ROUGHNESS AND THERMAL RESISTANCE OF THE BOUNDARY BETWEEN A SOLID AND LIQUID HELIUM

I. N. ADAMENKO and I. M. FUKS

Khar'kov State University; Institute of Radiophysics and Electronics of the Ukrainian Academy of Sciences

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We calculate the heat transfer coefficient of a rough boundary between liquid helium and a solid. We show that when the wavelength of the phonon is of the order of the characteristic dimensions of the roughness, there is a unique spatial resonance. It follows from the results that the roughnesses can significantly alter both the magnitude of the heat-transfer coefficient and its temperature dependence.

INTRODUCTION

THE temperature jump on the boundary between a solid and HeII was discovered by Kapitza back in 1941[1]. A theoretical explanation of the observed phenomenon was given by Khalatnikov[2]. According to[2], the thermal jump is due to the obstacles encountered by the phonons as they go from the liquid helium into the solid (and back), this being connected with the low impedance of the boundary. The heat flux Q from liquid helium into the solid can be written in the form[2]

$$Q=\frac{\hbar c_1^2}{(2\pi)^2}\int_0^\infty n\left(\frac{\hbar c_1}{T}k\right)k^3\,dk\int_0^1 P(k,\vartheta)\cos\vartheta\,d\cos\vartheta. \tag{1}$$

Here n is Planck's function, c_1 is the velocity of sound in helium, T is the temperature in energy units, and $P(k, \vartheta)$ is the coefficient of penetration of a phonon with momentum $\hbar k$, incident from He on the separation boundary at an angle ϑ, into the solid. From this we obtain for the heat-transfer coefficient of a plane boundary

$$\dot{Q}_0=\frac{\Delta Q_0}{\Delta T}=\frac{2\pi^2\rho_1 c_1 F_0}{15\hbar^3\rho_2 c_2^3}T^3, \tag{2}$$

where ΔQ_0 is the resultant heat flux, ΔT is the temperature jump, c_2 is the sound velocity in the solid, ρ_1 and ρ_2 are respectively the densities of the helium and of the solid, and $F_0 \approx 1$ is a certain function of the elastic constants of the solid.

The phenomenon in question is not a unique property of the superfluid liquid. As follows from Khalatnikov's theory, all that is required for the existence of a temperature jump is the presence of two media with a low impedance of the separation boundary. Thus, for example, an analogous effect was also observed for the boundary between a solid and He^3 [3]. To be sure, in this case it is important to take into account, in the temperature region below 0.2°K, the collective excitations of the liquid He^3—the so-called zero sound. The heat-transfer coefficient has here, as before, a cubic dependence on the temperature[4].

Numerous experiments have shown that the temperature dependence of the heat-transfer coefficient deviates from cubic, and different authors have obtained different values even for the same materials. As to the absolute values of $\dot{Q}$, the discrepancies between them are even larger. For example, Fairbank and Wilks[5] obtained at $0.3 \le T \le 1$°K a heat transfer coefficient $\dot{Q} \approx 2.2\times10^{-2}T^2$[W/cm²°K] for a Cu–HeII boundary. Kuan Wei-yen[6] obtained $\dot{Q} \approx 4.8\times10^{-2}T^{2.6}$ in the temperature interval 0.57–2.075°K. Anderson, Connolly and Wheatley[7], who performed measurements from 0.08 to 0.9°K, obtained a heat-transfer coefficient that behaved like $8.3\times10^{-2}T^4$ at 0.6°K, and then changed to $2.8\times10^{-2}T^{3.5}$ in the region of 0.09°K, whereas substitution of all the constants in formula (2) yields for copper $\dot{Q}_0 \approx 2\times10^{-3}T^3$. In addition, all the known experiments (see, for example[6,8,9]) point to a weaker dependence of the heat-transfer coefficient on the external pressure than that which follows from formula (2).

Interest in the problem of the temperature jump is due apparently not only to the interest in the study of the physics of helium, the solid state, and surfaces, but also to a number of applications of this phenomenon, for example the production of infralow temperatures. The causes of the discrepancies between some experimental data and others and the deviations from Khalatnikov's theory were considered in a number of papers. Thus, in[8] the weak dependence of the temperature jump on the pressure is attributed to the existence at the surface of the solid of a denser layer of helium produced under the influence of the Van der Waals forces. The thickness of the denser layer is of the order of 15 Å. The latter is obviously not very sensitive to the action of external pressure. The increased value of the impedance of the boundary also explains why experiments reveal larger values of $\dot{Q}$ than predicted by formula (2). According to the calculations of[8], allowance for the denser layer should lead to a steeper dependence of the heat-transfer coefficient on the temperature ($\dot{Q} \sim T^{4.2}$).

Another reason for the discrepancy between theory and experiment is apparently, as indicated in[6], the presence of an amorphous layer (the Beilby layer) on the surface of the solid. The strongly deformed Beilby layer has a thickness of the order of 10^{-6} cm. On the other hand, the total thickness of the transition zone of the deformed metal is obviously of the order of 10^{-5} cm.

Finally, the real surface of a solid is not absolutely smooth. For helium phonons, the wavelength of which is $\lambda \approx 4\times10^{-7}T^{-1}$ cm, even very well-finished surfaces[1)] are sufficiently rough. The role of the roughness has been noted in practically all experimental investigations, starting with Kapitza's first work[1]. As to the theoretical estimates of the contribution of the roughness, they were made, insofar as is known, only in the papers by Little[10,11]. Thus, when the phonon wavelength λ is much smaller than the roughness dimensions l, then the microscopic area is the true area S. The latter, for steep roughnesses, can differ greatly from the area S_0 of the geometrical (smooth) surface. Thus, in this case allowance for the roughness leads to an increase of the heat-transfer coefficient compared with formula (2) if in the latter, as usual, ΔQ_0 is taken to mean the heat flux per unit of an absolutely smooth geometric surface. On the other hand, when $\lambda \gg l$, then obviously the effective area is the geometric one. In the general case $Q = \gamma(\lambda)Q_0$ (when $\lambda \ll l$ we have $\chi(\lambda) = S/S_0$, and when $\lambda \gg l$ we have $\chi(\lambda) = 1$).

At intermediate temperatures, when the phonon wavelength is comparable with the dimensions of the roughnesses, it follows from Little's calculations[11] that the heat transfer coefficient decreases ($\chi(\lambda) < 1$ when $\lambda \sim l$). This rather unexpected result is a consequence of the assumption made by Little that the principal role in phonon scattering from a rough surface is played by the heights of the surfaces and not by their inclinations. In the calculation, the inclinations of the surface were in general assumed to be equal to zero, whereas the difference between the heat-transfer coefficient of a rough surface and a smooth one turned out in the final result of[1] to be of exactly the same order as the inclinations of the surface.

In the present paper we have considered consistently the role of the roughnesses in the thermal resistance of the boundary. We show that the roughness can greatly change both the magnitude and the temperature of the heat-transfer coefficient, namely, the exponent can be both smaller and larger than 3.

1. DIFFRACTION OF PHONONS BY AN UNEVEN SURFACE

To calculate the penetration coefficient $P(k, \vartheta)$ in the case of a rough surface, let us consider the problem of the diffraction of a plane monochromatic sound wave $\exp[i(\mathbf{k}\cdot\mathbf{R} - \omega t)]$ incident from liquid He on the boundary with the solid. The equation of the surface separating the helium (medium 1) from the solid (medium 2) is specified in the form $z = \zeta(\mathbf{r})$. We shall henceforth take $\zeta(\mathbf{r})$ to mean a certain random function of two variables, $\mathbf{r} = \{x, y\}$, and its statistical characteristics are assumed to be known.

For simplicity we confine ourselves to an investigation of only the longitudinal waves arising in a solid when phonons are incident from the liquid helium on the boundary. Allowance for the transverse waves leads to an inessential change of some numerical coefficients and does not entail any fundamental difficulties, but the calculations become much more complicated and cumbersome. The potentials of the velocities in the first and second media will be denoted by $\varphi_1(\mathbf{r}, z)e^{-i\omega t}(z > \zeta)$ and $\varphi_2(\mathbf{r}, z)e^{-i\omega t}(z < \zeta)$. The potentials $\varphi_{1,2}(\mathbf{r}, z)$ satisfy the Helmholtz equations in media 1 and 2 and the boundary conditions on the surface that separates them:

$$\frac{\partial\varphi_1}{\partial N} - \frac{\partial\varphi_2}{\partial N} = 0, \qquad \rho_1\varphi_1 - \rho_2\varphi_2 = 0, \tag{3}$$

where N is the normal to the surface $z = \zeta(\mathbf{r})$. The problem consists of determining the potential $\varphi_2(\mathbf{r}, z)$ in the second medium, under the condition that a wave with potential $\varphi_{\text{inc}} = \exp(i\mathbf{k}\cdot\mathbf{R})$ is incident on the surface from the helium.

We assume first that the wavelength of the sound greatly exceeds the characteristic height of the roughness. The latter can be taken to equal the variance $\sigma = \sqrt{\langle\zeta^2(\mathbf{r})\rangle}$ of the deviations of the random function $\zeta(\mathbf{r})$ from the mean level $\langle\zeta(\mathbf{r})\rangle = 0$. Here and throughout $\langle\ldots\rangle$ denotes averaging over the ensemble of the realizations of $\zeta(\mathbf{r})$.

If $(k\sigma)^2 \ll 1$ and the angles of inclination of the surface to the median plane $z = 0$ are sufficiently small ($\gamma^2 = \langle|\nabla_{\mathbf{r}}\zeta(\mathbf{r})|^2\rangle \ll 1$), then the boundary conditions (3) can be transferred from the surface $z = \zeta(\mathbf{r})$ to the median plane $z = 0$[12-15] by expanding $\varphi_{1,2}(\mathbf{r}, z)$ in powers of ζ and using the relation

$$\frac{\partial\varphi}{\partial N} = \left[\frac{\partial\varphi}{\partial z} - \mathbf{j}\nabla_{\mathbf{r}}\varphi\right](1 + j^2)^{-1/2}. \tag{4}$$

Confining ourselves in (3) to the quadratic terms in the small heights ζ and inclinations $\mathbf{j} = \nabla_{\mathbf{r}}\zeta$, we arrive at a system of effective boundary conditions for φ_1 and φ_2 on the plane $z = 0$:

$$\frac{\partial}{\partial z}(\varphi_1 - \varphi_2) = (\mathbf{j}\nabla)(\varphi_1 - \varphi_2) - \zeta\frac{\partial^2}{\partial z^2}(\varphi_1 - \varphi_2) + (\mathbf{j}\nabla)\zeta\frac{\partial}{\partial z}(\varphi_1 - \varphi_2) - \frac{\zeta^2}{2}\frac{\partial^3}{\partial z^3}(\varphi_1 - \varphi_2), \tag{5}$$

$$\rho_1\varphi_1 - \rho_2\varphi_2 = \zeta\frac{\partial}{\partial z}(\rho_2\varphi_2 - \rho_1\varphi_1) + \frac{\zeta^2}{2}\frac{\partial^2}{\partial z^2}(\rho_2\varphi_2 - \rho_1\varphi_1). \tag{6}$$

It will be more convenient in what follows to change over to the Fourier representation of the potentials $\varphi_{1,2}(\mathbf{r}, z)$ and of the random function $\zeta(\mathbf{r})$:

$$\varphi_1(\mathbf{r}, z \geqslant 0) = \varphi_{\text{inc}} + \varphi_{\text{ref}} = \iint_{-\infty}^{\infty} d^2\varkappa\,[\delta(\mathbf{k}_\perp - \varkappa)\,e^{-ik_z^{(1)}z} + \Phi_1(\varkappa)\,e^{i\varkappa_z^{(1)}z}]\,e^{i\varkappa r},^* \tag{7a}$$

$$\varphi_2(\mathbf{r}, z \leqslant 0) = \iint_{-\infty}^{\infty} d^2\varkappa\,\Phi_2(\varkappa)\,e^{i(\varkappa r - \varkappa_z^{(2)}z)}, \tag{7b}$$

$$\zeta(\mathbf{r}) = \iint_{-\infty}^{\infty} d^2\mathbf{q}\,\tilde{\zeta}(\mathbf{q})\,e^{i\mathbf{q}\mathbf{r}}, \tag{7c}$$

where $\kappa_z^{(1,2)} = \sqrt{k_{1,2}^2 - \kappa^2}$ ($\operatorname{Im}\kappa_z^{(1,2)} > 0$), $k_z^{(1,2)} = \sqrt{k_{1,2}^2 - k_\perp^2}$, and $\mathbf{k}_\perp$ is the projection of the vector $\mathbf{k}$ on the plane $z = 0$. The coefficients $\Phi_{1,2}(\kappa)$ in (7a) and (7b) determine the amplitudes of the reflected and refracted plane waves, the projections of the wave vectors of which on the plane $z = 0$ are equal to κ. The waves with $\kappa < k_1$ in medium 1 (reflected) and

1) At the highest, class-14 surface finish the roughness is $\sim 10^{-6}$ cm.

* $\kappa\mathbf{r} = \kappa\cdot\mathbf{r}$.

with $\kappa < k_2$ in medium 2 (refracted) are homogeneous plane waves and the normal components $\kappa_Z^{(1,2)}$ of their wave vectors are real. The opposite inequalities correspond to inhomogeneous waves with amplitudes that decrease exponentially away from the surface z = 0, and which thus carry no energy away from the bounary--these are surface waves.

Substituting (7a)—(7c) in the boundary conditions (5) and (6), we obtain a system of integral equations for the amplitudes $\Phi_{1,2}(\kappa)$:

$$\Phi_\alpha(\varkappa) = V_\alpha(\mathbf{k})\delta(\varkappa - \mathbf{k}_\perp) + iC_\alpha(\varkappa, \mathbf{k})\tilde{\zeta}(\varkappa - \mathbf{k}_\perp)$$
$$+ i\sum_{\beta=1}^{2}\iint_{-\infty}^{\infty} d^2\mathbf{q}\tilde{\zeta}(\varkappa - \mathbf{q})A_{\alpha\beta}(\varkappa, \mathbf{q})\Phi_\beta(\mathbf{q}) + B_\alpha(\varkappa, \mathbf{k})\iint_{-\infty}^{\infty} d^2\mathbf{q}\tilde{\zeta}(\mathbf{q})\tilde{\zeta}(\varkappa - \mathbf{q} - \mathbf{k}_\perp). \quad (8)$$

The indices α and β assume here two values 1 and 2, and the following notation is used (for $\alpha \neq \beta$):

$$A_{\alpha\alpha}(\varkappa, \mathbf{q}) = (-1)^\alpha[\rho_\alpha \varkappa_z^{(\beta)} q_z^{(\alpha)} + \rho_\beta(k_\alpha^2 - \varkappa\mathbf{q})]D^{-1}(\varkappa),$$

$$A_{\alpha\beta}(\varkappa, \mathbf{q}) = (-1)^\alpha \rho_\beta[\varkappa_z^{(\beta)} q_z^{(\beta)} - k_\beta^2 + \varkappa\mathbf{q}]D^{-1}(\varkappa),$$

$$C_\alpha(\varkappa, \mathbf{k}) = [-\rho_\beta(k_1^2 - \varkappa\mathbf{k}_\perp) + (-1)^\beta \rho_1 k_z^{(1)} \varkappa_z^{(\beta)}]D^{-1}(\varkappa),$$

$$B_\alpha(\varkappa, \mathbf{k}) = -\frac{\rho_1 k_z^{(1)}(k_1^2 - k_2^2)}{D(\varkappa)D(k)}[\rho_\beta k_z^{(2)} + (-1)^\alpha \rho_2 \varkappa_z^{(\beta)}],$$

$$D(\varkappa) = \rho_2\varkappa_z^{(1)} + \rho_1\varkappa_z^{(2)}.$$

Finally, $V_1(\mathbf{k})$ and $V_2(\mathbf{k})$ denote the Fresnel coefficients for reflection and refraction by a plane surface:

$$V_1(\mathbf{k}) = [\rho_2 k_z^{(1)} - \rho_1 k_z^{(2)}]D^{-1}(k), \qquad V_2(\mathbf{k}) = 2\rho_1 k_z^{(1)} D^{-1}(k). \quad (9)$$

The solution of the system of integral equations (8) is, accurate to terms quadratic in ζ,

$$\Phi_\alpha(\varkappa) = V_\alpha(\mathbf{k})\delta(\varkappa - \mathbf{k}_\perp) + iG_\alpha(\varkappa, \mathbf{k})\tilde{\zeta}(\varkappa - \mathbf{k}_\perp) - \sum_{\beta=1}^{2}\iint_{-\infty}^{\infty} d^2\mathbf{q}\tilde{\zeta}(\varkappa - \mathbf{q}) \cdot$$
$$\cdot\tilde{\zeta}(\mathbf{q} - \mathbf{k}_\perp)A_{\alpha\beta}(\varkappa, \mathbf{q})G_\beta(\mathbf{q}, \mathbf{k}) + B_\alpha(\varkappa, \mathbf{k})\iint_{-\infty}^{\infty} d^2\mathbf{q}\tilde{\zeta}(\mathbf{q})\tilde{\zeta}(\varkappa - \mathbf{q} - \mathbf{k}_\perp), \quad (10)$$

where

$$G_\alpha(\varkappa, \mathbf{k}) = C_\alpha(\varkappa, \mathbf{k}) + \sum_{\beta=1}^{2} A_{\alpha\beta}(\varkappa, \mathbf{k})V_\beta(\mathbf{k}).$$

2. HEAT FLOW THROUGH A ROUGH BOUNDARY

Let us now calculate the average energy flux entering into the solid. The energy through a unit area perpendicular to the z axis per unit time is

$$\overline{p(\mathbf{r}, z)v_z(\mathbf{r}, z)} = \frac{\omega\rho_2}{2}\operatorname{Re}\left\{i\frac{\partial\varphi_2(\mathbf{r}, z)}{\partial z}\varphi_2^*(\mathbf{r}, z)\right\} \quad (11)$$

(p—pressure, v_Z--normal component of sound-wave velocity in the solid, the bar denotes averaging over the time). This quantity, unlike in the case of a plane boundary, depends, generally speaking, on the coordinates $\mathbf{r}$ and z, i.e., on the position of the unit area. The total energy flowing per unit time through a plane parallel to the boundary is obtained from (11) by integrating with respect to $d^2\mathbf{r}$ and, naturally, does not depend on z. Changing over to the Fourier representation in accordance with formula (7b), we obtain the following formula from (11) for the energy flux $E\downarrow$ averaged over the plane $\{x, y\}$:

$$E_\downarrow = \frac{\omega\rho_2}{2}\lim_{S_0\to\infty}\frac{1}{S_0}\iint_{S_0} d^2\mathbf{r}\operatorname{Re}\left\{i\frac{\partial\varphi_2}{\partial z}\varphi_2^*\right\}$$
$$= \frac{\omega\rho_2}{2}\lim_{S_0\to\infty}\frac{(2\pi)^2}{S_0}\operatorname{Re}\iint_{-\infty}^{\infty} d^2\varkappa\varkappa_z^{(2)}|\Phi_2(\varkappa)|^2. \quad (12)$$

As expected, only homogeneous waves with $\kappa < k_2$ contribute to the energy flux in the solid; the energy of each of these waves is proportional to the square of the amplitude $\Phi_2(\kappa)$, so that formula (12) has a simple physical meaning. By normalizing $E\downarrow$ to the energy flux in the incident wave $E_{inc} = \rho_1\omega_Z^{(1)}/2$ and averaging over the ensemble of realizations of the random surface $\zeta(\mathbf{r})$, we obtain for the penetration coefficient $P(k, \vartheta)$ the expression

$$P(k, \vartheta) = \frac{\rho_2}{\rho_1 k_z^{(1)}}\lim_{S_0\to\infty}\frac{(2\pi)^2}{S_0}\operatorname{Re}\iint_{-\infty}^{\infty} d^2\varkappa\varkappa_z^{(2)}\langle|\Phi_2(\varkappa)|^2\rangle. \quad (13)$$

Substitution of $\Phi_2(\kappa)$ from (10) in this formula results in an average, over the ensemble, of products of the type $\tilde{\zeta}(\mathbf{q}_1)\tilde{\zeta}(\mathbf{q}_2)$. Assuming the random function $\zeta(\mathbf{r})$ to be spatially homogeneous (in the sense that all its statistical characteristics are independent of $\mathbf{r}$), we define the correlation function $W(\rho)$ as follows:

$$\langle\zeta(\mathbf{r})\zeta(\mathbf{r} + \rho)\rangle = \sigma^2 W(\rho). \quad (14)$$

It is easy to verify that $W(\rho) = W(-\rho) \leq 1$, $W(0) = 1$, and $W(|\rho| \to \infty) \to 0$.

Further, from the definition (7c), as a direct consequence of statistical homogeneity, we obtain the following relations:

$$\langle\tilde{\zeta}(\mathbf{q})\tilde{\zeta}(\mathbf{q}')\rangle = \delta(\mathbf{q} + \mathbf{q}')\sigma^2\tilde{W}(\mathbf{q}), \quad \langle\tilde{\zeta}(\mathbf{q})\tilde{\zeta}^*(\mathbf{q}')\rangle = \delta(\mathbf{q} - \mathbf{q}')\sigma^2\tilde{W}(\mathbf{q}), \quad (15)$$

where $\tilde{W}(\mathbf{q})$ is the so-called "spatial spectrum" of the surface—the Fourier transform of the correlation function:

$$\tilde{W}(\mathbf{q}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} e^{-i\mathbf{q}\rho}W(\rho)\,d^2\rho. \quad (16)$$

Using the already noted properties of the correlation function, we can verify that $\tilde{W}(\mathbf{q})$ is an even, real, and nonnegative function of the two-dimensional vector $\mathbf{q}$. Recognizing, finally, that for a two-dimensional δ function we have by definition $\delta(0) = S_0/(2\pi)^2$, we obtain ultimately from (13)

$$P(k, \vartheta) = \frac{\rho_2}{\rho_1 k_z^{(1)}}\operatorname{Re}\Bigg\{k_z^{(2)}\Bigg[|V_2(\mathbf{k})|^2 - 2\sigma^2\operatorname{Re}V_2^*(\mathbf{k})$$
$$\times\iint_{-\infty}^{\infty} d^2\varkappa\tilde{W}(\mathbf{k}_\perp - \varkappa)\,[A_{21}(\mathbf{k}, \varkappa)G_1(\varkappa, \mathbf{k}) + A_{22}(\mathbf{k}, \varkappa)G_2(\varkappa, \mathbf{k})] \quad (17)$$
$$+ 2\sigma^2\operatorname{Re}V_2^*(\mathbf{k})B_2(\mathbf{k}, \mathbf{k})\Bigg] + \sigma^2\iint_{-\infty}^{\infty} d^2\varkappa\varkappa_z^{(2)}\tilde{W}(\varkappa - \mathbf{k}_\perp)|G(\varkappa, \mathbf{k})|^2\Bigg\}.$$

Formula (17), in conjunction with (1), solves, in general form, the problem of the heat flux Q through a rough surface separating liquid helium from a solid, with low $(k\sigma)^2 \ll 1$ and gently sloping $(\gamma^2 \ll 1)$ roughnesses. For the actual calculation of the quadratures it is necessary to know the correlation function of the roughnesses, the explicit form of which can be determined by the surface-finish method—grinding, electric polishing, etc. Under sufficiently general assumptions, the surface can be regarded as statistically isotropic, and the correlation function as Gaussian:

$$W(\rho) = e^{-\rho^2/l^2}, \quad \tilde{W}(\mathbf{q}) = \frac{l^2}{4\pi}e^{-q^2l^2/4}. \quad (18)$$

In this case l has the meaning of the characteristic dimension (length) of the roughnesses, which turns out to be connected with the root-mean-squared height of the roughnesses σ and their inclination γ by the relation $\gamma = 2\sigma/l$. It should be noted that for a semiquantitative description of the influence of the roughnesses and the heat exchange between helium and a solid, it suffices to know only these two parameters, γ and l. The explicit form of the correlation function is of little importance in this case and affects the values of constants of the order of unity, which enter in the final results. Such a stability of the final results against variations of the type of the function $W(\rho)$ is due to the fact that the quantities of interest to us (the penetration coefficient $P(k, \vartheta)$ and the heat flux Q) contain not the function $W(\rho)$ itself, but integrals of this function (see (1) and (17)). On the other hand, specification of the correlation function in the form (18) makes it possible to express the integrals contained in (7) in terms of tabulated functions.

Using the smallness of the parameters $c_1/c_2 \ll 1$ and $\rho_1/\rho_2 \ll 1$ and omitting the rather cumbersome derivations, we present directly the result for the integral of $P(k, \vartheta)$ with respect to $d\vartheta$, which enters in formula (1):

$$F(k) \equiv \frac{1}{2}\frac{\rho_2}{\rho_1}\left(\frac{c_2}{c_1}\right)^3 \int_0^1 P(k,\vartheta)\cos\vartheta\, d\cos\vartheta = \frac{2}{3}\left[1+\frac{1}{2}\gamma^2\psi\left(\frac{kl}{2}\right)\right], \tag{19}$$

where

$$\psi(x) = 2\left\{xe^{-x^2}\int_0^x e^{t^2}\,dt + x^4 e^{-x^2/2}\left[K_0\left(\frac{x^2}{2}\right)+K_1\left(\frac{x^2}{2}\right)\right]\right\},$$

and K_0 and K_1 are McDonald functions of orders 0 and 1. $F(k)$ is proportional to the probability, averaged over all the incidence angles ϑ, of penetration, into the solid, of a phonon with momentum $\hbar k$ incident on the separation boundary from the liquid helium, As seen from (19), the increment to this probability due to the roughnesses on the surface is always positive and has a rather sharp maximum at $kl \approx 2.5$, ($\psi_{max} \approx 4.5$). In regions far from the maximum, the increment to the penetration probability has the following asymptotic forms:

$$\psi(x \ll 1) \approx 6x^2, \qquad \psi(x \gg 1) \approx 1 + 1/2x^2$$
$$x = \tfrac{1}{2}kl. \tag{20}$$

Averaging of $kF(k)$ with the phonon distribution function over the wavelength, i.e., calculation of the integral with respect to dk in formula (1), leads to the following dependence of the heat flux on the temperature:

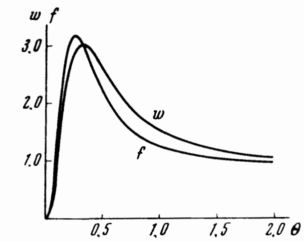
$$Q(T)/Q_0(T) = 1 + \tfrac{1}{2}\gamma^2 w(\Theta), \tag{21}$$

where $Q_0(T)$ is the heat flux from the helium into the solid for an ideally smooth surface[2],

$$Q_0(T) = \pi^2\rho_1 c_1 F_0 T^4 / 30\hbar^3\rho_2 c_2^3, \tag{22}$$

and $F_0 = 2/3$ in the simplified model considered here (without allowance for the transverse waves in the solid). The temperature dependence of the additional heat flux connected with the scattering of the phonons by the roughnesses is given by the function

$$w(\Theta) = \frac{240}{(2\pi\Theta)^4}\int_0^\infty n\left(\frac{x}{\Theta}\right)\psi(x)x^3\,dx, \tag{23}$$



where the dimensionless temperature $\Theta = lT/2\hbar c_1$ is introduced. A plot of $w(\Theta)$ is shown in the figure.

Thus, at a temperature $\Theta \approx 0.3$, when the thermal-phonon wavelength $\lambda \approx 2\hbar c_1/T$ is of the order of the linear dimensions l of the roughnesses, a unique spatial resonance takes place[2)]. At $\Theta \ll 1$, the additional heat flux increases in proportion to the square of the temperature, $w(\Theta \ll 1) \approx 113\,\Theta^2$, while in the region of high temperatures we have $w(\Theta \gg 1) \approx 1 + 0.127/\Theta^2$. Recognizing that when $\gamma^2 \ll 1$ the ratio of the area S of the rough surface to the area S_0 of its projection on the plane $z = 0$ is

$$\frac{S}{S_0} = \left\langle\frac{1}{N_z}\right\rangle = \left\langle\sqrt{1+(\nabla\zeta)^2}\right\rangle \approx 1 + \frac{\gamma^2}{2} \tag{24}$$

we can write the high-temperature asymptotic expression for $Q(T)$ in the form

$$Q(T) = Q_0(T)S/S_0. \tag{25}$$

At a small temperature difference ΔT between the liquid helium and the solid, the resultant heat flux is $\Delta Q = \dot{Q}\Delta T$. From (21) we obtain for the heat-transfer coefficient $\dot{Q}$

$$\dot{Q}/\dot{Q}_0 = 1 + \tfrac{1}{2}\gamma^2 f(\Theta),$$
$$f(\Theta) = w(\Theta) + \frac{\Theta}{4}\frac{dw(\Theta)}{d\Theta} = \frac{60}{(2\pi)^4\Theta^5}\int_0^\infty \frac{e^{x/\Theta}x^4}{(e^{x/\Theta}-1)^2}\psi(x)\,dx. \tag{26}$$

Here $\dot{Q}_0$ is the heat transfer coefficient of the smooth surface (2). As seen from the plot of the function $f(\Theta)$ in the figure, the maximum difference between the thermal conductivity of a smooth surface and that of a rough one occurs, as before, at a temperature $\Theta \approx 0.3$, corresponding to spatial resonance between the thermal phonons and the roughnesses. In the limiting cases of low and high temperatures we have

$$f(\Theta \ll 1) \approx 169\Theta^2, \quad f(\Theta \gg 1) \approx 1 + 0.0635\Theta^{-2}. \tag{27}$$

3. GENERALIZATION TO VERY ROUGH SURFACES

As follows from the foregoing results, the reasons for the increase of the heat flux through a rough surface compared with a flat helium-solid boundary are different at high and at low temperatures Θ. When $\Theta > 1$ the increase of the heat flux is due to the increase of the effective area of the surface resulting from the roughnesses of the boundary. On the other hand, when $\Theta \lesssim 1$, and the wavelength of the thermal phonons becomes comparable with the linear dimension l of the roughnesses, the scattering of the phonons by the surface becomes nearly diffuse. This causes

2) The possibility of resonant scattering was first noted in [13], where the diffraction of radio waves by a statistically rough surface was considered.

the phonons with $\sin\vartheta > c_1/c_2$, which make no contribution to the heat flux in the case of a smooth surface, owing to total internal reflection from the boundary, now to acquire a finite probability of becoming scattered by the roughnesses and carrying away energy from the surface into the interior of the solid. At still lower temperatures ($\Theta \ll 1$) the phonon wavelength increases and the roughnesses of the surface play an ever decreasing role—the reflection and refraction of the phonons occur in the same manner as for a plane surface.

The calculations in the preceding sections pertain, strictly speaking, only to the case of gently sloping roughnesses, with heights smaller than the thermal-phonon wavelengths. Although the latter limitation was used by us (in the expansion of the boundary conditions (3) and in the solution of the integral equation (8) for the Fourier amplitudes of the refracted waves), the results are actually independent of the height of the roughnesses, provided the latter are sufficiently gently sloping. Indeed, in the temperature region $\Theta \lesssim 1$ ($kl \lesssim 1$), where the influence of the roughnesses is the strongest, the inequality $(k\sigma)^2 \ll 1$ is always satisfied, since $\gamma^2 = (2\sigma/l)^2 \ll 1$. At higher temperatures, when $kl \gg 1$, scattering of the phonons by the surface obeys the laws of geometrical optics and the relative increase of the heat flux is completely independent of the temperature (and consequently of the ratio of the phonon wavelength to the height of the roughness) and is governed only by the effective area of the surface.

We note that this result holds true for roughnesses of arbitrary slope (not necessarily gently sloping), provided the characteristic curvature radii a of the surface greatly exceed the phonon wavelength, i.e., in the temperature region defined by the inequality $(ka)^{1/3} \gg 1$. In this case, to calculate the field that penetrates into the second medium, we can use the geometrical-optics approximation—at each point of the surface the refraction and reflection occur as if by the plane tangent to the surface at the given point.

If a plane wave $\varphi_{\text{inc}} = \exp(i\mathbf{k}\cdot\mathbf{R})$ is incident on the surface from the helium, then we have for the field φ_2 on the surface S

$$\varphi_2(\mathbf{r}) = V_2(\mathbf{k})e^{i\mathbf{kr}}, \quad \partial\varphi_2(\mathbf{r})/\partial N = -iV_2(\mathbf{k})\left[k_2^2 - k_1^2 + (\mathbf{Nk})^2\right]^{1/2}e^{i\mathbf{kr}},$$
$$\mathbf{r}\in S, \quad V_2(\mathbf{k}) \approx 2\rho_1/\rho_2. \tag{28}$$

From this we obtain for the energy flux from the surface into the solid

$$E_\downarrow = \frac{\omega\rho_2}{2}\lim_{S_0\to\infty}\frac{1}{S_0}\iint_S dS\,\mathrm{Re}\left\{i\frac{\partial\varphi_2}{\partial N}\varphi_2^*\right\} = \tag{29}$$
$$= \frac{2\omega\rho_1^2}{\rho_2 S_0}\mathrm{Re}\iint_S dS\sqrt{k_2^2 - k_1^2 - (\mathbf{Nk})^2}.$$

Normalizing this quantity to the flux of the incident energy and integrating over the angle ϑ, we obtain $F = F_0 S/S_0$. As a result we obtain formula (25), which thus is valid in the region of relatively high temperatures without limitations on the height and slopes of the roughnesses, for the total heat flux from the helium into the solid.

With decreasing temperature, the wavelength of the thermal phonons becomes comparable with the linear dimensions of the roughnesses and the scattering becomes more and more diffuse. However, whereas at $kl \approx 1$ and $\gamma^2 \ll 1$ the degree of diffuseness always remains small for small surface inclination angles, when $\gamma^2 \gtrsim 1$ and $kl \approx k\sigma \approx 1$ the degree of diffuseness can be arbitrarily large. Unfortunately at present we have not even a phenomenological theory of wave diffraction by an uneven surface with steep slopes and linear roughness dimensions comparable with the wavelength. It is only known from general consideration that in the limiting case of a very rough surface the radiation is scattered in accordance with Lambert's law (equal energies are scattered into equal solid angles in all directions). In this respect, such a roughness is equivalent to an absolutely black body—a phonon incident on its surface is reflected by the roughness in such a way that it loses completely the "memory" of its initial direction. Obviously, in this case not only the reflected radiation is isotropic, but also that penetrating into the solid, i.e., we can postulate for the Fourier amplitudes of the penetrating field (compare with (10)) the following distribution:

$$|\Phi_2(\varkappa)|^2 = \begin{cases} \dfrac{S}{(2\pi)^2}\left(2\dfrac{\rho_1}{\rho_2}\right)^2\dfrac{1}{\pi k_2^2}, & \varkappa < k_2 \\ 0 & \varkappa > k_2 \end{cases}. \tag{30}$$

Substituting this distribution in (13), we obtain

$$P(k,\vartheta) = \frac{8}{3}\frac{\rho_1}{\rho_2}\frac{k_2}{k_1\cos\vartheta}, \quad F = \frac{4}{3}\left(\frac{c_2}{c_1}\right)^2. \tag{31}$$

Thus, the thermal conductivity of an uneven surface at a temperature $\Theta \approx 0.3$, corresponding to maximum diffuseness of the scattering, can exceed in the limiting case of a very rough surface the thermal conductivity of a plane separation boundary by a factor $2(c_2/c_1)^2 \approx 10^2$. Such a possibility can be realized, at least partially if not fully, in the case of steep roughnesses $\gamma^2 \gtrsim 1$. The physical cause of the anomalously large thermal conductivity in this case is quite simple. In the case of an ideally smooth surface, the phonon incident at an angle $\sin\vartheta > c_1/c_2$ excites in the solid an inhomogeneous wave, the normal component of whose wave vector $k_z^{(2)} = k_2(1 - k_1^2 k_2^{-2}\sin^2\vartheta)^{1/2}$ is an imaginary quantity, so that such a wave does not carry energy away from the surface. As follows from (9), all that changes with the angle ϑ in the reflection coefficient $V_1(\mathbf{k})$ is the phase, while the modulus remains constant and equal to unity. But if the surface is rough, then such an inhomogeneous wave propagating along the boundary can become transformed, as it is scattered from the roughnesses, into a volume wave and carry away energy from the surface. This case is perfectly analogous to a phenomenon known in optics, whereby the surface wave produced on total internal reflection is scattered by inhomogeneities (see, for example,[16]).

In the case of gently sloping roughnesses, such an effect is certainly small and therefore calculations performed as above by perturbation theory lead to an additional heat flux, which is small to the extent that the parameter γ^2 is small. In the opposite limiting case of a very rough surface with $\gamma^2 \sim 1$ at a temperature $\Theta \approx 0.3$, one should actually expect no inhomogeneous waves to occur, and there is a near-unity probability that diffuse scattering of the inhomogeneous wave and its transformation into a volume wave occur at distances on the order of the dimension of the roughnesses (which at resonance is precisely equal to the wavelength of the surface phonon). The number of phonons transporting energy into the solid increases thus by an amount equal to the ratio of the solid angle $\sin \vartheta < c_1/c_2$ to the solid angle of the entire hemisphere, i.e., by a factor $2(c_2/c_1)^2$.

We emphasize once more that such an anomalous increase of the thermal conductivity pertains only to the limiting case of completely diffuse scattering. We note, however, that the existence of a surface that satisfies the last requirement completely is impossible for the same reason that the existence of an absolutely black body is impossible. Thus the increase of the thermal conductivity of the boundary by a factor $2(c_2/c_1)^2$ is the upper limit of the effect of roughnesses under real conditions.

4. DISCUSSION OF RESULTS

We note that by using the perturbation method without any fundamental changes it would be possible, of course, to take into account the contribution of the transverse waves to the heat-transfer coefficient. The complexity of the calculations is then greatly increased[3)]. It is clear, however, that no fundamental differences can arise between the results obtained above and those obtained by solving the more general problem, for when account is taken of the transverse waves, the heat fluxes through the smooth and through the rough surfaces increase by equal degrees. A transverse wave appears in the solid in addition to the longitudinal wave investigated by us, and leads, roughly, to a doubling of the heat flux through the boundary. One can hope that the formula (26) obtained above, which takes into account the influence of the roughnesses on the thermal resistance of the boundary, remains valid also in the general case, accurate to insignificant changes of the explicit form of the function $f(\Theta)$.

Particular notice should be taken of the role of the roughnesses in the transformation of the surface Rayleigh waves into volume waves, and to the ensuing increase of the heat flux. Calculations in[17] show that even in the case of small roughnesses the damping of the Rayleigh waves by scattering from the roughnesses and by transformation into volume waves can be quite appreciable. This mechanism of transformation of Rayleigh waves, unlike the mechanism connected with the conduction electrons[18], is not sensitive to the state of the metal--normal or superconducting. It is possible that the predominance of the first of the two mechanisms explains why a metal had the same thermal resistance in the normal and in the superconducting states in certain experiments (see, for example[6]).

Thus, allowance for the roughnesses of the boundary always leads to an increase of the heat-transfer coefficient compared with (2). As to the temperature dependence, it can be either weaker or stronger than cubic, depending on the location of the temperature interval in question relative to the maximum (see the figure). Thus, for $l \approx 10^{-6}$ cm one should expect the maximum at $T \approx 0.1°K$. It should be particularly emphasized here that when speaking of roughnesses, it should be kept in mind that the surfaces can have many parameters, i.e., besides the large-scale roughnesses obtained during the surface finishing of the sample, there can exist also small-scale roughnesses due to the microstructure of the boundary. At a suitable slope, it may turn out that it is the latter that play the principal role.

We note also that, as follows from (31), the roughnesses can also change the dependence of the heat-transfer coefficient on the pressure, compared with the dependence given by formula (2) (where $F = F_0 \approx 1$), since the speed of sound in liquid helium changes noticeably with changing external pressure.

A more concrete comparison of the foregoing calculations and the conclusions that follow from them with the available experimental data is made very difficult by a number of factors. First, there are at present no experiments in a sufficiently wide temperature interval, so that it is impossible to trace completely the change of the temperature dependence of the heat-transfer coefficient. Second, usually little is known of the surface state of the sample. Third, besides the roughnesses, other heat-transfer mechanisms, not accounted for by Khalatnikov's theory (and mentioned at the beginning of the paper) can make a noticeable contribution to the heat flux. All these mechanisms, like the boundary roughnesses, increase the heat-transfer coefficient. As to the temperature dependence of the latter, insofar as we know, only roughnesses can lead to a weaker dependence compared with that predicted by formula (2).

We note finally that when infralow temperatures are to be obtained, the presence of steep "resonant" roughnesses on the separation boundary should lead to a sharp decrease of the thermal resistance of the boundary.

In conclusion, we consider it our pleasant duty to thank M. I. Kaganov and É. A. Kaner for valuable advice and remarks made during the course of the work.

[3)]As the first step towards solving this problem, one can use the results of [14], where the scattering of a plane acoustic wave by the surface of a solid with two-dimensional roughnesses having generators perpendicular to the incidence plane, was investigated in first-order perturbation theory.

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