \sqrt{\lambda}\rho.$$

Here $K_{i\nu}(z)$ is the Macdonald function. From (5), (7), and (13) follows the probability of radiation of a classical particle

$$\frac{1}{2}(dW_{11} + dW_{22})_{\text{кл}} = \frac{e^{\pi\nu}}{4\pi^3E^2}\Big\{p_\perp^2K_{i\nu}^2(z) + (p_\perp^2 + m^2) \times \Big[K_{i\nu}'^2(z) - \frac{\nu^2}{z^2}K_{i\nu}^2(z)\Big]\Big\}\frac{d^3k}{k_0}, \tag{14}$$

which agrees with the previously obtained relation[15]. The probability of radiation with spin flip vanishes in the classical limit.

In the other limiting case, when $B = eE/m^2 \to 0$, and the quantities

$$\chi = \frac{\sqrt{(eF_{\mu\nu}p_\nu)^2}}{m^3}, \quad \chi' = \frac{\sqrt{(eF_{\mu\nu}p_\nu')^2}}{m^3},$$
$$\varkappa = \frac{\sqrt{(eF_{\mu\nu}k_\nu)^2}}{m^3}, \quad \tau = \frac{eF_{\mu\nu}^*p_\mu p_\nu'}{\varkappa m^4}, \tag{15}$$

where

$$F_{\mu\nu}^* = {}^1\!/_2 i\varepsilon_{\mu\nu\sigma\lambda}F_{\sigma\lambda}, \quad F_{\mu\nu} = \partial A_\nu/\partial x_\mu - \partial A_\mu/\partial x_\nu,$$

remain finite, we obtain, as would be expected, for the probability of emission per unit proper time

$$\frac{dW}{ds} = \frac{1}{2}\sum_{r'r}\frac{dW_{n'n}}{ds}$$

an expression that coincides with the corresponding value for a crossed field (i.e., for a field with $\mathbf{EH} = 0$ and $\mathbf{E} = \mathbf{H}$)[6].

Let us now stop briefly to discuss the inverse process $\gamma + e \to e$ and assume that the initial particles are not polarized. Then

$$dW = \frac{1}{4}\sum_{e', r', r} |M_{n, n'}|^2 = Q\left\{\frac{1}{2}\sum_{r'r} |\quad|^2_{r'r}\right\}\frac{\delta_{\mathbf{p}, \mathbf{p}'+\mathbf{k}}}{2V}. \tag{16}$$

Here $\delta_{\mathbf{p},\mathbf{p}'+\mathbf{k}}$ is the Kronecker symbol; $\{\ \}$ is the same as in formula (8). The substitution $k_\mu \to -k$ has led to the replacement $\rho \to -\rho$, and since

$$I_{00}(\rho) \equiv I_{00}(\rho, \lambda', \lambda) = I_{00}(-\rho, \lambda, \lambda'), \tag{17}$$

the expression in the curly brackets in (8) remains unchanged; formula (16) can be recast in a more customary form by introducing in the righthand side the factors $n_\gamma V$ and $n_e V$, i.e., the total number of initial particles. Then the probability per unit volume is $V^{-1}dW \propto n_e n_\gamma$ and does not depend on V.

In concluding this section, we present the probability of emission of a photon by a scalar particle:

$$dW = \frac{\exp\{-\pi(\lambda + \lambda')/4\}}{8\pi^3 E^2 (1 + e^{-\pi\lambda})(1 + e^{-\pi\lambda'})} \tag{18}$$
$$\times \left\{\left[\frac{(F_{\mu\nu}{}^* (p + p')_\nu)^2}{-8\mathscr{F}} - \frac{4\nu^2 e^2 \mathscr{F}^2}{(F_{\mu\nu}{}^* k_\nu)^2}\right] |I_{1/2\,1/2}|^2 + eE|I'_{1/2\,1/2}|^2\right\}\frac{d^3k}{k_0}.$$

Here $\mathscr{F} = (1/4)F^2_{\mu\nu}$ and $I_{1/2\ 1/2}$ is obtained from $I_{00}(\rho)$ in (A.12) by making the substitutions $\lambda' \to \lambda' - i$ and $\lambda \to \lambda - i$.

3. PAIR PRODUCTION BY A PHOTON

The matrix element for the production of a pair by a photon is

$$M_{n'n} = \int {}^+\bar{\psi}_{n'}(x'')\beta G(x'', x) e\hat{A}'(x) G(x, x')\beta^- \psi_n(x') d^3x''\, d^4x\, d^3x'$$
$$= N_{n'}N_n \int {}_+\bar{\psi}_{n'}(x) e\hat{A}'(x)\, {}_-\psi_n(x) d^4x; \quad t''\ t' \to \infty$$
$$A' = \frac{e_\mu' e^{ikx}}{\sqrt{2Vk_0}}, \quad n' = r', \mathbf{p}', \quad n = r, -\mathbf{p}, \quad r', r = 1, 2. \tag{19}$$

We treat the encountered integrals with respect to t in the same manner as the integrals $J_{j'j}$ in the Appendix. In place of $I_{j'j}(\rho)$, defined in (A.10), we have

$$R_{j'j}(\rho) = \int_{-\infty}^{\infty} D_{-i\lambda'/2-j'}((1 + i)\xi) D_{-i\lambda/2-j}((1 + i)\xi) e^{i\rho\xi} d\xi, \tag{20}$$
$$j', j = 0, 1.$$

Only large negative ξ, i.e., large positive t, contribute to the asymptotic value of R_{00} as $\rho \to 0$. Just as for $I_{00}(\rho)$, we obtain

$$R_{00}(\rho) = \sqrt{\pi}\exp\left\{-\frac{i\pi}{4} - \frac{\zeta}{2}\right\}$$
$$\times \left[\frac{\Gamma(-i\nu)}{\Gamma(i\lambda'/2)}\exp\left\{-\frac{\pi\lambda'}{8} - \frac{3\pi\lambda}{8}\right\}\left(\frac{i\rho}{\sqrt{2}}\right)^{i\nu}\Phi\left(\frac{i\lambda}{2}, 1 + i\nu; \zeta\right)\right.$$
$$\left.+ \frac{\Gamma(i\nu)}{\Gamma(i\lambda/2)}\exp\left\{-\frac{\pi\lambda}{8} - \frac{3\pi\lambda'}{8}\right\}\left(\frac{i\rho}{\sqrt{2}}\right)^{-i\nu}\Phi\left(\frac{i\lambda'}{2}, 1 - i\nu; \zeta\right)\right]$$
$$= \sqrt{\pi}\exp\left\{-i\frac{\pi}{4} + \frac{\pi\lambda'}{8} - \frac{5\pi\lambda}{8} - \frac{\zeta}{2} + i\nu\ln\frac{\rho}{\sqrt{2}}\right\} \tag{21}$$
$$\times \Psi\left(\frac{i\lambda}{2}, 1 + i\nu; e^{2\pi i}\zeta\right);$$
$$\zeta = -\frac{i\rho^2}{2} = -\frac{ik_\perp^2}{2eE}. \tag{21'}$$

Assuming the photon to be unpolarized, we obtain for the differential probability of pair production by a photon

$$dW_{n'n} = \frac{1}{2}\sum_{e'} |M_{n'n}|^2 \frac{Vd^3p}{(2\pi)^3} = \frac{1}{2}Q|\quad|^2_{r'r}\frac{d^3p}{(2\pi)^3}, \tag{22}$$

where $\mathbf{k} = \mathbf{p} + \mathbf{p}'$ and

$$|\quad|_{11}^2 = p_\perp'^2 |R_{00}|^2 + {}^1/_4 p_\perp^2 \lambda'^2 |R_{11}|^2$$
$$+ \lambda' \operatorname{Re}\{[(p_1' + ip_2')(-p\ + ip_2) + m^2]R_{01}R_{10}^*\},$$
$$|\quad|_{22}^2 = p_\perp'^2 |R_{00}|^2 + {}^1/_4 p_\perp^2 \lambda'^2 |R_{11}|^2$$
$$+ \lambda' \operatorname{Re}\{[(p_1' - ip_2')(-p_1 - ip_2) + m^2]R_{01}R_{10}^*\},$$
$$|\quad|_{12}^2 = |\quad|_{21}^2 = m^2\{|R_{00}|^2 + {}^1/_4\lambda'^2|R_{11}|^2 + \lambda' \operatorname{Im} R_{00}R_{11}^*\}. \tag{23}$$

The $R_{j'j}$ are defined in (20). In terms of $\Psi(i\lambda/2, 1 + i\nu; e^{2\pi i}\zeta$ we have

$$\frac{1}{2}\sum_{r'r} |\quad|^2_{r'r} = \pi\exp\left\{\frac{\pi\lambda'}{4} - \frac{5\pi\lambda}{4}\right\}\left\{(p_\perp^2 + p_\perp'^2)|\Psi|^2\right.$$
$$+ [eE\rho^2 - 2(p_1p_1' + p_2p_2' - m^2)]\frac{\rho^2}{\lambda}|\Psi'|^2$$
$$\left.+ [-p_\perp^2\rho^2 + 2\nu(p_1p_1' + p_2p_2' - m^2)]\frac{2}{\lambda}\operatorname{Re}\Psi'\Psi^*\right\}, \tag{24}$$
$$|\quad|^2_{12} = |\quad|^2_{21} = m^2\pi\exp\left\{\frac{\pi\lambda'}{4} - \frac{5\pi\lambda}{4}\right\}\frac{\rho^4}{\lambda^2}|\Psi'|^2,$$
$$\rho = k_\perp/\sqrt{eE}.$$

In the case under consideration we can change over to a system where $k_3 = 0$. Then the transition from the analog of formula (A.1) to (20) becomes trivial, since

$$\xi' = -\sqrt{eE}\left(t - \frac{p_3'}{eE}\right) = \xi = -\sqrt{eE}\left(t + \frac{p_3}{eE}\right),$$
$$p_3 + p_3' = k_3 = 0.$$

Equally obvious is the relation

$$\int dp_3 = eET. \tag{25}$$

Introducing the total number of photons in the field, $n_\gamma V$, into the right-hand side of (22) and using (25), we obtain a probability proportional to VT.

Let us consider now the limiting case of soft photons. As expected, the probability of observing electrons in different spin states is low. When $\nu \ll \lambda$ and $\lambda \gg 1$ we have

$$\frac{1}{2}\sum_{r', r=1}^{2} |\quad|^2_{r'r} \approx \frac{1}{2}(|\quad|^2_{11} + |\quad|^2_{22}) \approx$$
$$\approx 2\exp\left\{-\pi\frac{\lambda + \lambda,}{4}\right\}\lambda\left\{p_\perp^2|K_{i\nu}|^2\right.$$
$$\left.+ (p_\perp^2 + m^2)\left[|K_{i\nu}'|^2 - \frac{\nu^2}{z^2}|K_{i\nu}|^2\right]\right\}, \tag{26}$$

where $z = \sqrt{\lambda\rho}$ and $K_{i\nu} = K_{i\nu}(e^{i\pi}z)$ is the Macdonald function. According to formulas (26) and (22), we can, roughly speaking, assume that the field generates a pair with a probability $e^{-\pi\lambda}$, and this pair then absorbs a soft photon.

Just as in the case of photon emission by an electron, we can consider the limit $B = eE/m^2 \to 0$. To find the asymptotic form of $R_{00}(\rho)$ in this case it is convenient to use the following integral representation of the function Φ (see[16]):

$$\Phi\left(\frac{i\lambda'}{2}, 1 - i\nu; \zeta\right) = \frac{1}{2\pi i}\Gamma\left(1 - \frac{i\lambda'}{2}\right)\Gamma(1 - i\nu)\Gamma^{-1}\left(1 - \frac{i\lambda}{2}\right)\int_c \frac{e^{f(t)}}{t}dt,$$
$$f(t) = -{}^1/_2 i[\rho^2 t - \lambda'\ln(-t) + \lambda\ln(1 - t)]. \tag{27}$$

The contour C begins at the point $t = 1$, goes counterclockwise around $t = 0$, and returns to the initial point. Only the second term in (21) gives a nonvanishing contribution. Expanding $f(t)$ at the saddle point t_0 in a Taylor series and retaining terms up to $f'''(t_0)$ inclusive, we obtain from (21) and (27)

$$R_{00}(\rho)\,|_{B\to 0,\,\chi+\chi'\approx\varkappa} \approx \sqrt{\pi}\left(\frac{2\sqrt{\lambda\lambda'}}{\sqrt{\lambda}+\sqrt{\lambda'}}\right)^{1/3} e^{\theta+i\pi/4}v(y),$$

$$\theta = -\frac{i\pi}{4}+\frac{\pi}{8}(\lambda+\lambda')+\frac{i\lambda'}{4}\left(1-\ln\frac{\lambda'}{2}\right)+\frac{i\lambda}{4}\left(1-\ln\frac{\lambda}{2}\right),$$

$$y = \left(\frac{2\sqrt{\lambda\lambda'}}{\sqrt{\lambda}+\sqrt{\lambda'}}\right)^{1/3}(\sqrt{\lambda}+\sqrt{\lambda'}-\rho), \tag{28}$$

$$v(y) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left\{i\left(ty+\frac{t^3}{3}\right)\right\}dt.$$

A direct calculation leads to the probability given in[6] for the formation of a pair in a crossed field. The next asymptotic term of such an expansion in terms of the parameter B was investigated by Narozhnyĭ[17].

A process inverse to that considered here is single-photon annihilation. It is easy to verify that the probability of this process is given by the same formula (22), but without the statistical weight $Vd^3p/(2\pi)^3$ and the factor ½ from averaging over the photon polarization.

4. EMISSION OF A PHOTON BY A PAIR

The Feynman diagram of this process differs from the diagram for the production of a pair by a photon only in the sign of the four-momentum k_μ. This means that in expressions (23) only $R_{j'j}(\rho)$ changes. From the procedure described in the Appendix (see formula (A.3)) it is clear that the reversal of the sign of k_0 leads to a change in the sign of ρ in formula (20), and this is equivalent to the substitution $\rho \to e^{-i\pi}\rho$ in formula (21). We then obtain in the usual manner

$$dW_{n'n} = \sum_{e'}|M_{n'n}|^2\frac{V^2d^3pd^3k}{(2\pi)^6} = Q|\;\;|^2_{r'r}\frac{Vd^3pd^3k}{(2\pi)^6} \tag{29}$$
$$\mathbf{p}+\mathbf{p}'+\mathbf{k}=0.$$

Here $|\;\;|^2_{r'r}$ is obtained from (23) by means of the substitution $\rho \to e^{-i\pi}\rho$. To obtain the total probability, the integrations in (29) are best carried out in a cylindrical coordinate system

$$d^3pd^3k = p_\perp dp_\perp d\varphi dp_3k_\perp dk_\perp d\varphi' dk_3.$$

If we write

$$\int_0^{2\pi}d\varphi\int_0^{2\pi}d\varphi'\ldots = \frac{1}{2}\int_{-2\pi}^{2\pi}d\psi'\int_{\psi'}^{4\pi-\psi'}d\psi\ldots, \tag{30}$$

where $\psi = \varphi + \varphi'$ and $\psi' = \varphi - \varphi'$, then the integral with respect to ψ can be calculated in elementary fashion. The integration with respect to p_3 reduces to the use of a correspondence of the type (25). Finally, the divergence of the integral with respect to k_3 must be understood in the same sense as in formula (9).

The probability of the inverse process, i.e., annihilation of a pair with absorption of an unpolarized photon, is given by the expression

$$dW_{n'n} = {}^1/_2Q|\;\;|^2_{r'r}\delta_{\mathbf{k}+\mathbf{p}+\mathbf{p}',0}/V. \tag{31}$$

Here $|\;\;|^2_{r'r}$ is the same as in formula (29).

In conclusion, the author thanks V. I. Ritus for useful remarks.

APPENDIX

The probability of emission of a photon by an electron is expressed in terms of the integrals

$$J_{j'j} = \int_{-\infty}^{\infty}D_{-i\lambda'/2-j'}((1+i)\xi')D_{-i\lambda/2-j}(-(1+i)\xi)e^{ik_0t}dt, \quad j',j=0,1;$$

$$\xi' = -\sqrt{eE}\left(t-\frac{p_3'}{eE}\right) = u-u_0', \quad \xi = -\sqrt{eE}\left(t-\frac{p_3}{eE}\right) = u-u_0. \tag{A.1}$$

It suffices to consider J_{00}, since the remaining integrals are obtained from it by shifts of λ and λ'. Comparing the probabilities of the emission of a classical particle in a magnetic[18] and an electric[15] field with the quantum theory of radiation in a magnetic field[19], we expect to be able to express J_{00} in terms of a confluent hypergeometric function Ψ.

We proceed now to calculate the integral J_{00}. By means of the substitutions

$$v = u-(u_0+u_0')/2, \quad v_3 = u_0-u_0' = (p_3'-p_3)/\sqrt{eE}$$

we obtain

$$J_{00} = \frac{1}{\sqrt{eE}}\exp\left\{-\frac{ik_0}{\sqrt{eE}}\frac{u_0+u_0'}{2}\right\}\int_{-\infty}^{\infty}D_{-i\lambda'/2}\left((1+i)\left(v+\frac{v_3}{2}\right)\right)$$
$$\times D_{-i\lambda/2}\left(-(1+i)\left(v-\frac{v_3}{2}\right)\right)\exp\left\{-\frac{ik_0}{\sqrt{eE}}v\right\}dv. \tag{A.2}$$

It is then convenient to investigate in lieu of (A.2) the more general integral

$$J(\rho,\varphi) = \int_{-\infty}^{\infty}f_{\Lambda'}\left(v+\frac{v_3}{2}\right)f_\Lambda\left(v-\frac{v_3}{2}\right)e^{-iv_0v}dv, \tag{A.3}$$
$$v_0 = \rho\,\mathrm{ch}\,\varphi, \quad v_3 = \rho\,\mathrm{sh}\,\varphi, \quad \rho^2 = v_0^2-v_3^2,$$

where $f_\Lambda(\xi)$ is the solution of the equation

$$\left(\frac{d^2}{d\xi^2}+\xi^2+\Lambda\right)f_\Lambda(\xi)=0. \tag{A.4}$$

It is easy to verify that

$$\partial J(\rho,\varphi)/\partial\varphi = {}^1/_2i(\Lambda'-\Lambda)J(\rho,\varphi),$$

i.e.,

$$J(\rho,\varphi) = \exp\left\{i\frac{\Lambda'-\Lambda}{2}\varphi\right\}J(\rho,0). \tag{A.5}$$

Thus, the problem has reduced to a calculation of the integral

$$I(\rho) = J(\rho,0) = \int_{-\infty}^{\infty}f_{\Lambda'}(v)f_\Lambda(v)e^{-i\rho v}dv,$$
$$\rho = \begin{cases}\sqrt{v_0^2-v_3^2} & \text{for } v_0>0\\ -\sqrt{v_0^2-v_3^2} & \text{for } v_0<0\end{cases} \tag{A.6}$$

In analogous fashion, using (A.4) and integration by parts, we can set up an equation for $I(\rho)$[1)]:

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\frac{(\Lambda'-\Lambda)^2}{4\rho^2}+\frac{\rho^2}{4}-\frac{\Lambda'+\Lambda}{2}\right]I(\rho)=0. \tag{A.7}$$

[1)]It is surprising that Eq. (A.7) actually coincides with the radial Klein-Gordon equation, if the electric field is described by the potential $A_\mu = (0, 0, -½E\tau\cosh\varphi, -½E\tau\sinh\varphi)$, $\tau^2 = t^2-x_3^2$, $\tanh\varphi = x_3/t$. In this case the quantity $(k_0^2-k_3^2)eE$ corresponds to $eE(t^2-x_3^2)$.

This equation can be easily reduced to the equation for the confluent hypergeometric function (see, for example,[20]), so that $I(\rho)$ should have the form

$$I(\rho)=e^{-\zeta/2}\left\{C_1\zeta^{i\beta}\Phi\left(\frac{i\Lambda}{2}+\frac{1}{2},1+2i\beta;\zeta\right)+C_2\zeta^{-i\beta}\Phi\left(\frac{i\Lambda'}{2}+\frac{1}{2},1-2i\beta;\zeta\right)\right\}, \quad (A.8)$$

where $\zeta=-i\rho^2/2$, $\beta=(\Lambda-\Lambda')/4$ and $\Phi(\alpha,\gamma,\zeta)$ is the confluent hypergeometric function.

We now turn to the integral $I_{00}(\rho)$. Using (A.5) and (A.8) with $\Lambda=\lambda+i$, $\Lambda'=\lambda'+i$, we obtain

$$J_{00}=\frac{1}{\sqrt{eE}}\exp\left\{-\frac{ik_0}{\sqrt{eE}}\frac{u_0+u_0'}{2}-i\nu\varphi\right\}I_{00}(\rho),$$
$$\nu=\frac{\lambda-\lambda'}{2},\quad \operatorname{th}\varphi=\frac{v_3}{v_0},\quad \rho=\frac{k_\perp}{\sqrt{eE}}; \quad (A.9)$$

$$I_{j'j}(\rho)=\int_{-\infty}^{\infty}D_{-i\lambda'/2-j}((1-i)\xi)D_{-i\lambda/2-j}(-(1+i)\xi)e^{-i\rho\xi}d\xi,\quad j',j=0,1, \quad (A.10)$$

with

$$I_{00}(\rho)=e^{-\zeta/2}\left\{C_1\zeta^{i\nu/2}\Phi\left(\frac{i\lambda}{2},1+i\nu;\zeta\right)+C_2\zeta^{-i\nu/2}\Phi\left(\frac{i\lambda'}{2},1-i\nu;\zeta\right)\right\}, \quad (A.11)$$

where C_1 and C_2 are determined by the behavior of the integral (A.10) with $j'=j=0$ as $\rho\to 0$. We finally obtain

$$I_{00}(\rho)=\sqrt{\pi}\exp\left\{-\frac{i\pi}{4}-\frac{\zeta}{2}\right\}\times\left[\frac{\Gamma(-i\nu)}{\Gamma(i\lambda'/2)}e^{\pi\nu/4}\left(\frac{-i\rho}{\sqrt{2}}\right)^{i\nu}\Phi\left(\frac{i\lambda}{2},1+i\nu;\zeta\right)+\frac{\Gamma(i\nu)}{\Gamma(i\lambda/2)}e^{-\pi\nu/4}\left(\frac{i\rho}{\sqrt{2}}\right)^{-i\nu}\Phi\left(\frac{i\lambda'}{2},1-i\nu;\zeta\right)\right]$$
$$=\sqrt{\pi}\exp\left\{-\frac{i\pi}{4}+\frac{3\pi\nu}{4}-\frac{\zeta}{2}+i\nu\ln\frac{\rho}{\sqrt{2}}\right\}\Psi\left(\frac{i\lambda}{2},1+i\nu;\zeta\right). \quad (A.12)$$

It can be verified that the relations between $J_{j'j}$ turn out to be precisely those needed for gauge invariance of the matrix element.

[1] L. D. Landau and E. M. Lifshits, Sov. Phys. **6**, 244 (1934).
[2] H. Bethe and W. Heitler, Proc. Roy. Soc., **A146**, 83, 1934. F. Sauter, Ann. der Phys., **20**, 404 (1934).
[3] J. M. Jauch and F. Rohrlich, The theory of Photons and Electrons, Addison-Wesley, Cambridge, 1955.
[4] H. Bethe and L. Maximon, Phys. Rev. **93**, 768 (1954).
[5] H. R. Reiss, J. Math. Phys., **3**, 59 (1962).
[6] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776 (1964); **52**, 1707 (1967) [Sov. Phys.-JETP **19**, 529 (1964); **25**, 1135 (1967)].
[7] J. Schwinger, Phys. Rev., **82**, 664 (1951).
[8] A. I. Nikishov, Zh. Eksp. Teor. Fiz. **57**, 1210 (1969) [Sov. Phys.-JETP **30**, 660 (1970)].
[9] F. V. Bunkin and I. I. Tugov, Dok. Akad. Nauk SSSR **187**, 541 (1969) [Sov. Phys.-Dokl. **7**, 678 (1969)].
[10] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya élektrodinamika, (Quantum Electrodynamics) Fizmatgiz, 1959. [Interscience, 1965].
[11] R. P. Feynman, Phys. Rev., **76**, 749 (1949).
[12] N. B. Narozhnyĭ and A. I. Nikishov, Yad. Fiz. **11**, 1072 (1970) [Sov. J. Nucl. Phys. **11**, 596 (1970)].
[13] R. P. Feynman, Phys. Rev., **76**, 769 (1949).
[14] Higher Transcendental Function 1, McGraw-Hill Book Company, 1953.
[15] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **56**, 2035 (1969) [Sov. Phys.-JETP **29**, 1093 (1969)].
[16] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics) Fizmatgiz, 1963 [Addison-Wesley, 1965].
[17] N. B. Narozhnyĭ, Zh. Eksp. Teor. Fiz. **54**, 676 (1968) [Sov. Phys.-JETP **27**, 360 (1968)].
[18] L. D. Landau and E. M. Lifshitz, Teoriya polya, Fizmatgiz, 1962, Sec. 74 [Classical Theory of Fields, Addison-Wesley, 1965].
[19] N. P. Klepikov, Zh. Eksp. Teor. Fiz. **26**, 19 (1954).
[20] V. I. Kogan and V. M. Galitskiĭ, Sbornik zadach po kvantovoĭ mekhanike (Collection of Problems in Quantum Mechanics) Gostekhizdat, 1956, p. 312.

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