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COMPLETE SYSTEM OF HYDRODYNAMIC EQUATIONS FOR GASES WITH ROTATIONAL DEGREES OF FREEDOM

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Integration of the kinetic equation with a nonlocal collision integral leads to a system of hydrodynamic equations, including the equation for the transport of the polarization of rotational angular momenta of molecules. It is shown that the latter has independent significance in strongly nonstationary processes, and also under stationary conditions if a magnetic field is applied the direction of which does not coincide with the direction of rotation of the gas as a whole. Phenomena analogous to the Einstein-de Haas and the Barnett effects are discussed on the basis of the system of equations obtained in this paper.

1. Our object is to derive the hydrodynamic equations (HE) for molecular gases from the kinetic equation (KE).

Within the framework of the kinetic theory of gases the usual HE system is a macroscopic expression of the laws of conservation of the number of particles, of momentum and of energy, i.e., of the conservation laws which hold for each collision. However, there exists still another conserved quantity—the total moment of momentum. In the usual HE system (cf., for example,[1]) the corresponding transport equation is absent. In fact this corresponds to an explicit or an implicit assumption that the motion of the particles is completely described by the translational degrees of freedom. In this case the law of conservation of angular momentum is a direct consequence of the law of conservation of momentum. However, in making the transition to a gas with rotational degrees of freedom the total angular momentum—the sum of the rotational and the orbital angular momenta of the molecules—is an independent additive integral of the motion. In the general case this must lead to a new independent equation in the HE system which describes the transport of angular momentum.

The problem of the macroscopic equation for the transfer of angular momentum has been raised by a number of authors (cf.,[2]), but they did not succeed in obtaining an equation which would describe a natural, from the physical point of view, exchange between the rotational and the orbital parts of the angular momentum of the gas.

In hydrodynamics of liquids this problem is solved phenomenologically[3,4], an equation for the transport of angular momentum is introduced formally into the HE system and it is shown that a connection between the internal rotation and the translational motion exists only in the case if one introduces the antisymmetric part of the viscous stress tensor the explicit form of which is obtained from the principle of increase of entropy.

In the present paper it is shown that the complete HE system can be obtained from the KE only by taking into account the nonlocality of the collision integral. An analysis of the HE system that we have constructed taken together with an estimate of the kinetic coefficients shows that under ordinary conditions the rotational angular momentum rapidly relaxes to its thermodynamic equilibrium value. At the same time the equation for the transport of angular momentum loses its independence and the HE system assumes its usual form. As a result the complete HE system plays a role primarily in describing strongly nonstationary processes. Nevertheless, in the presence of an external field which acts directly on the rotational degrees of freedom the equation for the transport of angular momentum retains its validity even under stationary conditions and this leads to a number of interesting phenomena.

2. For the sake of definiteness we consider a diatomic gas in a constant magnetic field **B**, and we take the rotational degrees of freedom into account in a classical manner.

The variation with time of the distribution function f for a diatomic gas is described by the kinetic equation[5]

$$\frac{\partial f}{\partial t}+\nabla(\mathbf{v}f)+\frac{\partial}{\partial \mathbf{M}}(\dot{\mathbf{M}}f)=-J. \qquad (2.1)$$

Here J is the collision integral, $\mathbf{p} = m\mathbf{v}$ and **M** are the momentum and the rotational angular momentum of the particles, $\mathbf{M} = \boldsymbol{\mu} \times \mathbf{B}$.

For a paramagnetic molecule the magnetic moment of the molecule $\boldsymbol{\mu}$ depends on the discrete variable σ characterizing the possible components of the spin along the direction of the angular momentum. For example, for oxygen

$$\boldsymbol{\mu}=\gamma(\sigma)\mathbf{M}, \quad \gamma(\pm 1)=2\mu_0/M, \quad \gamma(0)=-2\mu_0\hbar/M^2 \qquad (2.2)$$

(μ_0 is the Bohr magneton). The distribution function is normalized by

$$\langle f\rangle=n \qquad (2.3)$$

(n is the particle number density). Here and below we utilize the following notation:

$$\langle Af\rangle=\sum_\sigma\int d\mathbf{p}\,d\mathbf{M}\,Af, \quad \bar{A}=\langle Af\rangle/n,$$

$$\mathbf{V}=\bar{\mathbf{v}}, \quad \mathbf{u}=\mathbf{v}-\mathbf{V},$$

$$\frac{d\bar{A}}{dt}=\frac{\partial\bar{A}}{\partial t}+(\mathbf{V}\nabla)\bar{A}^c, \qquad \nabla\langle A^c f\rangle\equiv\langle A^c\nabla f\rangle. \qquad (2.4)$$

An element of volume in the rotational angular momentum space for a diatomic gas is equal to

$$d\mathbf{M}=M\,dM\,d\cos\theta_M\,d\varphi_M. \qquad (2.5)$$

An equation for the transport of a quantity A is obtained from the kinetic equation by multiplying it by this quantity and by integrating over all the variables except time and space coordinates. With the aid of the equation of continuity

$$\partial n / \partial t + \nabla (n\mathbf{V}) = 0 \tag{2.6}$$

the equation for the transport of the quantity A is brought to the form

$$n\frac{d\bar{A}}{dt} = \left\langle \frac{\partial A}{\partial \mathbf{M}}[\boldsymbol{\mu}\mathbf{B}]f \right\rangle - \nabla \langle \mathbf{u}A^{c}f\rangle - \langle AJ\rangle. \tag*{(2.7)*}$$

In (2.7) the first term on the right hand side is the effect of the magnetic field which differs from zero only if A depends on the rotational angular momentum. The second term is the flux of the quantity A in the system of coordinates in which the given element of volume of gas is at rest. The last term ⟨AJ⟩ describes the change in the quantity A due to collisions which is summed over all the molecules contained within a unit volume.

The infinite system of differential equations (2.7) constructed for the complete system A of functions of p and M, is equivalent to the integro-differential kinetic equation (2.1). In this system a distinctive role is played by equations in which the quantity A is one of the additive integrals of the motion, i.e., momentum, total angular momentum of the molecule, and energy:

$$A_1 = \mathbf{p}, \quad A_2 = \mathbf{M} + [\mathbf{rp}], \quad A_3 = E = \frac{p^2}{2m} + \frac{M^2}{2I} - \mu B. \tag{2.8}$$

(In the presence of a magnetic field the total angular momentum remains an integral of the motion in a collision since during a collision time the change in it due to precession is negligibly small and this change is completely taken into account by the left hand side of the kinetic equation.)

In the kinetic theory of gases the collision integral is usually chosen in the local form due to Boltzmann J_B. The following equations hold for the Boltzmann integral

$$\langle \mathbf{p}J_B\rangle = 0, \quad \langle EJ_B\rangle = 0, \tag{2.9}$$

which physically mean that the momentum and the energy per unit volume of the gas do not change as a result of collisions. On the other hand, the quantity $\langle \mathbf{M}J_B\rangle$ differs from zero, since in a single collision the sum of the rotational angular momenta of the colliding molecules is not conserved, since there exists no restriction on an exchange between their rotational and orbital angular momenta. But the Boltzmann integral is a local operator in the sense that its value at the point r is determined by the value of the distribution function at the same point of coordinate space. Therefore, from the first of equations (2.9) it follows immediately that

$$\langle [\mathbf{rp}]J_B\rangle = [\mathbf{r}, \langle \mathbf{p}J_B\rangle] = 0. \tag{2.10}$$

Thus, for the Boltzmann integral the quantity $\langle(\mathbf{M} + \mathbf{r} \times \mathbf{p})J_B\rangle$ differs from zero. From this it becomes clear that in order to take into account the fact that the quantity which is conserved in a collision is not the orbital angular momentum (as in (2.10)), but only the total angular momentum of the colliding molecules, one should give up the approximate representation of the collision integral in local form.

The nonlocality of the collision integral is associated in the first instance with the fact that the range of interaction between the molecules has although a small (compared to the mean free path λ), but nevertheless finite value a. The nonpoint nature of the interaction is assumed already in J_B which is seen particularly clearly when the probability of collisions is expressed in terms of the impact parameter. But the collision integral assumes the form J_B only in the first nonvanishing approximation with respect to a/λ. In order to take into account in transport phenomena corrections in terms of the ratio a/λ the collision integral should be regarded as a nonlocal operator. This is well known from the theory of dense gases (cf., the papers by Enskog[6] and by Bogolyubov[7]).

In the case of a finite range of interaction the quantity which is strictly conserved in a collision is the sum of the additive integral A over all the molecules of the gas, i.e.,

$$\int d\mathbf{r}\langle AJ\rangle = 0. \tag{2.11}$$

The corresponding quantity ⟨AJ⟩ per unit volume may be different from zero due to collisions which occur at the boundary of this volume. According to (2.11) one can only assert that the integrand in (2.11) is a divergence of a certain flux which is directly related to the collisions

$$\langle AJ\rangle = -\nabla a. \tag{2.12}$$

For rarified gases the flux a is much smaller than the flux ⟨uAf⟩ (cf., (2.7)) and the former should be taken into account only if this leads, as in our case, to qualitatively new phenomena.

3. We substitute into (2.7) in succession the integrals of the motion (2.8):

$$\begin{gathered}
mn\frac{d}{dt}V_j = -\nabla_i\langle mu_iu_jf\rangle - \langle p_jJ\rangle, \\
n\frac{d}{dt}(\overline{\mathbf{M}} + m[\mathbf{r}^c\mathbf{V}])_j = n[\overline{\boldsymbol{\mu}}\mathbf{B}]_j \\
- \nabla_i\{\langle u_iM_jf\rangle + \langle u_i[\mathbf{r}^c, m\mathbf{u}]_jf\rangle - \langle(\mathbf{M} + [\mathbf{rp}])_jJ\rangle\}, \\
n\frac{d\bar{E}}{dt} = -\nabla_i\langle u_iEf\rangle - \langle EJ\rangle.
\end{gathered} \tag{3.1}$$

In the equation for the transport of the total angular momentum (the second of equations (3.1)) it has been taken into account that in accordance with the derivation of (2.7) the operator ∇ does not operate on the quantity r × p. The terms of this equation associated with the orbital angular momentum of the molecules can be eliminated with the aid of the equation for the transport of momentum (the first of equations (3.1)). As a result we obtain the equation for the transport of rotational angular momentum

$$n\frac{d}{dt}\overline{M}_j = n[\overline{\boldsymbol{\mu}}\mathbf{B}]_j - \nabla_i\langle u_iM_jf\rangle - \langle M_jJ\rangle. \tag{3.2}$$

In accordance with (2.12) we have

$$\begin{gathered}
\langle p_jJ\rangle = -\boldsymbol{\nabla}_i a_{ij}, \\
\langle(\mathbf{M} + [\mathbf{rp}])_jJ\rangle = -\boldsymbol{\nabla}_i b_{ij}, \\
\langle EJ\rangle = -\nabla_i c_i.
\end{gathered} \tag{3.3}$$

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*$[\boldsymbol{\mu}\mathbf{B}] \equiv \boldsymbol{\mu} \times \mathbf{B}$.

With the aid of the first two equations in (3.3) we can express a term in equation (3.2) in terms of the tensors a and b:

$$\langle M_j J\rangle = -2\sigma_j^{(1)} - \nabla_i b_{ij}',$$

$$2\sigma_j^{(1)} = e_{jkl}a_{kl}; \qquad b_{ij}' = b_{ij} - a_{il}e_{ljk}x_k. \tag{3.4}$$

The last equation in the system (3.1)—the equation for the transport of energy—can be put into a different form by transforming it into an equation for the transport of heat, or more accurately, of entropy. For the local value of the entropy of a nonequilibrium system such as a gas which performs a hydrodynamic, i.e., a large scale motion, it is customary to choose a quantity which in its form coincides with the entropy of an equilibrium system. In the Appendix it is shown that the entropy $\overline{S}$ defined in this manner referred to a single molecule satisfies the equation

$$n\frac{d\overline{S}}{dt} = \frac{1}{T}\left\{n\frac{d\overline{E}}{dt} - mn\mathbf{V}\frac{d\mathbf{V}}{dt} - (\mathbf{\Omega}_M - \mathbf{\Omega}_B)\frac{d\mathbf{M}}{dt}\right\} - \frac{dn}{dt}. \tag{3.5}$$

Here and further we shall use the special notation:

$$\mathbf{\Omega}_M = {}^3/_2\overline{\mathbf{M}}/I, \qquad \mathbf{\Omega}_B = (\mu\overline{\mathbf{M}}/2IT)\mathbf{B}. \tag{3.6}$$

We shall call the first quantity Ω_M the average frequency or the polarization of the rotation of the molecule. The factor 3/2 is associated with the specific form of the element of volume dM for a diatomic molecule (cf., (2.5)). The quantity $\mathbf{\Omega}_B$ is the effective rate of precession of the rotational angular momentum in a magnetic field.

Substituting equations (3.1)—(3.4) into expression (3.5) we obtain after simple rearrangements of terms the general equation for the transport of heat:

$$n\frac{d\overline{S}}{dt} = \frac{1}{T}\{-\nabla_i Q_i + \sigma_{ij}\nabla_i V_j + \beta_{ij}\nabla_i\Omega_{Mj} - 2(\mathbf{\Omega}_M - \mathbf{\Omega}_B)\boldsymbol{\sigma}^{(1)}\}. \tag{3.7}$$

The new quantities appearing in this equation are defined by the following formulas

$$\begin{gathered}\sigma_{ij} = -\langle m u_i u_j f\rangle + a_{ij} + p\delta_{ij}, \quad p = nT,\\ \beta_{ij} = -\langle u_i M_j f\rangle + b_{ij}',\\ Q_i = \langle u_i E f\rangle - c_i + \sigma_{ij}V_j + \beta_{ij}(\Omega_M - \Omega_B)_j - pV_i.\end{gathered} \tag{3.8}$$

The pseudovector $\boldsymbol{\sigma}^{(1)}$ in accordance with (3.4) and (3.8) is a dual vector to the antisymmetric part of the viscous stress tensor σ_{ij}.

With the aid of notations (3.8) the equations for the transport of momentum and of rotational angular momentum can be written in the form

$$\begin{gathered}mn\frac{d}{dt}V_j = -\nabla_j p + \nabla_i\sigma_{ij},\\ n\frac{d}{dt}\overline{M}_j = n[\overline{\mu}\mathbf{B}] + \nabla_i\beta_{ij} + 2\sigma_j^{(1)}.\end{gathered} \tag{3.9}$$

These equations together with the equation of continuity (2.6) and the equation for the transport of heat (3.7) form a complete HE system. This system is a closed system if the fluxes of σ, β and Q can be considered as known as soon as the values of n, T, V and M are given at each point of the gas. The latter quantities completely determine the state of the gas only in the case of a large scale hydrodynamic motion, when the deviation from thermal equilibrium is known to be small. In the next section we shall express the dissipative fluxes in terms of the local characteristics of the gas, assuming that the conditions for the applicability of a hydrodynamic treatment are satisfied.

In generalizing to the case of a gas of polyatomic molecules the HE system retains its form and only the relations (3.6) are altered:

$$\mathbf{\Omega}_M = 3\overline{\mathbf{M}}/(I_A + I_B + I_C), \quad \mathbf{\Omega}_B = \mathbf{B}(\overline{\mu\mathbf{M}})/(I_A + I_B + I_C)T. \tag{3.6$'$}$$

Here I_A, I_B, I_C are the principal moments of inertia.

4. In the complete HE system the dissipative processes are described by the tensor σ, the pseudotensor β and the vector Q. These quantities must vanish when the gas is in thermal equilibrium. For small deviations from thermal equilibrium the fluxes of σ, β and Q are linear functions of parameters characterizing the degree of deviation of the state of the gas from the equilibrium state. Such parameters, in the first instance, are the gradients of the velocity $\nabla_i V_j$ and of the temperature ∇T. An analogous obvious characteristic of the deviation from equilibrium is the inhomogeneity of the average rotational angular momentum, i.e., $\nabla_i \Omega_{Mj}$.

Moreover, one should take into account that in the case of thermal equilibrium in a rotating gas the angular velocity of rotation of the gas

$$\mathbf{\Omega}_V = {}^1/_2\,\mathrm{rot}\,\mathbf{V} \tag{4.1}$$

uniquely determines the value of the average rotational angular momentum or, what is the same, the value of Ω_M (cf., (A.3) and (A.8) in the Appendix). Therefore in the absence of thermal equilibrium a new macroscopic quantity arises—the difference $\delta\mathbf{\Omega}$ between the equilibrium and the actual values of the frequency of rotation of the molecules Ω_M which determines an additional independent dissipation channel.

The linear relation between the dissipative fluxes of σ, β and Q and the parameters of the deviation from thermal equilibrium ∇V, $\nabla\Omega_M$, ∇T and $\delta\Omega$ is of a complicated tensor nature if the properties of the gas are anisotropic. For example, in the presence of a magnetic field heat conductivity is, generally speaking, a second rank tensor, while viscosity is a fourth rank tensor[5]. In the present paper we will not consider such effects. For an isotropic medium the relationship between the dissipative fluxes and the parameters of deviation from equilibrium reduces to a simple proportionality between the irreducible parts of these quantities.

We decompose the fluxes of σ and β into irreducible parts:

$$\begin{gathered}\sigma_{ij} = \sigma^{(0)}\delta_{ij} + e_{ijk}\sigma_k^{(1)} + \sigma_{ij}^{(2)},\\ \beta_{ij} = \beta^{(0)}\delta_{ij} + e_{ijk}\beta_k^{(1)} + \beta_{ij}^{(2)}.\end{gathered} \tag{4.2}$$

Here $\sigma^{(0)}$ is a scalar which determines the spherical part of the tensor σ_{ij}; $\boldsymbol{\sigma}^{(1)}$ is a pseudovector, dual to the antisymmetric part of the tensor σ_{ij}; $\sigma_{ij}^{(2)}$ is the irreducible tensor of the second rank:

$$\sigma_{ij}^{(2)} = \{\sigma_{ij}\} = {}^1/_2(\sigma_{ij} + \sigma_{ji} - {}^2/_3\delta_{ij}\sigma_{ii}). \tag{4.3}$$

The decomposition of the pseudotensor β_{ij} is of an analogous nature.

The irreducible characteristics of the deviation of the state of the gas from the equilibrium state are the scalar div V, the pseudoscalar div $\mathbf{\Omega}_M$, the vectors ∇T and curl $\mathbf{\Omega}_M$, the pseudovector $\delta\mathbf{\Omega}$, the tensor $\{\nabla_i V_j\}$

and the pseudotensor $\{\nabla_i \Omega_{Mj}\}$ (the latter quantities are constructed in accordance with the psescription (4.3)). We note that the pseudovector dual to the antisymmetric part of the tensor $\nabla_i V_j$, i.e. (4.1), is absent from the above sequence since this expression does not have to vanish in the state of thermal equilibrium. Thus, from considerations of symmetry we obtain

$$\sigma^{(0)} = \zeta \operatorname{div} \mathbf{V}, \qquad \sigma^{(1)} = {}^1\!/_2 \xi \delta \Omega, \qquad \sigma_{ij}^{(2)} = 2\eta \{\nabla_i V_j\},$$
$$\beta^{(0)} = \beta_0 \operatorname{div} \Omega_M, \qquad \beta^{(1)} = \beta_1 \operatorname{rot} \Omega_M, \qquad \beta_{ij}^{(2)} = 2\beta_2 \{\nabla_i \Omega_{Mj}\}, \tag{4.4}$$
$$\mathbf{Q} = -\varkappa \nabla T.$$

In principle, on the right hand side of the expression for the heat flux **Q** a term could appear proportional to curl Ω_M, while on the right hand side of the expression for $\beta^{(1)}$ a term could appear containing ∇T. However, one can show that the corresponding kinetic coefficients are equal to zero, at least in the case when the probability of collisions is an even function of the rotational angular momenta.

The principal contributions to the fluxes of σ, β and Q are made by the usual terms of the form $\langle uAf \rangle$ which are calculated by the well known methods of kinetic theory with the collision integral in local form. From these calculations we obtain the majority of the kinetic coefficients which in order of magnitude are equal to:

$$\eta = p/\nu_V, \quad \zeta \approx p/\nu_M, \quad \varkappa \approx \eta/m, \quad \beta_n \approx I\varkappa \tag{4.5}$$

(ν_V and ν_M are the inverse relaxation times for the translational and the rotational degrees of freedom).

The fluxes a, b′ and c (cf., (3.3), (3.8)) arise only when the nonlocality of the collision integral is taken into account and give, benerally speaking, small corrections to the kinetic coefficients (4.5). These corrections are proportional to the ratio $(a/\lambda)^2$. (At first sight it might appear that the pseudotensor b′ is not small since, in accordance with the definition (3.8), this tensor contains the spatial coordinate x explicitly. However this explicit dependence on x must be compensated identically by the tensor b (without a prime). This is connected with the fact that the form of the HE system (3.9) must be invariant with respect to the choice of the origin of coordinates, and, consequently, all the quantities (in particular, β_{ij}) must not depend explicitly on the spatial coordinates.) Thus, we can neglect the fluxes a, b′ and c each time when we are dealing with corrections to the usual dissipative fluxes. An exception is made by the antisymmetric part of the flux a_{ij} (or $\sigma^{(1)}$), for which there is no analogue in the usual tensor for the transport of momentum $\langle u_i p_j f \rangle$.

5. In order to obtain the flux $\sigma^{(1)}$ one must evaluate (cf., (3.3), (3.4)) the quantity $\langle \mathbf{p} J \rangle$. For the evaluation of the latter quantity one can utilize the method of Enskog who calculated the corrections to the kinetic coefficients for dense gases[6], or one can construct a generalized method of moments in which the spatial coordinate is treated on the same basis as v and M. But one can avoid these complicated calculations and reduce the problem to the customary methods of the theory of rarified gases if one utilizes the relation (3.4) written in the form

$$-2\sigma^{(1)} = \langle MJ \rangle. \tag{5.1}$$

Here we have neglected the quantity $\nabla b'$. Indeed, the pseudotensor b′ is in its properties completely analogous to the pseudotensor β and, consequently, is constructed from first derivatives of Ω_M, while $\nabla b'$ is constructed of second derivatives of Ω_M. But taking into account in the viscosity tensor of the second derivatives of the average velocity of rotation of the molecules takes us far beyond the framework of applicability of HE.

We note that although the right hand side of (5.1) is different from zero already when the collision integral is in local form, the quantity $\langle \mathbf{M} J \rangle$ itself contains a small factor—the moment of inertia I—which is proportional to a^2 (cf., below (5.5)). On the left hand side the corresponding smallness arises as a result of the fact that $\langle \mathbf{p} J \rangle$ differs from zero only when the nonlocal structure of the collision integral is taken into account.

We evaluate the quantity $\langle \mathbf{M} J \rangle$ in the one-moment approximation. We assume that the state of the gas is close to the state of thermal equilibrium:

$$f = f_0(1+\chi), \qquad \chi \ll 1,$$
$$f_0 = \exp\left\{\frac{1}{T}[F - E + \mathbf{V}_0\mathbf{p} + \mathbf{\Omega}_0(\mathbf{M} + [\mathbf{rp}])\right\}. \tag{5.2}$$

In order to describe the situation in which the rotational angular momentum is not in equilibrium with the orbital angular momentum we assume that

$$\chi = \boldsymbol{\chi}\mathbf{M}/T, \tag{5.3}$$

where χ is a small constant pseudovector. In the case of small deviations from equilibrium the collision integral can be regarded as a linear integral operator:

$$J\{f_0\} = 0, \quad J\{f\} = f_0 \hat{\Omega} \chi, \tag{5.4}$$

while the diagonal matrix elements

$$\nu_n = \langle \varphi_n f_0 \hat{\Omega} \varphi_n \rangle$$

have, roughly speaking, the meaning of inverse relaxation times of the quantities φ_n if the latter are normalized to unity:

$$\langle \varphi_n f_0 \varphi_{n'} \rangle = \delta_{nn'}.$$

Substituting (5.3) into (5.4) we obtain

$$\langle M_j J \rangle = \frac{1}{T} \langle M_j f_0 \hat{\Omega} M_i \rangle \chi_i = \frac{2}{3} n I \nu_M \chi_j. \tag{5.5}$$

The vector $\boldsymbol{\chi}$ can be expressed in terms of the macroscopic characteristics of the gas, and this is most simply done by noting that the distribution (5.2), (5.3) coincides including the notation with the distribution (A.1). As a result of this we have

$$\boldsymbol{\chi} = \Omega_M - \Omega_B - \Omega_V. \tag{5.6}$$

Comparing (5.5) and (5.6) with the expressions (4.4) and (5.1) which determine the quantity $\sigma^{(1)}$ we obtain

$$\sigma^{(1)} = {}^1\!/_2 \xi \delta \Omega, \quad \delta\Omega = \Omega_V - \Omega_M - \Omega_B, \quad \xi = {}^2\!/_3 n I \nu_M. \tag{5.7}$$

Thus, the antisymmetric part of the viscous stress tensor, i.e., $\sigma^{(1)}$, has a form which is in complete agreement with the prediction on the basis of general considerations and describes a peculiar dissipative process associated with the exchange of angular momentum between the translational and the rotational degrees of freedom of the molecules.

We emphasize that the pseudovector $\sigma^{(1)}$ plays a double role. On the one hand it enters the equation for the transport of momentum (3.9) (cf., (4.2)) characterizing the effect of the internal rotation of the molecules on the circulation of the gas. This effect is a very weak one. The ratio of the coefficient ξ to the coefficient of first viscosity is in order of magnitude equal to

$$\xi / \eta \approx (\nu_V / \nu_M)(a / \lambda)^2. \tag{5.8}$$

From this point of view the ratio (5.8) appears as a manifestation of the smallness of nonlocal effects in a rarified gas, i.e., the smallness of the quantity $\langle \mathbf{p} \mathbf{J} \rangle$. On the other hand, in the equation for the transport of rotational angular momentum (3.9) the quantity $\sigma^{(1)}$ determines the effect of the translational degrees of freedom on the rotational degrees of freedom. This effect is of decisive significance. Indeed, when $\boldsymbol{\Omega}_M$ deviates from its thermodynamic equilibrium value it can be easily seen that in the equations for the transport of angular momentum the dissipative terms $\nabla\beta$ and $\sigma^{(1)}$ are in the ratio of

$$(\nu_V / \nu_M)(\lambda / L)^2. \tag{5.9}$$

Here L is a length over which $\boldsymbol{\Omega}_M$ changes in a significant manner. The ratio (5.9) is small since the inequality

$$\lambda / L \ll 1 \tag{5.10}$$

is a necessary condition for the applicability of hydrodynamic considerations.

6. Thus, the complete HE system for a diatomic gas in a constant magnetic field has the form (cf., (2.6), (3.7), (3.9), (4.4))

$$\partial n / \partial t + \nabla (\mathbf{V} n) = 0, \tag{I}$$

$$\begin{aligned} mn\frac{d\mathbf{V}}{dt} = &-\operatorname{grad} p + \eta\Delta\mathbf{V} + \left(\zeta + \frac{1}{3}\eta\right)\operatorname{grad}\operatorname{div}\mathbf{V} \\ &+ \frac{1}{2}\xi \operatorname{rot}(\boldsymbol{\Omega}_M - \boldsymbol{\Omega}_B - \boldsymbol{\Omega}_V), \end{aligned} \tag{II}$$

$$\begin{aligned} \frac{d\boldsymbol{\Omega}_M}{dt} = &[\boldsymbol{\Omega}_M \boldsymbol{\Omega}_B] + \frac{3}{2In}\{(\beta_1 + \beta_2)\Delta\boldsymbol{\Omega}_M \\ &+ (\beta_0 - \beta_1 + {}^1/_3\beta_2)\operatorname{grad}\operatorname{div}\boldsymbol{\Omega}_M\} - \nu_M(\boldsymbol{\Omega}_M - \boldsymbol{\Omega}_B - \boldsymbol{\Omega}_V), \end{aligned} \tag{III}$$

$$\begin{aligned} n\frac{d\bar{S}}{dt} = \frac{1}{T}\Big\{ &\varkappa\Delta T + \frac{1}{2}\eta\left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j} - \frac{2}{3}\delta_{ij}\operatorname{div}\mathbf{V}\right)^2 + \zeta(\operatorname{div}\mathbf{V})^2 \\ &+ \frac{1}{2}\beta_2\left(\frac{\partial\Omega_{Mj}}{\partial x_i} + \frac{\partial\Omega_{Mi}}{\partial x_j} - \frac{2}{3}\delta_{ij}\operatorname{div}\boldsymbol{\Omega}_M\right)^2 + \beta_1(\operatorname{rot}\boldsymbol{\Omega}_M)^2 \\ &+ \beta_0(\operatorname{div}\boldsymbol{\Omega}_M)^2 + \xi(\boldsymbol{\Omega}_M - \boldsymbol{\Omega}_B - \boldsymbol{\Omega}_V)^2\Big\}. \end{aligned} \tag{IV}$$

We note first of all that the kinetic coefficients β_n and ξ (just as η, ζ, κ) are essentially positive quantities within the framework of kinetic theory. Therefore, the entropy of the gas in accordance with (IV) can only grow with time as it ought to.

Assume that there is no magnetic field present. In order to understand how within the framework of the system IV there occurs a transfer of angular momentum between the rotational and the translational degrees of freedom of the gas we consider that the following model problem.

We assume that at the initial instant we are given a perturbation of vortex type

$$\begin{gathered} \operatorname{div}\mathbf{V} = 0, \quad \boldsymbol{\Omega}_V = {}^1/_2 \operatorname{rot}\mathbf{V} = \varphi\boldsymbol{\Omega}_0 \sin kr / r, \\ \boldsymbol{\Omega}_M = \psi\boldsymbol{\Omega}_0 \sin kr / r, \quad k\lambda \ll 1. \end{gathered} \tag{6.1}$$

(Such a formulation of the problem enables us to avoid the solution with boundary conditions on the walls of the vessel in which the gas is placed.) We neglect nonlinear effects (i.e., the difference between dA/dt and ∂A/∂t, and also, in accordance with (5.9), the dissipative processes associated with the kinetic coefficients β_n. The HE equations (II) and (III) for the given problem can be written in the form

$$\begin{gathered} \partial\boldsymbol{\Omega}_V / \partial t = \nu_V\lambda^2\Delta\boldsymbol{\Omega}_V - \nu_M a^2\Delta(\boldsymbol{\Omega}_M - \boldsymbol{\Omega}_V), \\ \partial\boldsymbol{\Omega}_M / \partial t = -\nu_M(\boldsymbol{\Omega}_M - \boldsymbol{\Omega}_V), \quad a^2 = 2I / 3m. \end{gathered} \tag{6.2}$$

In order to obtain the first of these equations we took the curl of both sides of equation (II) and utilized the explicit form for the kinetic coefficients η and ξ (4.5), (5.7).

The form of the perturbation (6.1) is especially chosen so that the solution of the system (6.2) will reduce to the determination of the time dependence of the quantities φ and ψ:

$$\begin{gathered} \varphi(t) = \varphi(0)e^{-\nu t} + (ak)^2[\psi(0) - \varphi(0)]\,(e^{-\nu t} - e^{-\nu_M t}), \\ \psi(t) = \psi(0)e^{-\nu_M t} + \{\varphi(0) + (ak)^2[\psi(0) - \varphi(0)]\}(e^{-\nu t} - e^{-\nu_M t}), \quad \nu = \nu_V(\lambda k)^2. \end{gathered} \tag{6.3}$$

Evidently the quantity ν^{-1} is the usual characteristic damping time due to the first viscosity. If t is large compared to the relaxation time for the rotational angular momentum ν_M^{-1}, then expressions (6.3) assume the form

$$\varphi(t) = \psi(t) = \{\varphi(0) + (ak)^2[\psi(0) - \varphi(0)]\}e^{-\nu t}. \tag{6.4}$$

We see that during a time of the order of ν_M^{-1} the vortex comes to a state of local equilibrium, so that $\boldsymbol{\Omega}_V = \boldsymbol{\Omega}_M$. At the same time, when at the initial instant the rotating gas had no polarization of the rotational angular momentum $\psi(0) = 0$, after local equilibrium has been established (but for $t \ll \nu^{-1}$) the angular velocity of the gas turns out to be altered by a very small amount of the order of $(ak)^2$ where $L = k^{-1}$ is the dimension of the vortex. Conversely, if at the initial moment the gas was not rotating, while the polarization (created, for example, by an inhomogeneous magnetic field which is switched off at time t = 0) differs from zero in a restricted region of space (of size L), then after a time of the order of ν_M^{-1} a very slow rotation of the gas arises with an angular velocity proportional to $(a/L)^2$. It is easy to understand the reason for the small influence that the polarization of the rotational angular momenta of the molecules has on the angular velocity of rotation of the gas: the moment of inertia of the vortex is of order of magnitude mL^5, while the total moment of inertia of the molecules is ma^2L^3. Therefore, in the case of equilibrium ($\boldsymbol{\Omega}_V = \boldsymbol{\Omega}_M$) the ratio of the angular momentum of translational motion to the total angular momentum of rotational motion of the molecules is equal to $(L/a)^2$, i.e., it is very great if the vortex has macroscopic dimensions.

We note that if we had attempted to solve the problem of the transfer of angular momentum from the translational degrees of freedom to the rotational ones without taking into account the finite size of the vortex, i.e., on the assumption that the angular velocity $\boldsymbol{\Omega}_V$ does not depend on the spatial coordinates, then we would at once have obtained from (6.2) that $\boldsymbol{\Omega}_V = \text{const}$, $\boldsymbol{\Omega}_M = \boldsymbol{\Omega}_M(0)e^{-\nu_M t} + \boldsymbol{\Omega}_V(1 - e^{-\nu_M t})$, i.e., the rotation of the gas as a whole is completely independent of the state of internal motion! It is clear that this fact is associated with the condition for the independence of $\mathbf{\Omega}_V$ from the coordinates being physically equivalent to the assumption of a vortex of infinitely large size.

Thus, the terms of HE proportional to the kinetic coefficient ξ enable one to trace the dynamics of the transition process from the state in which the angular velocity of translational motion $\mathbf{\Omega}_V$ and the polarization of the rotational angular momentum $\mathbf{\Omega}_M$ are specified independently, to the local equilibrium state in which

$$\mathbf{\Omega}_M = \mathbf{\Omega}_V = {}^1\!/_2 \operatorname{rot} \mathbf{V}. \tag{6.5}$$

However, for the majority of classical hydrodynamic problems the characteristic times are great compared to the time for the establishment of local equilibrium, and for these problems the terms of HE proportional to ξ do not play any role. Moreover, since at the same time equation (6.5) holds, then equation (III) loses its independent meaning, and in the law of increase of entropy (IV) terms quadratic in $\partial\mathbf{\Omega}_M/\partial x$ are in fact quadratic with respect to the second derivatives of the velocity and must therefore be neglected. We arrive at the conclusion that for the majority of hydrodynamic problems the HE system has its usual form in which the rotational degrees of freedom affect only the values of the kinetic coefficients.

Let the gas now be placed in a constant magnetic field. (The inhomogeneity of the magnetic field can be easily taken into account by adding to equation (II) the force $F_j = n\overline{\mu}_i \nabla_j B_i$). The transport equation (III) taking into account the inequality (5.9) in a magnetic field has the form

$$\partial\mathbf{\Omega}_M / \partial t = [\mathbf{\Omega}_M\mathbf{\Omega}_B] - \nu_M(\mathbf{\Omega}_M - \mathbf{\Omega}_V - \mathbf{\Omega}_B),$$

where the angular velocity, in accordance with the foregoing, can be regarded as a given quantity. The stationary solution of this equation is

$$\mathbf{\Omega}_M = \mathbf{\Omega}_V[\nu_M^2 / (\nu_M^2 + \mathbf{\Omega}_B^2)] + [\mathbf{\Omega}_V\mathbf{\Omega}_B][\nu_M / (\nu_M^2 + \mathbf{\Omega}_B^2)]$$
$$+ \mathbf{\Omega}_B[(\nu_M^2 + \mathbf{\Omega}_B^2 + \mathbf{\Omega}_V\mathbf{\Omega}_B) / (\nu_M^2 + \mathbf{\Omega}_B^2)].$$

At the same time, in accordance with equation IV

$$nTd\bar{S} / dt = \xi(\mathbf{\Omega}_M - \mathbf{\Omega}_B - \mathbf{\Omega}_V)^2 = \xi[\mathbf{\Omega}_V\mathbf{\Omega}_B]^2 / (\nu_M^2 + \mathbf{\Omega}_B^2).$$

This quantity is quite small. However, it is of significance in principle, demonstrating that the rotation of a gas in a constant magnetic field occurs with liberation of heat if the directions of the magnetic field and of the rotation do not coincide.

We assume that the gas as a whole is not rotating. Then the polarization of the rotational angular momentum is established at the level

$$\mathbf{\Omega}_M = \mathbf{\Omega}_B.$$

For oxygen (cf., (2.2), (3.6)) we have

$$\mathbf{\Omega}_B = -(\mu_0\hbar / IT)\mathbf{B}.$$

For a nonparamagnetic gas this quantity is smaller by an order of magnitude.

If the magnetic field is switched off, then the moment of momentum which is concentrated in the presence of a magnetic field in the rotational degrees of freedom, during a time ν_M^{-1} goes over practically entirely into the translational degrees of freedom, and the gas acquires a small angular velocity $\mathbf{\Omega}_V \approx \mathbf{\Omega}_B(a/L)^2$. This effect is an analog of the Einstein–de Haas effect for ferromagnets.

Evidently the inverse effect will also exist (an analogue of the Barnett effect)—the appearance of a magnetic moment in the gas in the presence of rapid rotation of the gas as a whole (cf., (6.5), (A.2))

$$\bar{\mu} = (\overline{\mu M} / 3T)\mathbf{\Omega}_V.$$

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APPENDIX

Let the state of the gas be described by the distribution

$$f = \exp\left\{\frac{1}{T}(F - E + \mathbf{V}\mathbf{p} + \mathbf{\Omega}\mathbf{M})\right\}. \tag{A.1}$$

For this distribution in the linear approximation with respect to the parameters $\mathbf{V}$, $\mathbf{\Omega}$ and the magnitude of the magnetic field $\mathbf{B}$ we have

$$\begin{gathered} n = 2(2\pi mT)^{3/2}(2\pi IT)e^{F/T}, \\ \bar{E} = 5T/2, \ \bar{\mathbf{v}} = \mathbf{V}, \ \bar{\mathbf{M}} = {}^2\!/_3 I\mathbf{\Omega} + (\overline{\mu \mathbf{M}} / 3T)\mathbf{B}, \\ \bar{\boldsymbol{\mu}} = (\overline{\boldsymbol{\mu}\mathbf{M}}\Omega + \overline{\boldsymbol{\mu}^2}\mathbf{B})/3T. \end{gathered} \tag{A.2}$$

If in the distribution (A.1) the parameters T, F and $\mathbf{\Omega}$ do not depend on the coordinates, while

$$\mathbf{V} = \mathbf{V}_0 + [\mathbf{\Omega}\mathbf{r}], \ \mathbf{V}_0 = \text{const}, \tag{A.3}$$

then the logarithm of this distribution can be decomposed into a sum of additive integrals of the motion (2.8). In this case the distribution (A.1) must make the collision integral vanish, and in this sense can be called the equilibrium distribution, with the vectors $\mathbf{V}_0$ and $\mathbf{\Omega}$ determining the velocity and the angular velocity of rotation of the gas as a whole.

We substitute the distribution (A.1) into the statistical definition of entropy calculated per molecule,

$$\bar{S} = -\overline{\ln f} = -(F - \bar{E} + \mathbf{V}\bar{\mathbf{p}} + \mathbf{\Omega}\bar{\mathbf{M}}) / T. \tag{A.4}$$

The differential of this expression is

$$d\bar{S} = \frac{dT}{T^2}(F - \bar{E} + \mathbf{V}\bar{\mathbf{p}} + \mathbf{\Omega}\bar{\mathbf{M}}) - \frac{1}{T}d(F - \bar{E} + \mathbf{V}\bar{\mathbf{p}} + \mathbf{\Omega}\bar{\mathbf{M}}). \tag{A.5}$$

Moreover, from the definition of the particle number density it follows that

$$n = \langle f \rangle, \quad dn/\bar{n} = -\frac{dT}{T^2}(F - \bar{E} + \mathbf{V}\bar{\mathbf{p}} + \mathbf{\Omega}\bar{\mathbf{M}}) + \frac{1}{T}(dF + \bar{\mathbf{p}}d\mathbf{V} + \bar{\mathbf{M}}d\mathbf{\Omega}). \tag{A.6}$$

Adding (A.5) and (A.6), we obtain

$$d\bar{S} + \frac{dn}{n} = -\frac{1}{T}(-d\bar{E} + \mathbf{V}d\bar{\mathbf{p}} + \mathbf{\Omega}d\bar{\mathbf{M}}). \tag{A.7}$$

This relation leads to equation (3.5) in the text of the article if with the aid of (A.2) one expresses the parameter $\mathbf{\Omega}$ in terms of the average value of the rotational angular momentum.

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