EFFICIENCY OF COLLISIONS BETWEEN ELECTRONS AND LONG-WAVE PHONONS IN POLYVALENT METALS

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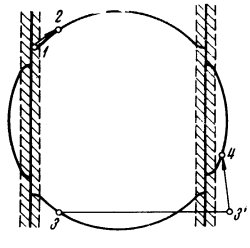
The role of "effective" electron collisions with long-wave phonons in the vicinity of the Bragg planes is considered for metals which can be described by the weak pseudopotential model. It is shown that the contribution of the "effective" collisions to the electric conductivity for $\omega\tau \gg 1$ is of major importance at comparatively low temperatures and frequencies and leads to a change in the frequency and temperature dependences of the electric conductivity. In the case of static electric conductivity, the "effective" electron-phonon collisions determine the temperature dependence of the resistance at low temperatures for not very pure metals, where the collisions between electrons and impurities are predominant.

As is well known, the collisions of electrons with phonons of small momentum provide for the static electric resistance of pure metals at low temperatures. The rapid decrease in the electric resistance with temperature (according to the T^5 law) is explained first by the decrease in the number of phonons in the metal and, second, by the decrease in the effectiveness of the individual collision. The low efficiency of collisions of electrons with thermal phonons at low temperatures is connected with the fact that each collision changes the velocity of the electron slightly and, consequently, has little effect on the electric current.

A similar situation holds for the electric conductivity in a high-frequency electric field. For field frequencies $\omega\tau \gg 1$, the electric conductivity is proportional to the effective frequency of collisions of the electrons with phonons[1)] $1/\tau = \nu$, which falls off with temperature according to the law T^5 in the temperature range $\hbar\omega \ll T \ll \Theta$ (Θ is the Debye temperature). If the energy of the electromagnetic quantum exceeds the temperature spreading out of the Fermi surface ($\hbar\omega \gg T$), processes begin to dominate in which the electron, absorbing the electromagnetic quantum, emits a phonon. In these processes, the role of the temperature is played by the quantity $\hbar\omega$ and, as was shown by Gurzhi,[1] the effective collision frequency is $\nu \propto \omega^5$ (for $T \ll \hbar\omega \ll \Theta$).

We emphasize once again that the dependences shown above follow from the assumed low efficiency of the individual collision of the electron with a long-wave phonon; in other words, it is assumed that the velocity of the electron is changed materially in momentum space over distances of the order of Θ/s (s is the velocity of sound). This assumption may not be satisfied, however, for different special electron dispersion laws. In such cases, the results are valid only for such low temperatures and frequencies for which the quantities T/v_F and $\hbar\omega/v_F$ are much smaller than those distances in p space over which the velocity of the electron changes appreciably. At the present time, there are researches (see, for example, [2-4]) in which the temperature dependences of the static electric resistance have been determined; these differ from T^5 at not very low temperatures. It will be shown below that for metals described by the weak pseudopotential method (see [5]) the assumption of the low efficiency of the collisions is not valid for all the electrons on the Fermi surface.

We consider a metal the Fourier components of whose pseudopotential V_g (**g** is the corresponding vector of the reciprocal lattice) satisfy the condition $V_g/\epsilon_F \ll 1$ and for which the Fermi surface intersects the Bragg planes (see the drawing). The pseudopotential of the lattice has a significant effect on the dispersion relation of the electrons only in the narrow layer of p space close to the Bragg plane, the width of which is of the order of gV_g/ϵ_F (the shaded regions on the drawing); therefore the velocity of the electrons with quasimomentum outside this layer differs little from the velocity of the free electron. For the same Bragg plane, the component of the velocity normal to it is equal to zero. Therefore, a collision with a phonon which takes the electron from the shaded portion of the drawing strongly changes its velocity for an insignificant quasimomentum change ($\Delta v \approx v_F$ for $\Delta p \gtrsim gV_g/\epsilon_F$). In other words, the collisions which bring the electron from the Bragg region (transition $1 \to 2$ on the drawing) are "efficient," and the same, it seems, applies to the opposite transitions. Although only electrons from a narrow layer close to the Bragg plane can take part in the "effective" collisions, it will be shown below that such processes can give a decisive contribution to the high-frequency electrical conductivity.



Intersection of the Fermi surface with a pair of Bragg planes. $1 \to 2$ and $3 \to 4$ are transitions for "effective" collisions, $3 \to 3'$ is the reciprocal lattice vector.

[1)] We refer to that part of the electric conductivity which is connected with volume collisions of the electrons.

Collisions with a Umklapp[2] (the transition 3 → 4 in the drawing) are another type of collision with low-momentum phonons that strongly change the electron velocity. We note that in order for such collisions to be "effective," the initial state of the electron should not lie in the shaded region.

For the determination of the contribution of the aforementioned collisions to the electrical conductivity of the metal, it is first necessary to calculate the matrix element of the electron-phonon collision in the weak pseudopotential model.

1. THE MATRIX ELEMENT OF ELECTRON-PHONON INTERACTION

In the fixed-ion approximation, the matrix element of the electron-phonon interaction is equal to[6]

$$B_{\mathbf{p}\mathbf{p}_1\mathbf{q}\lambda} = \sum_{(\mathbf{p}'=\mathbf{p}+\mathbf{g})} c^{*}_{\mathbf{p}\mathbf{p}'} c_{\mathbf{p}_1\mathbf{p}_1'} A_{\mathbf{p}'\mathbf{p}_1'\mathbf{q}\lambda}, \tag{1}$$

$$A_{\mathbf{p}\mathbf{p}_1\mathbf{q}\lambda} = i\alpha_{\mathbf{q}\lambda}\mathbf{e}_{\mathbf{q}\lambda}(\mathbf{p}_1 - \mathbf{p})\, u_{|\mathbf{p}_1-\mathbf{p}|}\delta_{\mathbf{p}-\mathbf{p}_1+\mathbf{q}+\mathbf{g}}, \quad \alpha_{\mathbf{q}\lambda} = (2\hbar\Omega\rho\omega_{\mathbf{q}\lambda})^{-1/2},$$

where $A_{\mathbf{p}\mathbf{p}_1\mathbf{q}\lambda}$ is the matrix element in the plane wave representation, $c_{\mathbf{p},\mathbf{p}+\mathbf{g}}$ are the expansion coefficients of the Bloch wave function in plane waves, $\mathbf{e}_{\mathbf{q}\lambda}$ is the polarization vector of the phonon, $\omega_{\mathbf{q}\lambda}$ the frequency of the phonon with momentum $\mathbf{q}$ and polarization λ (we shall omit the index λ in what follows), u_p the Fourier transform of the pseudopotential of a single ion, Ω the volume of the metal, and ρ its density. Far from the Bragg planes, the Bloch wave function differs slightly from a plane wave and in this case $B_{\mathbf{p}\mathbf{p}_1\mathbf{q}} \approx A_{\mathbf{p}\mathbf{p}_1\mathbf{q}}$. For example, if the state $\mathbf{p}$ is located near one of the Bragg planes, then it is necessary to leave the two terms in the sum (1) which contain $c_{\mathbf{p}\mathbf{p}}$ and $c_{\mathbf{p},\mathbf{p}+\mathbf{g}}$, where $\mathbf{g}$ is the corresponding vector of the reciprocal lattice. We shall not consider the regions of $\mathbf{p}$ space close to the interaction of several Bragg planes; such regions make a contribution to the conductivity of higher order of smallness in the parameter V_g/ϵ_F. The quantities $c_{\mathbf{p}\mathbf{p}}$ and $c_{\mathbf{p},\mathbf{p}+\mathbf{g}}$ are easily computed from perturbation theory (by using the secular equation):

$$c_{\mathbf{p}\mathbf{p}} = \{2(1 + x^2 - \sqrt{x^4 + x^2})\}^{-1/2}, \quad x = \frac{g(g/2 - p_\perp)}{2mV_g},$$

$$c_{\mathbf{p},\mathbf{p}+\mathbf{g}} = -\operatorname{sign} x\{2(1 + x^2 + \sqrt{x^4 + x_2})\}^{-1/2},$$

where $p_\perp$ is the quasimomentum component perpendicular to the Bragg plane.

Let us consider in more detail the case of interaction of electrons with phonons of low momentum $q \ll p_F$, a case that is important for what follows. In this case, $\mathbf{p}_i - \mathbf{p} \approx \mathbf{g}$ and the Fourier transform of the pseudopotential $u_{|\mathbf{g}+\mathbf{q}|}$ can be expanded in powers of $\mathbf{q}$ near this value of the argument:

$$u_{|g+q|} = u_g + \frac{du_g}{dp}\frac{\mathbf{g}}{g}\mathbf{q} + \dots .$$

As is known from the theory of the pseudopotential,[5] the Fourier transform u_p is a rapidly decaying function of p: $u_0 \approx \epsilon_F$, $u_g = V_g$; consequently, $u_g/u_0 \approx V_g/\epsilon_F \ll 1$. Therefore, for $q/p_F \gg V_g/\epsilon_F$ terms containing u_g can be omitted in (1) in comparison with the term containing u_0. In the general case, it is impossible,to speak of terms containing the derivative du_g/dp, since in real situations, the value of this derivative changes in the limits $V_g/g \lesssim du_g/dp \lesssim \epsilon_F/g$.

Thus, for the matrix element of electron-phonon interaction, we have for $p_F V_g/\epsilon_F \ll q \ll p_F$:

$$A = A^{I} + A^{II},$$

$$A^{I}_{\mathbf{p}\mathbf{p}_1\mathbf{q}} = i\alpha_{\mathbf{q}}(\mathbf{e}_{\mathbf{q}}\mathbf{q})u_0\delta_{\mathbf{p}-\mathbf{p}_1+\mathbf{q}}, \tag{2}$$

$$A^{II}_{\mathbf{p}\mathbf{p}_1\mathbf{q}} = i\alpha_{\mathbf{q}}\frac{(\mathbf{e}_{\mathbf{q}}\mathbf{g})}{g}\frac{du_g}{dp}(\mathbf{q}\mathbf{g})\delta_{\mathbf{p}-\mathbf{p}_1+\mathbf{q}+\mathbf{g}}, \quad \mathbf{g} \neq 0, \tag{3}$$

and correspondingly, $B = B^{I} + B^{II}$.

We note that the probability of collisions with change-over, contained in B^{I}, is not small only near the Bragg planes.

For simplicity, we assume that the expressions (2) and (3) are valid even for $q \approx p_F$, which, as it is not difficult to show, does not significantly affect the results given below.

2. HIGH-FREQUENCY ELECTRICAL CONDUCTIVITY OF METALS

In this section, we shall discuss the high-frequency electrical conductivity of metals described by the weak pseudopotential model, the Fermi surface of which intersects the Bragg plane. Such a situation occurs for many polyvalent metals.

For simplicity, we limit ourselves to the consideration of metals with a cubic lattice, the electrical conductivity of which is isotropic. Generalization to the anisotropic case does not present any difficulty. By solving the Gurzhi kinetic equation by means of the method of successive approximations,[1,7] it is not difficult to show that the electrical conductivity of the metal, which is associated with electron-phonon collisions in the range of frequencies $1/\tau \ll \omega \ll \omega_t$, where ω_t is the threshold frequency of the internal photoeffect, is determined by the expression

$$\sigma = \frac{2\pi e^2}{3\hbar^2\Omega\omega^3}\sum_{\mathbf{p}\mathbf{p}_1\mathbf{q}} (\mathbf{v}_{\mathbf{p}_1} - \mathbf{v}_{\mathbf{p}})^2 |B_{\mathbf{p}\mathbf{p}_1\mathbf{q}}|^2 [(n_{\mathbf{p}}\bar{n}_{\mathbf{p}_1}N_{\mathbf{q}} - \bar{n}_{\mathbf{p}}n_{\mathbf{p}_1}\bar{N}_{\mathbf{q}})$$

$$\times\delta(\varepsilon_{\mathbf{p}_1} - \varepsilon_{\mathbf{p}} - \hbar\omega_{\mathbf{q}} - \hbar\omega) + (n_{\mathbf{p}}\bar{n}_{\mathbf{p}_1}\bar{N}_{\mathbf{q}} - \bar{n}_{\mathbf{p}}n_{\mathbf{p}_1}N_{\mathbf{q}})\delta(\varepsilon_{\mathbf{p}_1} - \varepsilon_{\mathbf{p}} + \hbar\omega_{\mathbf{q}} - \hbar\omega)],$$

$$\bar{n}_{\mathbf{p}} = 1 - n_{\mathbf{p}}, \quad \bar{N}_{\mathbf{q}} = 1 + N_{\mathbf{q}}, \tag{4}$$

where $\mathbf{v}_p$ is the velocity of the electron, n_p and N_q are the equilibrium distribution functions of electrons and phonons, ϵ_p is the energy of the electron. The quantities ϵ_p and $\mathbf{v}_p$ in the weak pseudopotential model can be computed by perturbation theory (by using the secular equation):

$$\varepsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} + V_g x(1 - \sqrt{x^{-2} + 1}), \quad x = \frac{g(g/2 - p_\perp)}{2mV_g},$$

$$\mathbf{v}_{\mathbf{p}} = \frac{\mathbf{p}}{m} + \frac{\mathbf{g}}{2m}\left(1 - \frac{1}{\sqrt{x^{-2} + 1}}\right).$$

In the following, we shall use the isotropic Debye model for the phonons.

[2] The "expanded band" scheme is used. By a collision with Umklapp, we mean a collision with nonconservation of quasimomentum. In the "reduced band" scheme, some of such collisions (for example, the transition 3 → 4 in the drawing) must be regarded as interband transitions without Umklapp.

We shall first determine σ_0—the electrical conductivity without account of the contribution of the Bragg regions of **p** space and without account of changeovers (i.e., without the term B^{II} in the matrix element). For this purpose, we can use the free electron model. We introduce the effective collision frequency of the electron, defining it by the equality

$$\sigma = e^2 N\nu / m\omega^2, \tag{5}$$

where N is the density of valence electrons. Simple calculation gives

$$\nu_0 = \gamma\varphi\left(\frac{T}{\Theta}, \frac{\omega}{\omega_D}\right); \quad \gamma = \frac{q_D{}^5\omega_D{}^2 m}{3(2\pi)^3\rho s\hbar^5 N},$$
$$\varphi(x,y) = \int_0^1 \Phi(x,y,t)\,t^4\,dt, \tag{6}$$
$$\Phi(x,y,t) = \frac{2}{e^{t/x}-1} + \frac{t/y-1}{e^{(t-y)/x}-1} - \frac{t/y+1}{e^{(t+y)/x}-1},$$

where q_D is the limiting momentum of the phonons, ω_D the Debye frequency. This expression, with accuracy to within a coefficient, agrees with the result from [1]. The asymptotes of the dependence of the effective frequency of collisions on ω and T for $\omega \ll \omega_D$, $T \ll \Theta$ have the form

$$\nu_0 = \begin{cases} 248\gamma(T/\Theta)^5 & \text{for} \quad T \gg \hbar\omega \\ \frac{1}{30}\gamma(\omega/\omega_D)^5 & \text{for} \quad T \ll \hbar\omega. \end{cases} \tag{7}$$

We now determine the contribution to the electrical conductivity of the "effective" collisions, i.e., collisions with a large change in the velocity of the electron. The contribution of these processes divides into two parts: σ_I and σ_{II}, corresponding to the parts of the matrix elements B^I and B^{II}. Thus the total electrical conductivity is equal to $\sigma = \sigma_0 + \sigma_I + \sigma_{II}$.

The electrical conductivity σ_I is determined by processes for which the initial or final state of the electron lies in the Bragg region. Actually, the changeover-free collision is effective if the electron before or after the collisions is in the shaded region on the drawing; in this same case, the probability of collision with changeover is not small (it is determined by the matrix element B^I. Calculation of σ_I is materially simplified if one of the following conditions is satisfied: $T/\Theta \gg V_g/\epsilon_F$ or $\omega/\omega_D \gg V_g/\epsilon_F$. In this case, we obtain

$$\nu_I = \sum_g n_g \nu_I{}^g, \quad \nu_I{}^g = \gamma\frac{\pi g V_g m}{2q_D{}^2 p_F}\psi_I\left(\frac{T}{\Theta}, \frac{\omega}{\omega_D}\right),$$
$$\psi_I(x,y) = \int_0^1 \Phi(x,y,t)\,t^2\,dt, \tag{8}$$

where n_g is the number of physically equivalent Bragg planes corresponding to a given $|\mathbf{g}|$. For $\omega < \omega_D$ and $T \ll \Theta$, we have, for the quantity ν_I^g,

$$\nu_I{}^g = \begin{cases} 22.7\gamma\frac{gV_g m}{q_D{}^2 p_F}\left(\frac{T}{\Theta}\right)^3 & \text{for} \quad T \gg \hbar\omega, \\ \gamma\frac{\pi g V_g m}{24 q_D{}^2 p_F}\left(\frac{\omega}{\omega_D}\right)^3 & \text{for} \quad T \ll \hbar\omega. \end{cases} \tag{9}$$

The fundamental contribution to the effective frequen- ν_{II}^g

$$\nu_{II} = \sum_g n_g \nu_{II}{}^g, \quad \nu_{II}{}^g = \frac{\gamma g^3 (du_g/dp)^2}{3q_D u_0{}^2}\psi_{II}\left(\frac{T}{\Theta}, \frac{\omega}{\omega_D}\right),$$

$$\psi_{II}(x,y) = \int_0^1 \Phi(x,y,t)\,t^3\,dt. \tag{10}$$

For $T \ll \Theta$ and $\omega \ll \omega_D$ the asymptotes of the collision frequency ν_{II}^g have the form

$$\nu_{II}{}^g = \begin{cases} \frac{8\pi^4\gamma g^3 (du_g/dp)^2}{45 q_D u_0{}^2}\left(\frac{T}{\Theta}\right)^4 & \text{for} \quad T \gg \hbar\omega \\ \frac{\gamma g^3 (du_g/dp)^2}{60 q_D u_0{}^2}\left(\frac{\omega}{\omega_D}\right)^4 & \text{for} \quad T \ll \hbar\omega. \end{cases} \tag{11}$$

The results given are not difficult to explain from intuitive physical considerations. For definiteness, we limit ourselves to the case $T \gg \hbar\omega$; moreover, we shall consider that the temperature momentum of the phonons is much greater than the width of the Bragg region. Under such conditions, the effective collision frequency for electrons from the Bragg region differs from the usual value $\nu_0 \approx \gamma(T/\Theta)^5$ by the absence of the factor $(T/\Theta)^2$, which is connected with the low effectiveness of the single collisions. This refers both to the Umklapp-free transitions from the Bragg region, and also to the collisions which are accompanied by Umklapp. Moreover, it is clear that one must add the factor S_B/S_F to the expression for ν_I, where S_F is the area of the Fermi surface and $S_B \approx p_F^2 V_g/\epsilon_F$ is the area of that part of it which falls in the Bragg region. Finally, we get $\nu_I \approx \gamma(V_g/\epsilon_F)(T/\Theta)^3$, which agrees with (9) in temperature dependence and order of magnitude.

The effective frequency ν_{II} is determined by the collisions with changeover of electrons with quasimomenta outside the Bragg region. As is seen from the drawing, the Umklapp processes are possible only from those states on the Fermi surface which are found at distances from the Bragg plane that are no larger than the temperature momentum of the phonon **q**. Therefore, the role of the area S_B in the given case will be played by the quantity $qp_F \approx p_F^2(T/\Theta)$ and as a result, we get $\nu_{II} \approx \gamma(T/\Theta)^4$.

It is seen from a comparison of Eqs. (9), (11), and (7) that at low temperatures and frequencies ($T \ll \Theta$ and $\omega \ll \omega_D$) the "effective" collisions make the principal contribution to the electrical conductivity of the metal that is associated with volume collisions. For example, for T/Θ, $\omega/\omega_D \approx V_g/\epsilon_F$, the ratio of the effective frequencies ν_I/ν_0 is of the order of ϵ_F/V_g.[3] We recall that "effective" collisions of the electrons are possible only with phonons whose momentum $q \gtrsim p_F(V_g/\epsilon_F)$. Therefore, for T/Θ and $\omega/\omega_D \ll V_g/\epsilon_F$, the number of such collisions is negligible and the total electrical conductivity is determined by Eq. (5) with $\nu = \nu_0$.

The relation of the effective collision frequencies ν_I and ν_{II} depends on the value of the ratio $(du_g/dp)^2/(u_0/g)^2$. For different metals the value of this quantity varies in the range from $(V_g/\epsilon_F)^2$ to unity.

We now discuss the possibility of the experimental discovery of the indicated dependences of the collision frequency on ω and T. As has already been pointed out, the contribution of the "effective" collisions is most

[3] In the comparison of the quantities ν_I and ν_{II} with it should be kept in mind that for most polyvalent metals, $n_g \approx 10$.

significant at low temperatures $T \ll \Theta$ and frequencies $\omega \ll \omega_D$. However, for sufficiently low frequencies, Eq. (5) for the electrical conductivity is invalid as the result of the anomalous skin effect. It was shown in the work of Motulevich [8] that the corrections to the surface impedance associated with the anomaly are small if the quantity $\frac{3}{8}(v_F/c)[1 + (n/\chi)^2]^{1/2}[1 + (\nu/\omega)^2]^{-1/2}$ (n and χ are the real and imaginary parts of the index of refraction) is smaller than or of the order of unity. For $\omega \gg \nu$, this condition leads to the following limitation on the frequency of the electromagnetic field: $\omega \gtrsim 10^{13}$ Hz.

Up to now we have considered the case of comparatively high frequencies of the electromagnetic field, for which $\omega\tau \gg 1$. We shall show that the dependences of the frequency of the electron-phonon collisions on the temperature, obtained above, are also valid for the static electrical conductivity if collisions of electrons with impurities are predominant. In this case, the small contribution to the distribution function of the electrons, which is associated with electron-phonon collisions, can be determined by iteration of the kinetic equation.

If we assume that each collision of an electron with an impurity is "effective" (i.e., the mean momentum transferred to the electron in the collision is of the order of p_F), then we get for the resistivity $\rho = 1/\sigma$,

$$\rho = m(\nu_{ei} + \nu_{ep}) / e^2 N.$$

In this formula, ν_{ei} is the frequency of collisions of electrons with impurities, and the frequency of the electron-phonon collisions $\nu_{ep}(T) = \nu_0 + \nu_I + \nu_{II}$, where the quantities ν_0, ν_I and ν_{II} are determined by Eqs. (6), (8), and (10) in their temperature dependence and their order of magnitude; here one must set $\omega = 0$ (exact agreement is obtained if the approximation of the relaxation time is applicable for collisions of electrons with impurities).

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