CONTRIBUTION TO THE THEORY OF BACK SCATTERING OF FAST ELECTRONS FROM A CONTINUOUS MEDIUM. I. NONRELATIVISTIC CASE

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We develop a quantum-mechanical approach to the theory of backward scattering of fast charged particles from continuous media. The differential back-scattering coefficient is represented in the form of a series, the n-th term of which describes n acts of elastic scattering through a large angle, with the particle experiencing multiple small-angle elastic and inelastic scattering between the collisions that lead to the large-angle scattering. It is shown that for not very large Z the first term of this series gives a reasonable approximation for the differential back-scattering coefficient, which agrees with the experimental results.

1. INTRODUCTION

WHEN a beam of fast electrons interacts with a thick target, some of the electrons are stopped and remain in the target, and some are scattered backwards. Among the back-scattered electrons there are always secondary electrons, but their energy is on the order of 10 electron volts, and they are eliminated in experiments by establishing a suitable energy threshold for the recording apparatus. Therefore the back-scattered electrons are usually defined as those having an energy exceeding 50 eV. Back scattering of fast electrons from solid amorphous bodies was investigated in sufficient detail experimentally by Dressel[1] and by Kanter[2]. Numerous attempts to construct an elementary theory on the basis of the diffusion model[3] or the model of single scattering[4] did not lead to noticeable success. On the other hand, the solution of the problem with the aid of the transport equation is made difficult by the considerable computational difficulties. Indeed, a determination of the angular and energy distributions of the back-scattered electrons with the aid of ordinary transport-theory methods calls for determinations of the energy and angular distribution functions of the electrons at each point inside the target.

However, the simplest problem is to obtain the angular and energy distributions of the back-scattered electrons without the need for considering the distributions inside the target. To this end, we develop here a quantum-mechanical approach that makes it possible to represent the differential back-scattering coefficient in the form of a series, each term of which describes a definite number of elastic scatterings through a large angle, $\theta \gg \kappa/p$ (where p is the electron momentum and $\kappa = me^2 Z^{-1/3}$ is the reciprocal Thomas-Fermi radius of the atom of the medium, with $\hbar = c = 1$). Then the electron experiences multiple small-angle elastic and inelastic scattering between the acts of large-angle scattering. The quantum-mechanical approach makes it possible to justify in the simplest manner the validity of such a representation, and also to show that at not very large Z in the nonrelativistic case, and at all Z at relativistic energies of the incident electrons, the differential back-scattering coefficient is well described by the first term of this series.

2. WAVE FUNCTION OF ELECTRON IN MATTER

The differential back-scattering coefficient is the ratio of the reflected electron flux per unit solid angle and unit energy interval to the incident flux. This quantity can be determined directly if one knows the wave function of the electron. Let us consider the Schrödinger equation for an electron interacting with the entire aggregate of the scattering target:

$$(\Delta + p_0^2)\Psi(\mathbf{r}, \dots \mathbf{R}_a \dots, \dots q_a \dots) = 2E \sum_a \{H^{(0)}(q_a) + U^{(0)}(\mathbf{r} - \mathbf{R}_a, q_a)\}\Psi(\mathbf{r}, \dots \mathbf{R}_a \dots, \dots q_a \dots), \quad (2.1)$$

where p_0 is the momentum of the incident particle, E is the total energy of the entire system (electron plus medium), $\mathbf{r}$ is the radius vector of the electron $H^{(0)}(q_a)$ is the Hamiltonian of the internal motion of the scatterers, and $U^{(0)}(\mathbf{r} - \mathbf{R}_a, q_a)$ is the Hamiltonian of the interaction of the electron with the scatterer.

We divide the interaction Hamiltonian into three terms:

$$U^{(0)}(\mathbf{r} - \mathbf{R}_a) = V_{in}(\mathbf{r} - \mathbf{R}_a, q_a) + V_e(\mathbf{r} - \mathbf{R}_a) + W_e(\mathbf{r} - \mathbf{R}_a), \quad (2.2)$$

where $V_{in}(\mathbf{r} - \mathbf{R}_a, q_a)$ is the inelastic-interaction potential, $V_e(\mathbf{r} - \mathbf{R}_a)$ is the potential of small-angle elastic scattering, and $W_e(\mathbf{r} - \mathbf{R}_a)$ is the potential of large-angle elastic scattering, which will henceforth be regarded as a perturbation.

The wave function Ψ_0 of the unperturbed problem satisfies the equation

$$(\Delta + p_0^2)\Psi_0(\mathbf{r}, \dots \mathbf{R}_a \dots, \dots q_a \dots) = 2E \sum_a \{H^{(0)}(q_a) + V_{in} + V_e\} \cdot \Psi_0(\mathbf{r}, \dots \mathbf{R}_a \dots, \dots q_a \dots). \quad (2.3)$$

The solution of this equation was investigated in[5]. In particular, the wave function Ψ_0 makes it possible to obtain different distributions with respect to the coordinates and momenta for particles experiencing multiple elastic scattering; allowance for the inelastic-interaction potential makes it possible to obtain the Landau distribution[6] for the particle energy losses in the medium.

We shall focus our attention on the investigation of the influence of the perturbing term $W_e(\mathbf{r} - \mathbf{R}_a)$. We have

$$\left\{\Delta + p_0^2 - 2E\sum_a (H^{(0)}(q_a) + V_{in}(\mathbf{r}-\mathbf{R}_a, q_a) + V_e(\mathbf{r}-\mathbf{R}_a))\right\} \times \Psi(\mathbf{r}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots) = 2E\sum_a W_e(\mathbf{r}-\mathbf{R}_a)\Psi(\mathbf{r}, \ldots \mathbf{R}_a, \ldots q_a). \quad (2.4)$$

We rewrite this equation in integral form

$$\Psi(\mathbf{r}_1 \ldots \mathbf{R}_a \ldots, q_a, \ldots) = \Psi_0(\mathbf{r}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots) + \int G_0(\mathbf{r}-\mathbf{x}, \ldots \mathbf{R}_a \ldots, q_a \ldots)\sum_a W_e(\mathbf{x}-\mathbf{R}_a) \times \Psi(\mathbf{x}, \ldots \mathbf{R}_{a_1} \ldots, \ldots q_a \ldots)d^3\mathbf{x}, \quad (2.5)$$

where $\Psi_0(\mathbf{r}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots)$ is the solution of (2.3) and $G_0(\mathbf{r} - \mathbf{x}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots)$ is the Green's function of the unperturbed equation (2.3).

We assume that the process of multiple scattering is incoherent, i.e., the average distance between the scattering centers is much larger than the scattering length:

$$n_0^{-1/3} \gg l_0, \quad (2.6)$$

where n_0 is the density of the atoms of the scattering medium.

Using the condition (2.6), we can rewrite (2.5) in the form[7]

$$\Psi(\mathbf{r}, \ldots, \mathbf{R}_a \ldots, \ldots q_a \ldots) = \Psi_0(\mathbf{r}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots) + \int G_0(\mathbf{r}-\mathbf{x}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots) \sum_a{}' \mathcal{F}_0(\mathbf{x}-\mathbf{R}_a)\Psi(\mathbf{x}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots)d^3\mathbf{x}, \quad (2.7)$$

where $\mathcal{F}_0(\mathbf{x} - \mathbf{R}_a)$ is the total amplitude for scattering by a single scatterer, and the primed summation sign means summation with respect to unequal atomic indices. Using expression (2.7) for $\Psi(\mathbf{r}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots)$, we can obtain the probability of the transition from the initial state with momentum $\mathbf{p}_0$ and energy E_0 to the final state $\mathbf{p}_f$ and E_f. This transition probability is determined by the usual rules of quantum mechanics.

3. DISTRIBUTION FUNCTION OF BACK-SCATTERED ELECTRONS

Let us consider a plane-parallel monoenergetic beam of electrons incident on a solid medium. The differential back-scattering coefficient is determined by the Wigner distribution function[8] $\mathcal{R}(\mathbf{p}_f, E_f, \mathbf{R} = 0; \mathbf{p}_0, E_0)$ on the surface of the target. Consequently, we have

$$\mathcal{R}(\mathbf{p}_f, E_f, \mathbf{R} = 0; \mathbf{p}_0, E_0) = S\delta\left[(E_0 - E_f) - \sum_a E_a\right](2\pi)^{-3}\int d^3\mathbf{r}e^{i\mathbf{p}_f\mathbf{r}}\left\{\int\prod_{a=1}^{N}(dq_a\Phi_{E_a}^*(q_a)) \times \Psi\left(\mathbf{R} - \frac{\mathbf{r}}{2}, \ldots \mathbf{R}_a \ldots, \ldots q_{a_1} \ldots\right)\right\}\left\{\prod_{a=1}^{N}(dq_a\Phi_{E_a}(q_a)) \times \Psi^*\left(\mathbf{R} + \frac{\mathbf{r}}{2}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots\right)\right\}, \quad (3.1)$$

where $\Phi_{Ea}(q_a)$ is the wave eigenfunction of the stationary state with energy E_a, pertaining to a scatterer located at the point $\mathbf{R}_a$, and S is the sum over all E_a.

Substituting expression (2.7) for the wave function of the system into (3.1), we get

$$\mathcal{R}(\mathbf{p}_f, E_f, \mathbf{R} = 0; \mathbf{p}_0, E_0) = S\delta\left[(E_0 - E_f) - \sum_{a=1}^{N} E_a\right](2\pi)^{-3}\int d^3\mathbf{r}e^{i\mathbf{p}_f\mathbf{r}}\left\{\int\prod_{a=1}^{N}[dq_a\Phi_{E_a}^*(q_a)] \times G_0\left(\mathbf{R} - \frac{\mathbf{r}}{2} - \mathbf{x}_{a_1} \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots\right)\sum_a{}'\mathcal{F}_0(\mathbf{x}-\mathbf{R}_a) \times \Psi(\mathbf{x}, \ldots \mathbf{R}_{a_1} \ldots, \ldots q_a \ldots)d^3\mathbf{x}\right\}\left\{\int\prod_{a=1}^{N}[dq_a\Phi_{E_a}(q_a)]G_0^*\left(\mathbf{R} + \frac{\mathbf{r}}{2} - \mathbf{y}, \ldots \ldots \mathbf{R}, \ldots, \ldots q_a \ldots\right)\sum_a{}'\mathcal{F}_0^*(\mathbf{y}-\mathbf{R}_a)\Psi^*(\mathbf{y}, \ldots, \mathbf{R}_a, \ldots \ldots q_a \ldots)d^3\mathbf{y}\right\} \quad (3.2)$$

The result depends on the concrete location of the scattering centers and must therefore be averaged over the distribution of the scatterer coordinates. Assuming the coordinates of the different scatterers to be independent, we can average in accordance with the formula

$$\langle\mathfrak{M}(\mathbf{R}_a)\rangle = \frac{1}{V}\int d^3\mathbf{R}_a\mathfrak{M}(\mathbf{R}_a). \quad (3.3)$$

Since we have subdivided the interaction with each isolated center into three processes, namely inelastic scattering and small-angle and large-angle ($\theta \gg \kappa/p$) elastic scattering, and since these processes are not coherent, it follows that the averaging of different types of interactions can be carried out independently. We first average over the distribution of the coordinates of the scatterers contained in $W_e(\mathbf{r} - \mathbf{R}_a)$. It should be noted that in averaging the interaction that determines the large-angle scattering, the amplitude for scattering by any individual atom should enter an even number of times, since it follows from the definition of the large-angle scattering amplitude that $\langle\mathcal{F}_0(\mathbf{r} - \mathbf{R}_a)\rangle \equiv 0$. Assuming condition (2.6) to be satisfied, and also assuming that the wavelength of the incident electrons is much smaller than the average distance $n_0^{-1/3} \gg 1/p_0$ between the atoms of the medium, we can show that expression (3.2) represents a series consisting of only ladder diagrams[7]. Each step of the ladder describes one act of large-angle scattering, and the number of steps in the ladder determines the number of large-angle scattering processes. In addition, it should be noted that inelastic and elastic (small-angle) interactions before and after each large-angle-scattering act are independent and separated in space. Therefore the averaging over the inelastic interactions and the elastic small-angle scatterings before and after each large-angle scattering act breaks up into independent averagings, and the result is the product of the mean values.

Thus, the result of the averaging is written in the form

$$\langle\mathcal{R}(\mathbf{p}_f, E_f, \mathbf{R} = 0, \mathbf{p}_0, E_0)\rangle = (2\pi)^{-3}\int d^3\mathbf{r}e^{i\mathbf{p}_f\mathbf{r}}S\delta\left[E_0 - E_f - \sum_{a=1}^{N}E_a\right] \times\left\langle G_0\left(\mathbf{R} - \frac{\mathbf{r}}{2} - \mathbf{x}\right)G_0\left(\mathbf{R} + \frac{\mathbf{r}}{2} - \mathbf{x}'\right)\right\rangle \times\left\langle\sum_a{}'\mathcal{F}_0(\mathbf{x}-\mathbf{R}_a)\mathcal{F}_0^*(\mathbf{x}'-\mathbf{R}_a)\right\rangle \times\langle\Psi(\mathbf{x}, \ldots \mathbf{R}_a \ldots, \ldots q_a \ldots)\Psi^*(\mathbf{x}'_1 \ldots \mathbf{R}_{b_1} \ldots, \ldots q_b \ldots)\rangle d^3\mathbf{x}\,d^3\mathbf{x}' \times\iint\prod_{a=1}^{\bar{N}}[dq_a\Phi_{E_a}(q_a)]\prod_{b=1}^{N}[dq_b\Phi_{E_b}^*(q_b)]. \quad (3.4)$$

Using the expression for the wave function (2.7), we can represent the differential back-scattering coefficient in the form of a series. Let us consider the first term of the series

$$\langle \mathscr{R}_1(\mathbf{p}_f, E_f, \mathbf{R}=0; \mathbf{p}_0, E_0)\rangle = (2\pi)^{-3}\int d^3\mathbf{r} e^{i\mathbf{p}_f\mathbf{r}} \times S\delta\left(E-E_f-\sum_{a=1}E_a\right)\int\prod_{c=1}^{N}[dq_c\Phi^*_{E_c}(q_c)]\prod_{d=1}^{N}[dq_d\Phi_{E_d}(q_d)] \left\langle G_0\left[\left(\mathbf{R}-\frac{\mathbf{r}}{2}\right)-\left(\mathbf{x}-\frac{\mathbf{r}'}{2}\right)\right]G_0^*\left[\left(\mathbf{R}+\frac{\mathbf{r}}{2}\right)-\left(\mathbf{x}+\frac{\mathbf{r}'}{2}\right)\right]\right\rangle e^{i\mathbf{q}\mathbf{r}'}d^3\mathbf{r}' \times |\mathscr{F}_0(\mathbf{p}-\mathbf{q})|^2 d^3\mathbf{q}\, d^3\mathbf{x}\, d^3\mathbf{p} e^{i\mathbf{p}\mathbf{r}''}\left\langle \Psi_0\left(\mathbf{R}-\frac{\mathbf{r}''}{2},\dots\mathbf{R}_a\dots,\dots q_a\dots\right) \times \Psi_0^*\left(\mathbf{R}+\frac{\mathbf{r}''}{2},\dots\mathbf{R}_b\dots,\dots q_b\dots\right)\right\rangle S\delta\left(E_0-E''-\sum_a E_a\right)dE''\int\prod_{a=1}^{N}[dq_a\Phi_{E_a}(q_a)]\prod_{b=1}^{N}[dq_b\Phi_{E_b}(q_b)]. \quad (3.5)$$

Using the general properties of the Wigner distribution function[8], we can show that (3.5) has a simple physical meaning. Indeed, the right-hand side of (3.5) is the product of three (classical) probabilities: the probability of absorbing an electron at the point x with momentum p and energy E″, the probability of elastic scattering in a state with momentum q, and the probability of transition from the phase state (x, q, E″) to the final state (R = 0, p_f, E_f). Consequently, the iteration series (2.7) for Ψ causes the differential back-scattering coefficient to be expressed as the sum of an infinite series, each term of which corresponds to a definite number of acts of scattering through a large angle, and as the particle moves between points corresponding to large-angle scattering processes, it experiences elastic and inelastic multiple scatterings, which can be described by the functions of Moliere[9] and Landau[6]

4. LIMITS OF APPLICABILITY OF THE CONSIDERED APPROACH

Let us investigate the relations between two neighboring terms of the series for $\mathscr{R}(p_f, E_f; p_0, E_0)$. It is easy to see that the ratio of the second term of the series to the first is

$$\frac{\mathscr{R}_2}{\mathscr{R}_1} \sim n_0 \frac{Z^2e^4}{4E^2_{0\,\mathrm{kin}}} L_{\mathrm{eff}}. \quad (4.1)$$

We consider the obtained estimate in two limiting cases: 1) $L_{\mathrm{eff}} \sim (n_0\sigma_{\mathrm{in}})^{-1}$, under the condition that interest attaches to back-scattered electrons with energy equal to the energy of the incident particles; 2) $L_{\mathrm{eff}} \sim l_m/2$, when interest attaches to the integrated (with respect to the energies) back-scattering coefficient.

In the first case relation (4.1) takes the form

$$\mathscr{R}_2/\mathscr{R}_1 \sim 2I_Z/4\pi E_{0\,\mathrm{kin}}, \quad (4.2)$$

where I_Z is the ionization potential of the medium. Thus, if the energy of the incident electrons is much larger than I_Z, then $\mathscr{R}_1$ gives a reasonable approximation for the differential back-scattering coefficient when the energies of the scattered electrons are close to the initial energy.

In the second limiting case, $L_{\mathrm{eff}} \sim l_m/2$. The value of l_m depends strongly on the initial energy of the incident electrons. If the incident electrons are nonrelativistic, then[10]

$$L_{\mathrm{eff}} \sim \frac{I^2}{8\pi Ze^4 n_0}\mathrm{Ei}\left(2\ln\frac{2E_0}{I}\right), \quad (4.3)$$

where $E_i(x)$ is the integral exponential function. For relativistic electrons[10] we have

$$L_{\mathrm{eff}} \sim E_0 m_e/4\pi n_0 Ze^4 \ln(2E_0/I_Z). \quad (4.3')$$

Substituting (4.3) and (4.3′) in (4.1) we obtain in the nonrelativistic case

$$\frac{\mathscr{R}_2}{\mathscr{R}_1} \sim \frac{Z}{16\pi\ln(2E_0/I_Z)} \quad (4.4)$$

and in the relativistic case

$$\frac{\mathscr{R}_2}{\mathscr{R}_1} \sim \frac{Zm_e}{16\pi E_0\ln(2E_0/I_Z)}. \quad (4.4')$$

Consequently, in the second limiting case R_1 gives the correct result at not very large values of Z (4.4), and at arbitrary Z for relativistic incident electrons.

It was noted above that $\mathscr{R}_1$ is the result of one act of large-angle scattering. It must be emphasized that this approximation describes not one collision in the target material. The total number of scatterings, even in the first limiting case (when the differential back-scattering coefficient is considered, with energies close to the energies of the incident electrons), amounts to

$$N \sim L_{\mathrm{eff}}/l_{\mathrm{el}} \sim 4ZE_{0\,\mathrm{kin}}l_Z/\varkappa^2 \gg 1. \quad (4.5)$$

In the second limiting case, the total number of electron collisions inside the target will be much larger. Therefore the considered approximation $\mathscr{R}_1$ is not a single-scattering approximation[4]. From this point of view, the proposed method has a number of advantages over Dashen's approach[11] to the back-scattering problem. Dashen's exact nonlinear integral equation for the differential back-scattering coefficient corresponds to the complete iteration series (3.4). However, the first approximation obtained by Dashen takes into account only multiple inelastic scattering, and ignores completely the multiple elastic scattering experienced by the electron before and after the large-angle scattering. The approximation $\mathscr{R}_1$ considered takes into account the effect of multiple elastic scattering. In addition, the proposed method can be used also to consider back scattering from targets of finite thickness, whereas Dashen's approach is applicable only to a semi-infinite target.

The proposed method also makes it possible to determine the limits of applicability of the "diffusion" approach. An important criterion for this is the mean square of the multiple-scattering angle accumulated over the effective length of the back scattering. For back-scattering electrons that lose little energy we have

$$\langle\theta^2\rangle_{L\,\mathrm{eff}} \sim \frac{ZI_Z}{E_{0\,\mathrm{kin}}}\ln(183Z^{-1/3}) \ll 1. \quad (4.6)$$

In considering the integrated (with respect to the energy) back-scattering coefficient we have in the nonrelativistic case

$$\langle\theta^2\rangle_{L\,\mathrm{eff}} \sim \frac{Z}{4\pi}\frac{\ln(183Z^{-1/3})}{\ln(2E_{0\,\mathrm{kin}}/I_Z)} \quad (4.7)$$

and in the relativistic case

$$\langle\theta^2\rangle_{L\,\mathrm{eff}} \sim \frac{Zm_e}{E_0}\frac{\ln(183Z^{-1/3})}{\ln(2E_{0\,\mathrm{kin}}/I_z)}. \tag{4.8}$$

Thus, only in the nonrelativistic case at large Z (5.7) can the multiple-scattering angle accumulated over the effective back-scattering length become comparable with unity. In this case the approximation $\mathscr{R}_1$ becomes unsatisfactory and it is necessary to consider the entire iteration series (3.4) for the electron back-scattering coefficient. Consequently, at not very large Z (and at relativistic energies of the incident electrons for all Z) we can confine ourselves to the approximation $\mathscr{R}_1$, and the next terms of the series (3.4) can be regarded as corrections.

5. CALCULATION OF THE DIFFERENTIAL BACK-SCATTERING COEFFICIENT OF FAST ELECTRONS IN THE APPROXIMATION $\mathscr{R}_1$

The differential back-scattering coefficient of fast electrons is determined, according to (3.5), by the relation

$$\mathscr{R}_1(\mathbf{p}_f, E_f; \mathbf{p}_0, E_0) = \int W(\mathbf{p}, E''; \mathbf{p}_0, E_0, t)\, d^3\mathbf{p}\, dE''\, dt \times \frac{d\sigma(\mathbf{p}-\mathbf{q})}{d\Omega} d^3\mathbf{q} W(\mathbf{p}_f, E_f; \mathbf{q}, E'', t), \tag{5.1}$$

where $W(\mathbf{p}, E''; \mathbf{p}_0, E_0, t)$ is the classical distribution function describing the probability of observing a charged particle at a depth t with momentum $\mathbf{p}$ and energy E'', if it had on the surface of the target a momentum $\mathbf{p}_0$ and an energy E_0. In the approximation in question, the function W is governed by two factors: multiple elastic small-angle scattering and multiple inelastic scattering. Since these interaction processes can be considered independently, the function W can be represented in the form of a product of the Moliere function[9] describing the multiple elastic scattering and the Landau function[6] describing the energy loss of the electron. For $d\sigma(\mathbf{p}-\mathbf{q})/d\Omega$ we use the usual elastic-scattering cross section determined by the Rutherford formula. When the foregoing is taken into account we obtain in the case of nonrelativistic energies of the incident electrons

$$\mathscr{R}_1(\mathbf{p}_f, E_f; \mathbf{p}_0, E_0) = \int \frac{1}{p^2\langle\theta^2\rangle_t \pi}\exp\left\{-\frac{(\mathbf{p}_0-\mathbf{p})^2}{p^2\langle\theta^2\rangle_t}\right\} d^3\mathbf{p}\frac{p}{m_e} \times \delta\left(\frac{p^2}{2m_e} - E''\right)(\pi\Delta^2E_t)^{-1/2}\exp\left\{-\frac{(E_0-E''-\Delta E_t)^2}{\Delta^2E_t}\right\} dE'' \times \frac{4Z^2e^4m_e^2n_0}{[(\mathbf{p}-\mathbf{q})^2+\varkappa^2]^2} d^3\mathbf{q}\cdot\frac{q}{m_e}\delta\left(\frac{\mathbf{q}^2}{2m_e} - E''\right) \times \frac{1}{q^2\langle\theta^2\rangle_{t'}\pi}\exp\left\{-\frac{(\mathbf{q}-\mathbf{p}_f)^2}{q^2\langle\theta^2\rangle_{t'}}\right\}\cdot(\pi\Delta^2E_t')^{-1/2}\exp\left\{-\frac{(E''-E_f-\Delta E_t')^2}{\Delta^2E_t'}\right\} dt, \tag{5.2}$$

where

$$\langle\theta^2\rangle_t = n_0\frac{Z^2e^4}{E_0E''}\pi\ln(183Z^{-1/3})\frac{t}{\cos(\widehat{\mathbf{n}_0\mathbf{p}})}. \tag{5.3}$$

Here $(\mathbf{n}_0\mathbf{p})$ is the angle between $\mathbf{p}$ and the normal to the target $\mathbf{n}_0$,

$$\Delta E_t = n_0\frac{2\pi Ze^4}{E_0}\ln\left(\frac{2E_0}{I_z}\right)\frac{t}{\cos(\widehat{\mathbf{n}_0\mathbf{p}})}, \tag{5.3'}$$

$$\Delta^2E_t = 2\pi n_0Ze^4\frac{t}{\cos(\widehat{\mathbf{n}_0\mathbf{p}})} \tag{5.3''}$$

and accordingly

$$\langle\theta^2\rangle_{t'} = n_0\frac{Z^2e^4}{E''E_f}\ln(183Z^{-1/3})\frac{t}{\cos(\widehat{\mathbf{n}_0\mathbf{q}})}, \tag{5.4}$$

$$\Delta E_{t'} = n_0\frac{2\pi Ze^4}{E''}\ln\left(\frac{2E''}{I_Z}\right)\frac{t}{\cos(\widehat{\mathbf{n}_0\mathbf{q}})}, \tag{5.4'}$$

$$\Delta^2E_t' = 2\pi n_0Ze^4\frac{t}{\cos(\widehat{\mathbf{n}_0\mathbf{q}})}. \tag{5.4''}$$

It should be noted that for relativistic incident electrons, formulas (5.3)–(5.3″) and (5.4)–(5.4″) are modified[10] because the decelerating ability decreases with increasing electron velocity. The question of the differential back-scattering coefficient of relativistic electrons will be considered separately.

Carrying out the necessary integrations in (5.2), which are elementary but quite laborious, we obtain

$$\mathscr{R}_1 = \frac{Z}{8\pi E_0\ln(2E_0/I_Z)}\left[(1-\cos\Theta) + \left(\ln\frac{E_0}{E_f}\right)\frac{Z\ln(183Z^{-1/3})}{4\pi\ln(2E_0/I_Z)}\right]^{-2} \times \frac{\cos(\widehat{\mathbf{n}_0\mathbf{p}_0})\cos(-\widehat{\mathbf{n}_0\mathbf{p}_f})}{\cos(\widehat{\mathbf{n}_0\mathbf{p}_0}) + (E_f/E_0)^2\cos(-\widehat{\mathbf{n}_0\mathbf{p}_f})}, \tag{5.5}$$

where Θ is the angle between $\mathbf{p}_0$ and $\mathbf{p}_f$. The result (5.5) goes over into Dashen's result[11] if one neglects multiple elastic scattering processes before and after the collision that leads to deflection through a large angle. This obviously is possible when $(E_0 - E_f)/E_0 \ll 1$ or at small Z, as follows directly from (5.5).

6. DISCUSSION OF RESULTS

Expression (5.5), within the applicability limits defined by (4.4) and (4.7), explains satisfactorily the main laws governing the back scattering of fast electrons and following from Dressel's[1] and Kanter's[2] experiment.

Thus, the dependence of the differential and integral back-scattering coefficients on the energy of the incoming particles E_0 is weak (logarithmic), as is confirmed by experiment. Let us integrate (5.5) with respect to the angles and energies, assuming the quantity $\eta = Z\ln(183Z^{-1/3})/4\pi\ln(2E_0/I_Z)$ to be small. We then obtain for the integrated back-scattering coefficient, in the case of normal incidence of the incoming particles, the expression

$$q = \left(\frac{\pi}{8}\right)^2\frac{Z}{\ln(4E_0^2/I_Z^2)}. \tag{6.1}$$

The solid curves of Fig. 1 show plots of q against E_0. As seen from the figure, for not too large Z the ex-

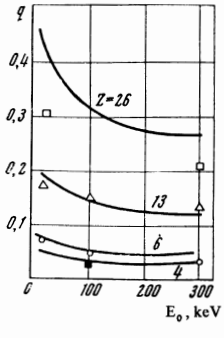

FIG. 1

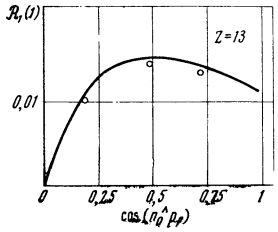

FIG. 2

perimental data[12] are in satisfactory agreement with the values of q obtained from (6.1).

An analysis of the energy distribution of the back-scattered electrons leads to the conclusion that the coefficient $\mathcal{R}_1$ has a maximum at an energy $E_f = \xi_0 E_0$, which is determined by the relation

$$\xi_0 = \left\{ \frac{\cos(\hat{p_0 n_0})}{\cos(-\hat{p_f n_0})} \frac{\eta}{|1-\cos\Theta-\eta|} \right\}^{1/2}, \quad \eta = \frac{z}{4\pi} \frac{\ln(183z^{-1/3})}{\ln(2E_0/I_z)}. \quad (6.2)$$

For small Z, the quantity determining the position of the maximum and its dependence on the atomic number of the target $\xi_0 \sim \sqrt{Z}$ are in satisfactory agreement with the experimental data[1,2].

The differential back-scattering coefficient at $E_f = E_0$ is also in satisfactory agreement with experiment. The solid curve in Fig. 2 shows a plot of $\mathcal{R}_1(E_f = E_0)$ against $\cos(n_0 p_f)$ for Z = 13. The experimental points[12,13] agree with the theoretical values within the limits of experimental error.

In analyzing the angular distribution of the back-scattered electrons, it must be borne in mind that owing to the finite cross sections of the incoming-electron beams and the detectors, the quantity actually measured is frequently not $\mathcal{R}$ but

$$\Phi(\mathbf{p}_f, E_f; \mathbf{p}_0, E_0) = \frac{\cos(-\hat{\mathbf{n}_0 \mathbf{p}_f})}{\cos(\hat{\mathbf{n}_0 \mathbf{p}_0})} \mathcal{R}(\mathbf{p}_f, E_f; \mathbf{p}_0, E_0). \quad (6.3)$$

Taking (6.3) and (5.5) into account, the angular distribution of the back-scattered electrons, integrated with respect to the energies, is given by

$$F[E_0, \cos(\hat{p_0 n_0}), \cos(\hat{p_f n_0})] = \frac{Z(1-\cos\Theta)^{-2}}{8\pi \ln(2E_0/I_Z)} \cos(\hat{p_f n_0}) \operatorname{arc\,tg} \sqrt{\frac{\cos(\hat{p_f n_0})}{\cos(\hat{p_0 n_0})}}. \quad (6.4)$$

Expression (6.4) correctly describes the main regularities of the angular distribution of the back-scattered flux. Thus, Fig. 3 shows polar distribution diagrams of the electrons scattered from an aluminum target in the (n_0, p_0) plane. As seen from the figure, the experimental and theoretical distributions are in satisfactory agreement. It follows from (6.4) that the maximum of the function F is located approximately

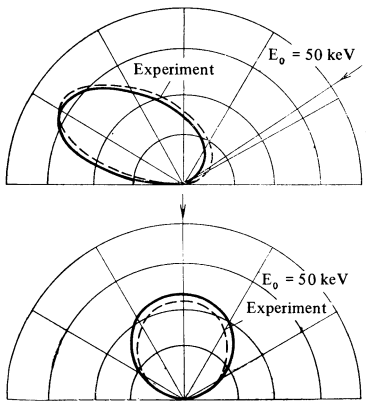

FIG. 3

at the point $(\hat{p_f n_0}) \approx (\hat{p_0 n_0})$, which at not very large Z is confirmed by the experimental data. It should be noted that for $(\hat{p_f n_0}) \sim \pi/2$ we have $F \sim \cos^2(p_f n_0)$, and for small scattering angles $F \sim \cos^{3/2}(p_f n_0)$.

It thus follows from the foregoing that expression (5.5) agrees with experiment in the region of its applicability. The results can be of definite use in connection with the extensive practical application of the phenomenon in question.

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232