COLLECTIVE SLOWING DOWN OF AN INTENSE BEAM OF RELATIVISTIC ELECTRONS IN A DENSE PLASMA TARGET

L. I. RUDAKOV

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The kinetic equation is used to investigate the instability of a relativistic beam of electrons in a plasma. It is shown that the statement made in [3], namely that the slowing down of an initially monochromatic beam occurs just as in the one-dimensional model, is valid only at not too intense beams, $(n'/n)^{1/3}\gamma \ll 1$ (n' and n are the particle concentrations in the beam and in the plasma, $\gamma = (1 - v^2/c^2)^{-1/2}$). The slowing down of intense beams and of beams having an initial particle-velocity scatter with respect to the angles $\overline{\theta}$ larger than $(n'/n)^{1/6}\gamma^{-1/2}$ may be the consequence of only the induced Cerenkov effect ("kinetic" instability). However, the quasi-one-dimensional character of the slowing down of the beam remains also in the case of a kinetic instability, so that the characteristic instability increment is determined by the initial $\overline{\theta}$. The role of the mechanism of nonlinear scattering of Langmuir oscillations is analyzed. A formula is obtained for the mean free path of a relativistic electron beam in a dense target. It is shown that two-stream instability may increase the effective frequency of plasma-electron scattering and by the same token increase the rate of heating of the plasma in the target as a result of ohmic dissipation of the back current and the associated slowing down of the beam.

1. INTRODUCTION

RECENT progress in the production of intense beams of relativistic electrons[1, 2] gives grounds for hoping that they can be used to heat a plasma to thermonuclear temperatures. It is therefore of interest to ascertain the efficiency of collective mechanisms of slowing down of relativistic electron beams in a plasma.

In a theoretical analysis of this phenomenon, the linear approximation, which indicates that the beam is unstable, does not answer the question fully. In the quasilinear approximation, which takes into account the reaction of the instability-induced oscillations on the beam, this problem was recently considered in a paper by Faĭnberg, Shapiro, and Shevchenko.[3] In the case of a sufficiently monoenergetic beam of electrons traveling in parallel and having not too high energies (see below), we can confine ourselves to this approximation.

The problem of using electron beams to heat a dense plasma to thermonuclear temperature leaves its unique imprint on the theoretical problem which we intend to consider. First, in a dense plasma, even of high temperature, one cannot neglect the dissipation of the oscillation energy as a result of Coulomb collisions. Second, to attain a high energy concentration in a beam entering into a target, it is desirable that the beam be prefocused. Focusing of an intense relativistic beam and its macroscopic stability in a plasma were investigated by the author, jointly with Ivanov, in [4]. However, radial focusing of the beam leads to an increase in the scatter of the transverse velocities of the beam particles. As we shall show below, the mean free path of the beam depends strongly on the initial value of this quantity. Unlike in the case considered in [3], the instability of the beam may be kinetic from the very beginning, i.e., a spectrum of Langmuir plasma oscillations with a sufficiently broad phase-velocity spectrum will be excited at phase resonance with each oscillation of a small group of beam particles.

In the case of the kinetic instability, an important role is played practically always by effects of nonlinear scattering of waves by plasma particles, which limit the amplitudes of the unstable oscillations and lead to an increase of the length of the collective slowing down of the beams.[5] Analysis of the nonlinear equations for a two-stream instability yields a formula for the mean free path of a beam of relativistic electrons in a dense plasma target.

2. INCREMENT OF TWO-STREAM INSTABILITY IN THE QUASILINEAR APPROXIMATION

In developing a quasilinear theory for the collective slowing down of a relativistic electron beam in a plasma, we shall use [3]. Some of our results duplicate the results of that reference. In addition, we construct in this section, a quasilinear theory of the instability of the beam with relatively large scatter of the transverse particle velocities.

In the absence of a field or in a relatively weak magnetic field, $H^2 \ll 4\pi nmc^2$, a beam with a particle concentration n' in a plasma having a much higher density n will build up longitudinal Langmuir oscillations most rapidly.[3] In the stage of the so-called "hydrodynamic" instability, when the beam can be regarded as "monochromatic," the oscillation growth increment δ for a relativistic beam depends strongly on the angle θ' between the direction of the wave vector $\mathbf{k}$ and the average beam velocity $\mathbf{v}_0$. This is a consequence of the anisotropy of the relativistic mass. The monochromaticity condition is of the form $\mathbf{k}\cdot\Delta\mathbf{v}$, where $\Delta\mathbf{v}$ is the velocity scatter in the beam. If this condition is satisfied, then the increment for the most unstable perturbations for which $\mathbf{k}\cdot\mathbf{v}_0 = \omega_p$, is equal to [3]

$$\delta \approx \omega_p\left(\frac{n'}{n\gamma}\right)^{1/3}\left(\frac{k_z^2}{k^2}\frac{1}{\gamma^2}+\frac{k_\perp^2}{k^2}\right)^{1/3}. \quad (1)$$

Here $\gamma = (1 - v_0^2/c^2)^{-1/2}$ is the relativistic factor, k_z and $k_\perp$ are the projections of the wave vector on directions parallel and perpendicular to the direction of the beam motion, and ω_p is the frequency of the Langmuir oscillations of the plasma.

As shown in [3], the predominant growth of the oscillations with $k_\perp \approx k$ leads to an increase in the scatter of the transverse velocities of the beam particles and to an appreciable decrease and change in the angular dependence of δ. The main laws governing this stage of the instability process can easily be traced qualitatively, both in the short stage of a monochromatic beam and in the "kinetic" stage, by using an estimate, obvious from the point of view of elementary processes, for the ratio of the rates of broadening of the beam-particle longitudinal ($\overline{\Delta p_z}$) and transverse ($\Delta p_\perp$) momentum distribution functions relative to their mean values:

$$\frac{d\overline{\Delta p_z}}{dt} \Big/ \frac{d\overline{\Delta p_\perp}}{dt} \approx \frac{k_z}{k_\perp}. \tag{2}$$

This estimate follows, for example, from formula (7) of the cited paper. We note that the increase of the scatter of the values of the longitudinal momentum of relativistic beam particles leads to a small scatter with respect to the longitudinal velocities, and may not violate the monochromaticity condition

$$k_z\overline{\Delta v_z} \ll \delta, \quad \overline{\Delta v_z} = -c\frac{\overline{p_\perp^2}}{p^2} + c\frac{\overline{\Delta p}}{p}\frac{1}{\gamma^2}. \tag{3}$$

It is violated principally because of the appearance of a transverse scatter $\Delta p_\perp$ and most readily for large $k_\perp$:

$$k_\perp c\overline{\Delta p_\perp}/p > \delta. \tag{3'}$$

Furthermore, perturbations with large $k_\perp \approx k$ are, according to (1), the most unstable and, in accordance with the estimate (2), lead to the appearance of a transverse momentum scatter.

In addition to formula (1), we present expressions for the increment during the stage of the kinetic instability:

$$\delta \approx \frac{\pi}{2}\frac{\omega_p^3}{k^2}\frac{n'}{n} m \int \left(\mathbf{k}\frac{\partial f}{\partial \mathbf{p}}\right)\delta(\omega - \mathbf{kv})\,d\mathbf{p}$$
$$= \pi\frac{\omega_p^3}{k^2 c}\frac{n'}{n} m \int p\,dp\,\theta\,d\theta \left[\left(\frac{\omega}{kc} - \frac{k_z}{k} - \frac{p-p_0}{p\gamma^2} - \frac{\theta^2}{2}\right)\frac{1}{\theta}\frac{\partial f}{\partial \theta}\right.$$
$$\left. + p\frac{\partial f}{\partial p}\right]\left[\theta'^2\theta^2 - \left(\frac{\omega}{kc} - \frac{k_z}{k} - \frac{p-p_0}{p\gamma^2} + \frac{\theta^2}{2}\right)^2\right]^{-1/2}, \tag{4}$$
$$\frac{\delta^2}{k^2c^2} \ll \max\left\{\theta'^2\overline{\theta}^2, \overline{\left(\frac{\theta^2}{2} + \frac{p-p_0}{p\gamma^2}\right)^2}\right\}.$$

Here $f(p, \theta)$ is the beam-particle distribution function. It is assumed to be independent of the azimuthal angle φ, and is normalized as follows: $\int f d^3p = 1$. In the last line, the increment has been rewritten in a form more convenient for the subsequent exposition, in the spherical coordinates (p, θ, φ) and (k, θ', φ'), with the polar axis along the average beam velocity $v_0 \approx c$ for small angles θ and θ'.

Violation of the monochromatic-beam approximation, when it is necessary to use formula (4) instead of (1), occurs the latest for angles $\theta' < \overline{\theta}$, where $\overline{\theta}^2 = \overline{\Delta p_\perp^2}/p^2$ is the average angle scatter in the beam.

The instability of the beam will be described by formula (4) even if the beam is monoenergetic, if $\overline{\theta}^2 \gg \delta/kc$. Such a situation can arise if the beam was focused on the path to the target, for example, by its own magnetic field.[4]

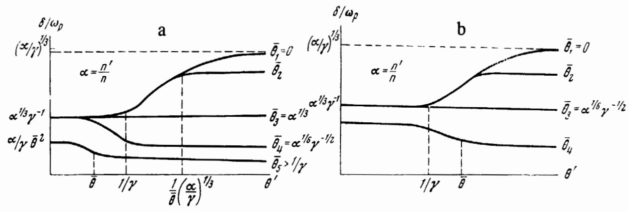

FIG. 1. Plots of the instability increment of a beam of relativistic electrons in a plasma, maximized with respect to the parameter $\omega - kc$, against the angle θ' at $\overline{\Delta p} \ll p\overline{\theta}^2\gamma^2$ for two cases: a—$(n'/n)^{1/3}\gamma \ll 1$, b—$(n'/n)^{1/3}\gamma \ll 1$.

In order to get an idea of the character of the $\delta(\theta')$ dependence, we present an expression obtained for the increment from (4) in the particular case when the distribution function is given by $f \sim \exp(-\theta^2/\overline{\theta}^2 - (p-p_0)^2/\overline{\Delta p^2})$:

$$\delta = \pi\omega_p \frac{n'}{n\gamma}\frac{2}{\overline{\theta}^4}\int \frac{dp}{\sqrt{\pi}\,\overline{\Delta p}} \exp\left\{-\frac{(p-p_0)^2}{\overline{\Delta p}^2} - \frac{\theta'^2 + 2y}{\overline{\theta}^2}\right\}\cdot$$
$$\cdot y\left\{\left(1 - \frac{\overline{\theta}^2}{2y} + \frac{(p-p_0)p}{(\overline{\Delta p})^2}\right) I_0(x) - \frac{\theta'}{\sqrt{2}\,y}\frac{dI_0}{dx}\right\} \tag{5}$$
$$= \omega_p\frac{n'}{n\gamma}\begin{cases} 2\pi\overline{\theta}^{-2}\exp\left(-\dfrac{2\Delta}{\overline{\theta}^2}\right)\left(\dfrac{2\Delta + \gamma^{-2}}{\overline{\theta}^2} - 1\right), & \theta' \ll \overline{\theta}, \\ \dfrac{\sqrt{\pi}}{\overline{\theta}^2}\dfrac{\theta'}{\overline{\theta}}\exp\left[-\dfrac{\theta'^2}{\overline{\theta}^2}\left(1 - \dfrac{\sqrt{2}\,\Delta}{\theta'}\right)^2\right]\left(\dfrac{\sqrt{2}\,\Delta}{\theta'} - 1\right), & \theta' \gg \overline{\theta}. \end{cases}$$

Here

$$x = \frac{2\theta'\sqrt{2y}}{\overline{\theta}^2}, \quad y = 1 - \frac{\omega}{kc} + \frac{p-p_0}{p_0\gamma^2} \qquad \Delta = 1 - \frac{\omega}{kc}$$

Here $\overline{\Delta p} \ll p\overline{\theta}^2\gamma^2$, $I_0(x)$ is a Bessel function.

In Fig. 1 are plotted, with the aid of formulas (1), (4), and (5), several curves showing the dependence of the increment δ, maximized with respect to the parameter $\omega - kc$, on the angle θ' at characteristic values of the mean scatter of the particle velocities in the beam relative to the angle θ. Figure 1a corresponds to the case $(n'/n)^{1/3}\gamma \ll 1$, and Fig. 1b to the case $(n'/n)^{1/3}\gamma \gg 1$.

The decrease of the increment δ compared with the case of a monochromatic beam is the result of the increase of the scatter $\overline{\theta}$ in the beam and of the transition of the instability into the kinetic stage. An instability of perturbations with small θ' can have a "hydrodynamic" character at the same time that one with large θ' has a kinetic character.

The kinetic instability in the quasilinear approximation, i.e., when the nonlinear interaction of the oscillations is insignificant, is described by the following system of equations:

$$\frac{\partial N_k}{\partial t} + \frac{\partial \omega}{\partial \mathbf{k}}\nabla N_k - \nabla\omega\frac{\partial N_k}{\partial \mathbf{k}} = 2\delta_k N_k - \frac{\nu_{ei}}{2}N_k, \tag{6}$$
$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = \frac{\partial}{\partial p_i}D_{ij}\frac{\partial f}{\partial p_j}, \quad D_{ij} = \pi e^2\int d^3k\frac{k_i k_j}{k^2}N_k\omega_k\delta(\omega - \mathbf{kv}), \tag{7}$$

ν_{ei} is the frequency of the Coulomb collisions, $w_k = N_k\omega_k$ is the spectral density of the oscillation energy. The increment δ is determined by the formula (4). We shall henceforth use Eq. (7) at small angles θ and θ':

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial z} = \frac{1}{p^2\theta}\frac{\partial}{\partial \theta}\theta D_\theta\frac{\partial f}{\partial \theta} + \frac{1}{p\theta}\frac{\partial}{\partial \theta}\theta D_{\theta p}\frac{\partial f}{\partial p}$$

$$+\frac{1}{p^2}\frac{\partial}{\partial p}pD_{\theta p}\frac{\partial f}{\partial \theta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D_p\frac{\partial f}{\partial p},$$

$$\left.\begin{matrix} D_\theta \\ D_{\theta p} \\ D_p \end{matrix}\right\} = 2\pi e^2\int\frac{k^2dk\theta' d\theta'\omega_k N_k(\theta',k)}{[\theta'^2\theta^2-(y-{}^1/_2(\theta'^2+\theta^2))^2]^{1/2}}\left\{\begin{matrix} \left[\left(y-\frac{\theta^2+\theta'^2}{2}\right)\frac{1}{\theta}-\theta\right]^2 \\ \left[\left(y-\frac{\theta^2+\theta'^2}{2}\right)\frac{1}{\theta}-\theta\right], \\ 1 \end{matrix}\right.$$

$$y=1-\frac{\omega}{kc}+\frac{p-p_0}{p\gamma^2}. \qquad (7')$$

In this section we shall discuss predominantly slowing down of a beam in time in a homogeneous plasma, neglecting the damping of the Langmuir oscillations as a result of Coulomb collisions.

If $(n'/n)^{1/3}\gamma \ll 1$, then during a time interval $t \lesssim (n'/n\gamma)^{-1/3}\cdot\omega_p^{-1}$, an initially monochromatic beam will excite oscillations with an increment that is practically constant in a large solid angle $\theta' > (n'/n\gamma)^{1/3}/\overline{\theta}(t)$, and with a deep minimum at $\theta' < 1/\gamma$. On the plateau of the increment, the instability is kinetic and can be described by a system of quasilinear equations. The influence of oscillations with smaller θ' on the evolution of the distribution function can be neglected during this stage, since these oscillations grow with a smaller increment and, furthermore, their relative contribution to the quasilinear diffusion coefficient is smaller by an amount equal to the ratio of the solid angles. For the same reason, the quasilinear broadening of the distribution function is determined by the oscillations with $k_\perp \approx k$ and leads, as is seen from (7), to an approximately uniform increase of $\overline{\theta} = \overline{\Delta p_\perp}/p$ and $\overline{\Delta p_z}$.

When $\overline{\theta}$ becomes larger than $(n'/n)^{1/3}$ but smaller than $1/\gamma$, the increment for the angles $\theta' > 1/\gamma$ becomes smaller by a factor $(n'/n)^{2/3}/\overline{\theta}^2$ than the increment at small angles, when the scatter of the velocity with respect to the angle in the beam still does not exert any influence. From that time on, oscillations with small θ' will be preferentially excited, and the phase of rapid slowing down of the beam by the hydrodynamic instability will set in, without an appreciable change of $\overline{\theta}$ and with a characteristic time $\omega_p t = (n/n')^{1/3}/\gamma$. While the longitudinal-velocity scatter reaches a value $k\overline{\Delta v_z} \approx \delta$ and the longitudinal-momentum scatter the value $\overline{\Delta p_z} \approx p\gamma^2\delta$, the beam energy lost to excitation of oscillations during this instability stage is

$$c\overline{\Delta p} \approx (n'/n)^{1/3}\gamma p_0 c \qquad (8)$$

and may amount to an appreciable fraction of the beam energy. This result was first reported in [3]. Further slowing down of the beam can be connected only with the kinetic instability.

If the beam is strongly relativistic $(n'/n)^{1/3}\gamma \gg 1$, then the process of slowing down of an initially monochromatic beam should proceed in a somewhat different manner. In this case the previously described initial stage of almost isotropic beam broadening with respect to $p_\perp$ and p will continue until the average angle scatter $\overline{\theta}$ reaches a value $\gamma^{-1/2}(n'/n)^{1/6}$. This scatter suffices to violate the monochromaticity condition for all angles θ', including small ones, unlike the case when $(n'/n)^{1/3}\gamma \ll 1$. The "hydrodynamic" instability does not play an essential role in this case, since the increase of $\overline{\theta}$ leads to an increase of the longitudinal-velocity scatter $\overline{\Delta v_z} = -c\overline{\theta}^2/2$, something not taken into account in [3].

As noted in the introduction, practical interest attaches also to a different formulation of the problem, when the beam injected into the plasma target has a noticeable angle scatter $\overline{\theta}_0$ resulting from the focusing of the beam. If $\overline{\theta}_0 > (n'/n)^{1/6}\gamma^{-1/2}$, then the beam can have only the kinetic instability described in the quasilinear approximation by Eqs. (6) and (7), and the main conclusions of [3] are not applicable. We shall therefore concentrate our attention on the case of a beam with a large angle scatter of the velocities. We shall leave out the narrow interval of $\overline{\theta}_0$ values $1/\gamma > \overline{\theta}_0 > (n'/n)^{1/6}\gamma^{-1/2}$, which is possible in the case $(n'/n)^{1/3}\gamma \ll 1$, and discuss in detail the case when $\overline{\theta}_0 > 1/\gamma$.

At such a large angle scatter of the velocities, the scatter of the momenta, even if $\overline{\Delta p} \approx p$, corresponds to a relatively small scatter of the angular velocities, and we shall neglect them compared with $\overline{\Delta v_z} = -c\overline{\theta}^2/2$, thereby greatly facilitating the analysis of the system (6) and (7). In this limiting case, the instability increment (4) also has a maximum at $\theta' = 0$, but it is not very strongly pronounced. For the smooth trial function used in (5), the maximum of the increment exceeds its value maximized with respect to the parameter $1-\omega/kc$, at large angles θ', by an amount smaller than 0.1δ. This result was obtained from formula (5), and the qualitative form of δ is shown by the lowest curves of Fig. 1. Such a maximum, of course, does not suffice, as in the case of a small angle scatter $\overline{\theta} < (n'/n)^{1/6}\gamma^{-1/2}$, to conclude that the oscillations predominantly excited are those with small θ', and that these determine the value of the diffusion coefficient and lead to the slowing down of the beam, without substantially increasing the initial angle scatter. This is nevertheless the case, but in order to verify this it is necessary to analyze attentively the properties of the system of quasilinear equations (6) and (7).

Let us rewrite expressions (4) for δ in a more convenient form, neglecting in it the terms with Δp:

$$\delta=\pi\omega_p\frac{n'}{n\gamma}p_0\int\theta d\theta\left[-\left(\Delta+\frac{\theta^2-\theta'_2}{2}\right)\frac{1}{\theta}\frac{\partial\Phi}{\partial\theta}-2\Phi\right]\left[\theta'^2\theta^2-\left(\Delta-\frac{\theta'^2+\theta^2}{2}\right)^2\right]^{-1/2}, \qquad (9)$$

$$\Phi(\theta)=\int_0^\infty fpdp, \qquad \Delta=1-\frac{\omega}{kc}.$$

We shall prove that quasilinear diffusion of the beam particles by oscillations with large θ', described by Eq. (7), leads to a function Φ with a steeper front. When $\theta' \ll \theta$, the expression for δ can be represented in the form

$$\delta=\pi\omega_p\frac{n'}{n\gamma}\pi p_0\int_0^\infty\theta d\theta\left[-\theta\frac{\partial\Phi}{\partial\theta}-2\Phi\right]\delta\left(\frac{\theta^2}{2}-\Delta\right)$$
$$=\frac{\pi}{2}\omega_p\frac{n'}{n\gamma}2\pi p_0\left[-4\Delta\frac{\partial\Phi}{\partial\theta^2}-2\Phi\right]_{\theta^2=2\Delta}. \qquad (10)$$

Thus, for a function Φ with a steep front the increment δ turns out to be much larger as $\theta'\to 0$ than when $\theta' \gg \overline{\theta}$:

$$\delta(x)=2\pi\omega_p\frac{n'}{n\gamma}\frac{1}{\overline{\theta}^2}\left(\frac{\theta^2}{x^2}-1\right)^{-1/2}, \quad x=\frac{\Delta-{}^1/_2\theta'^2}{\theta'}. \qquad (11)$$

A beam of relativistic electrons with $\overline{\theta}_0 > 1/\gamma$ and with an initially smooth function $\Phi(\theta)$ will excite oscillations in a wide solid angle θ' with approximately the same increment. The values of the diffusion coefficients in (7) will be determined in this case by the large angles θ'.

When $\overline{\theta} > 1/\gamma$ and when the momentum scatter is small, we can separate from (7) an equation that determines the evolution of the function $\Phi(\theta)$ under the influence of oscillations with $\theta' \gg \overline{\theta}$. To this end, we integrate this equation with respect to p, with a weight p, and use the fact that

$$\int p^3 f dp = p_0^2 \Phi\left(1 + O\left(\frac{\overline{\Delta p}}{p}\right)\right).$$

Then, accurate to terms of order $\overline{\theta}$ and $\overline{\Delta p}/p_0$, we obtain from (7)

$$\frac{\partial \Phi}{\partial t} = \pi \frac{\partial}{\partial \theta^2} \frac{8e^2}{\omega p_0^2} \int \frac{k^2 w_k x^2\, dk \theta'^2\, d\theta'}{(\theta^2 - x^2)^{1/2}} \frac{\partial \Phi}{\partial \theta^2}, \quad dk = k\theta'\, dx. \tag{12}$$

This equation, together with the equation

$$\frac{dw}{\partial t} = -2\pi\omega_p \frac{n'}{n\gamma} p_0 \int \frac{d\theta^2}{(\theta^2 - x^2)^{1/2}} \frac{\partial \Phi}{\partial \theta^2} xw, \tag{13}$$

which follows from (6) and (9), forms a closed system of equations.

A system of this type is obtained in the study of the slowing down of an ion beam in a plasma. This problem was investigated in [6]. Equations (12) and (13) have a self-similar solution that determines the time broadening of the function $\Phi(\theta)$. To find this solution, it is convenient to introduce the dimensionless self-similar variables and functions

$$\xi = \theta\left(2\pi t \omega_p \frac{n'}{n\gamma}\right)^{-1/2}, \quad \eta = x\left(2\pi t \omega_p \frac{n'}{n\gamma}\right)^{-1/2},$$
$$W(\eta) = \frac{1}{n' p_0 c} \int k^3 w \theta'^3\, d\theta', \quad \Psi = \Phi 2\pi t \omega_p \frac{n'}{n\gamma}. \tag{14}$$

In terms of these variables, Eqs. (12) and (13) can be integrated:

$$\Psi \xi^2 + 2 \int_0^{\xi} \frac{W \eta^2\, d\eta}{(\xi^2 - \eta^2)^{1/2}} \frac{\partial \Psi}{\partial \xi^2} = 0, \tag{12'}$$

$$W(\eta) = W_0 \exp\left[2 \int_{\eta}^{\infty} d\xi^2 \frac{\partial \Psi}{\partial \xi^2}\left(\arcsin\frac{\eta}{\xi} - \frac{\pi}{2}\right)\right]. \tag{13'}$$

It is seen from (12′) that when $\xi > \overline{\eta}$ the particle distribution function Ψ decreases with increasing ξ like $\exp(-\xi^5)$. Such a steep function can be replaced in the calculation of the integral of (13′) by the threshold $2\xi_0^{-2}\,\theta(\xi_0^2 - \xi^2)$. Such a replacement leads to an error not exceeding 20%. With this accuracy, we obtain from (13′)

$$W(\eta) = W_0 \begin{cases} \exp[-4\xi_0^{-2}(\arcsin(\eta/\xi_0) - \pi/2)], & \eta < \xi_0, \\ 1 & \eta > \xi_0, \end{cases} \tag{15}$$

W_0 is the initial noise level. If W_0 is of the order of the thermal-noise level, then the oscillation level increases during the course of the instability development by several orders of magnitude, so that the parameter $\ln[W(0)/W_0] = \Lambda$ can be regarded as large (for more details see [7]). The constant ξ_0 is determined by the upper line of formula (15): $\xi_0^2 = 2\pi/\Lambda$. If $\Lambda \gg 1$, then $\xi_0 \lesssim 1$, and then we can confine ourselves when substituting (15) into (12′) to the approximation

$$W(\eta) = W(0) \exp\{-4\Lambda^{3/2}\eta / (2\pi)^{3/2}\}. \tag{16}$$

This formula expresses a characteristic feature of quasilinear relaxation, which has already been noted in [7], namely that if the noise begins to increase from a very low level, then the time of quasilinear relaxation stretches out in comparison with the characteristic time of the problem by a factor $\Lambda = \ln(W_{max}/W_0)$, and the largest amplitude is possessed not by oscillations with an increment that is maximal at the given instant, but by oscillations whose increment was maximal much earlier. In this problem, the width of the noise distribution $\overline{\eta}$ turns out to be narrower by a factor Λ than the width of the particle distribution function, although the maximum of the increment corresponds to $\eta = \xi_0$.

In the approximation (16) we can easily obtain the function $\Psi(\xi)$ for $W(\eta)$ from (12′), and from the condition of the normalization of Ψ we obtain the value of W(0):

$$\Psi = 2\xi_0^{-2} \exp(-\xi^5/\xi_0^5), \quad W(0) = 8.4/\xi_0^4. \tag{17}$$

The results of our analysis of the process of the quasilinear evolution of the relativistic beam-electron distribution function

$$\Phi(\theta) = \int_0^{\infty} f(p, \theta) p\, dp$$

can be represented in terms of physical quantities in the form

$$p_0 \Phi(\theta) = \xi_0^{-2}\left(\pi\omega_p t \frac{n'}{n\gamma}\right)^{-1} \exp\left[-\frac{\theta^5}{\xi_0^5}\left(2\pi\omega_p t \frac{n'}{n\gamma}\right)^{-5/2}\right], \quad \xi_0 = \left(\frac{2\pi}{\Lambda}\right)^{1/2}, \tag{18}$$

$$\int k^3 w\left(\frac{\omega - k_z c}{kc\theta'}, \theta'\right) \theta'^3\, d\theta'$$
$$\approx 10 n' p_0 c \xi_0^{-4} \exp\left[-4\xi_0^{-3} \frac{\omega - k_z c}{kc\theta'}\left(2\pi\omega_p t \frac{n'}{n\gamma}\right)^{-1/2}\right]. \tag{19}$$

Thus, the time required for the function $\Phi(\theta)$ to double in width is equal to

$$t \approx \left(2\pi \frac{n'}{n\gamma}\omega_p\right)^{-1} \frac{\Lambda}{2\pi} \overline{\theta}^2. \tag{20}$$

Then the density of the noise energy increases to an approximate value

$$2\pi \int w \theta'^2\, d\theta'\, dx k^3 \approx 2\Lambda n' p_0 c \overline{\theta}. \tag{21}$$

During that time, the beam slows down and broadens in momentum by $\Delta p \approx 2\Lambda p_0 \overline{\theta}$.

After establishment of the solution (18), the increment $\delta(\theta')$ has a sharply pronounced maximum at $\theta' < \overline{\theta}$ (see formula (10)). Therefore the beam begins to excite predominantly oscillations with $\theta' < \overline{\theta}$. In the quasilinear theory, a difference of even a factor of two between the values of the increments leads to a difference of a factor e^{Λ} between the oscillation amplitudes. Consequently, the principal role in (7′) is played by the diffusion coefficient D_p. But since Eqs. (12) and (13) are not very sensitive to the form of the beam distribution function with respect to the momentum p, so long as $\overline{\Delta p} \ll p_0$, it follows that the mechanism analyzed by us for the formation of a steep front of the function $\Phi(\theta)$, which ensures a maximum of the increment at $\theta' \approx 0$, will remain in force during the course

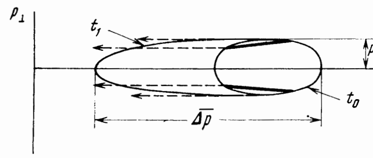

FIG. 2. Slowing down of a beam of relativistic electrons in a plasma. Dashed lines—trajectories along which the slowing down takes place. The light curves denote the regions of phase space occupied by the beam particles for two instants of time. The straight line separates the region of unstable angles.

of the beam slowing-down process, owing to excitation of oscillations with $\theta' < \bar{\theta}$. Conversely, the development of oscillations with small $\theta' < \theta$ will have little effect on $\Phi(\theta)$, since, according to an estimate that follows from (7), the broadenings $\overline{\Delta\theta}$ and $\overline{\Delta p}$ are connected by the relation

$$\frac{p^2}{\overline{\Delta p}^2}\overline{\Delta\theta}^2 \approx \frac{D_\theta}{D_p} \approx \bar{\theta}'^2$$

i.e., the broadening $\overline{\Delta\theta}$ due to oscillations with $\theta' < \bar{\theta}$ is smaller than $\bar{\theta}$. Under these conditions, the slowing down of a relativistic beam with a scatter $\bar{\theta} > 1/\gamma$ is described by the relatively simple system of equations ($\overline{\Delta p} \ll p_0$)

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p_z} D_p \frac{\partial f}{\partial p_z}, \quad D_p = \omega_p \frac{m}{4n} \int w_k \delta\left(\frac{kc-\omega}{kc} - \frac{\theta^2}{2}\right) d^3k, \tag{22}$$

$$\frac{dw_k}{w_k dt} = \pi\omega_p \frac{n'}{n\gamma} 2\pi p_0 \left[-4\frac{kc-\omega}{kc}\frac{\partial\Phi}{\partial\theta^2} - 2\Phi\right]_{\theta^2=2(kc-\omega)/kc}. \tag{23}$$

The function $\Phi(\theta)$ in (23) is determined here by (18). Given the diffusion coefficient, Eq. (22) can, in principle, be easily solved.

Figure 2 shows the two-dimensional diffusion picture. The dashed lines represent the lines along which the beam particles diffuse from the region of angles θ, on the steep slope of the function $\Phi(\theta)$, where phase resonance takes place with the narrow spectrum ($\Delta k/k \lesssim \bar{\theta}^2$) of the unstable oscillations. The particle velocities to the right of the solid curve fall on the plateau of the distribution (18) into the region of stable phase velocities of the oscillations with $\theta' < \bar{\theta}$. Equations (22) and (23) yield for the beam slowing-down time an estimate that coincides with (20).

Thus, the slowing-down time of a beam with a smeared-out angular distribution is determined by the initial angle scatter

$$t\omega_p \approx (n'/n\gamma)^{-1}\bar{\theta}_0^2, \quad \bar{\theta} > 1/\gamma. \tag{24}$$

This is a most important result of the quasilinear theory developed here.

In practice it is more important to estimate from the quasilinear theory the free path λ of a beam injected in stationary manner into a homogeneous plasma. In this case the noise accumulates up to a level $n'p_0mc^3/T_e$ (T_e is the plasma-electron temperature), and λ turns out to be smaller by a factor T_e/mc^2 than the result that follows from the simple estimate c/δ. For nonrelativistic beams, the solution of this problem can be found in [7]. We have

$$\lambda \approx \Lambda \frac{c}{\omega_p}\frac{n\gamma}{n'}\frac{T_e}{mc^2}\bar{\theta}_0^2. \tag{25}$$

3. INFLUENCE OF NONLINEAR EFFECTS OF OSCILLATION TRANSFORMATION BY UNSTABLE BEAMS

Tsytovich and Shapiro[5] (see also [8]) pointed out that the nonlinear effect of induced scattering of Langmuir oscillations by plasma particles can lead, even in weak beams, to a limitation of the noise level and to a lengthening of the slowing down time, and even to a complete suppression of the instability. We shall analyze below the degree to which such an effect can influence the collective slowing down of an intense relativistic beam in a dense plasma target.

Induced nonlinear scattering of Langmuir oscillations by plasma particles leads to a transfer of the oscillation energy into the region of larger phase velocities. If $\omega/k > c$, then the resonant interaction of the waves with the particles is no longer possible. But accumulation of oscillations in the region $\omega/k > c$ affects the growth of the unstable oscillations with $\omega/k < c$.

Let us include in (6) a term describing this nonlinear effect. According to [8], it is necessary to add to the right-hand side of (6) the term

$$2(\delta_n^e + \delta_n^i)N_k, \text{ where}$$

$$\delta_n^e \approx -\pi\omega_p \int \frac{w_k'}{nm(\omega/k)^2}\frac{(k^2-k'^2)v_{Te}}{|\mathbf{k}-\mathbf{k}'|^3\omega_p}\frac{(\mathbf{kk}')^2}{k'^2}\frac{[\mathbf{kk}']^2}{k^2k'^2}d^3k'.$$

$$\delta_n^i \approx -\pi\omega_p \frac{T_e/T_i}{(1+T_e/T_i)^2}\int \frac{w_{k'}}{nT_e}\frac{(k^2-k'^2)v_{Te}^2}{|\mathbf{k}-\mathbf{k}'|\omega_p v_{Ti}}\frac{(\mathbf{kk}')^2}{k^2k'^2} \times \exp\left\{-\frac{1}{2}\left[\frac{(k^2-k'^2)v_{Te}^2}{\omega_p|\mathbf{k}-\mathbf{k}'|v_{Ti}}\right]^2\right\}d^3k. \tag{26*}$$

The first nonlinear term in (26) takes into account the scattering of the oscillations by the plasma electrons ($v_{T\alpha} = (2T_\alpha/m_\alpha)^{1/2}$, $r_D = v_{Te}/\omega_p$). In order of magnitude, δ_n^e is equal to $w\omega(kr_D)^3/nE_e$. The role of scattering by ions is maximal if $\omega/k > v_{Te}^2/v_{Ti}$, i.e., if

$$T_i > \frac{T_e^2}{m(\omega/k)^2}\frac{M}{m}. \tag{27}$$

then $\delta_n^i \approx w\omega_p/nT$ ($T_e = T_i$). On the other hand, if the inequality of (27) is inverted, then the nonlinear scattering by ions is possible only for a small change of the magnitude of the wave vector

$$\Delta k/k < v_{Ti}\omega/kv_{Te}^2. \tag{28}$$

In this case the characteristic frequency of the spectral redistribution can be estimated at $\omega_p w(\Delta k/k)^2/nT$. Although in practice δ_n^i is larger than δ_n^e also in this case, the role of this process as a factor stabilizing the instability turns out to be negligible compared with scattering by electrons.

Nonlinear scattering of oscillations by electrons can suppress the stability almost completely if the oscillation energy density accumulated in the region $\omega/k > c$ is large

$$w/nT \gtrsim \delta/\omega_p(kr_D)^3. \tag{29}$$

This energy is in practice smaller than the limiting value for w, $n'\gamma m^2(\omega/k)^4/T_e$, which is established following stationary injection of a beam into a homogene-

*$(\mathbf{kk}') \equiv \mathbf{k}\cdot\mathbf{k}'$; $[\mathbf{kk}'] \equiv \mathbf{k}\times\mathbf{k}'$.

ous plasma. The transfer of the noise from the instability region $\omega/k < c$ into the region $\omega/k > c$ will, given satisfaction of the inequality inverse to (27), proceed in relay fashion via scattering of the oscillations by ions. After the oscillations with $\omega/k > c$ reach the level (29), the residual instability may be due to the fact that the oscillations are attenuated by Coulomb collisions. In an inhomogeneous plasma, their phase velocity decreases and the oscillations can become absorbed by the thermal electrons when the latter move at the group velocity in a direction opposite to the density gradient. This effect is taken into account by the second and third terms of the left side of Eq. (6). Finally, the energy concentration in the long-wave part of the spectrum counteracts a nonlinear effect of higher order, namely plasmon–plasmon scattering.[9]

Let us define the problem more concretely in accordance with the title of this article (a dense plasma target, where the Coulomb-collision frequency is high; a strongly radially inhomogeneous plasma resulting from large heat release when the beam is self-focused[4]). Under these conditions, the principal role is assumed by the first two dissipation effects. We shall assume that $T_i = T_e$ and the condition

$$T_e > mc^2 m/M, \tag{30}$$

is satisfied, i.e., $T_e > 0.5$ keV. Then the condition (28) is satisfied and the long-wave oscillations ($\omega/k > 2c$) can influence the transfer of the oscillations from the unstable region of phase velocities only via scattering by electrons. Scattering by ions only decreases the noise energy in the unstable region, and transfers it in relay fashion into the region $\omega/k > c$. It hardly changes the characteristic time of doubling of the noise energy density.

In accord with the results of the preceding section, the region of strong kinetic instability, where the main accumulation of the oscillations takes place, is narrow relative to the phase velocities $\Delta(\omega/kc) \approx \frac{1}{2}\overline{\theta}^2$ and to the solid angle $\theta' < \overline{\theta}$. After one act of scattering by ions, which changes the phase velocities of the oscillations in accord with (28) by an amount $\Delta(\omega/kc) \approx \sqrt{m/M}\, c/v_{Te}$, it falls into the nonresonant region $\omega/k > c$ if

$$\overline{\theta}^2 < \frac{c}{v_{Te}}\sqrt{\frac{m}{M}}. \tag{31}$$

We shall assume that this condition is satisfied.

The value of δ_n^i varies with the angle θ' relatively smoothly. Therefore, after several scattering acts, the oscillation spectrum can be regarded as isotropic. But isotropic noise corresponds to δ_n^i and δ_n^e that are independent of the angle. If this is so, then the nonlinear scattering of the oscillations by the plasma particles does not change the character of the angular distribution of the total instability increment. Consequently, the main conclusion of the preceding section, namely that the instability of the beam leads predominantly to deceleration of the beam—to an increase of the longitudinal momentum scatter—should hold true. The angle scatter $\overline{\theta}$ remains small and hardly changes during the slowing-down process.

We can now write down the energy balance, which connects the rate of oscillation generation by the instabilities to the rate of their damping in the region $\omega/k > c$:

$$2\delta w_{\rm I} = (\nu_{ei} + c/a)\, w_{\rm II}. \tag{32}$$

Here $w_{\rm I,\,II}$ are the energy densities in the regions $\omega/k < c$ and $\omega/k > c$, respectively, and a is the minimum characteristic dimension of the inhomogeneity of the plasma density. Oscillations in the region $\omega/k > c$ accumulate up to a level determined by the condition (29). Substituting the minimum value of $w_{\rm II}$ from (29) into (32), we obtain an estimate for the oscillation energy density in the region of the instability in the nonlinear regime:

$$w_{\rm I} = \left(\nu_{ei} + \frac{c}{a}\right)\omega_p^{-1}\left(\frac{c}{v_{Te}}\right)^3 nT_e. \tag{33}$$

If this estimate is substituted in Eq. (22) for f(p), then we obtain the following estimate for the mean free path L(p) of the beam of relativistic electrons:

$$L(p) = \frac{p_0^2 c}{D_p} \approx \gamma^2 \frac{v_{Te}}{\nu_{ei} + c/a}\overline{\theta}^2 \approx \lambda_c(p)\left(\frac{v_{Te}}{c}\right)^4 \overline{\theta}^2 \left(1 + \frac{c}{a\nu_{ei}}\right)^{-1}\gamma, \tag{34}$$

$\lambda_c(p)$ is the mean free path of the relativistic electrons in the plasma relative to Coulomb collisions. This formula was obtained for the kinetic instability under the following conditions:

$$\left(\frac{m}{M}\frac{c^2}{v_{Te}^2}\right)^{1/4} > \overline{\theta} > \frac{1}{\gamma},\quad T_i = T_e,\quad T_e > mc^2\frac{m}{M}.$$

If the last condition is not satisfied, then the beam instability will be suppressed by the scattering of the oscillations by the ions, and in lieu of (34) we obtain

$$L(p) = \lambda_c(p)\frac{v_{Te}}{c}\overline{\theta}^2 \exp\left(-\frac{M}{m}\frac{T_e}{mc^2}\right)\left(1 + \frac{c}{a\nu_{ei}}\right)^{-1}\gamma. \tag{35}$$

Comparing (34) and (35), we see that the free path L(p) is a nonmonotonic function of the plasma temperature. The minimum value of L(p) in the case $a \gg c/\nu_{ei}$ corresponds to a temperature $T_e \approx mc^2 m/M$:

$$L_{min}(p) = \lambda_c(p)\,(m/M)^2\overline{\theta}^2\gamma. \tag{36}$$

It must be borne in mind, however, that in the kinetic instability the Coulomb collisions of the plasma electrons with the ions decrease the increment, and there is no instability when $\nu_{ei} > 2\delta$.

We emphasize that in the case of instability of a monochromatic beam, the nonlinear effects do not suppress the instability, but intensify it ("explosive" instability), since the unstable oscillations have a negative energy.[10]

The inhomogeneity of the plasma density can stop the instability because the phase velocity of the Langmuir oscillations leaves the range of values corresponding to the maximum increment. This effect is taken into account by the principal, third term of (6). Even if one sees to it that the plasma target is sufficiently homogeneous, the target becomes heated during the course of slowing down of the beam, and its homogeneity is disturbed by the spread of the plasma. All these effects are analyzed in [11]. However, nonlinear spectral transfer of the oscillations, not taken into account in [11], should lead to an appreciable increase of the initial level w_0 of the oscillations, and to a decrease of the numerical factor $\ln(w_{max}/w_0)$. Therefore the criterion for neglecting the influence of the plasma density inhomogeneity on the beam instability is

$$\frac{1}{n}\frac{dn}{dz} < \frac{\omega_p}{c}\frac{n'}{n\gamma}. \tag{37}$$

In concluding this section, we present an interpolation formula for the length of collective slowing down of a beam of relativistic electrons in a plasma, at values $\overline{\theta}$, $(mc^2/Mv_{Te}^2)^{1/4} > \overline{\theta} > 1/\gamma$; this formula goes over in the corresponding limiting cases into formulas (25), (34), and (35) and takes into account the instability condition $\nu_{ei} < 2\delta$ and (37):

$$L(p) = \lambda_c(p)\frac{v_{Te}}{c}\theta^2\left[\left(\frac{v_{Te}}{c}\right)^3 + \exp\left(-\frac{T_e M}{m^2c^2}\right)\right]\left(1 + c\frac{\nabla n}{n\nu_{ei}}\right)^{-1}\gamma \times\left(1 - \frac{\nu_{ei}}{2\omega_p}\frac{n\gamma}{n'}\theta^2 - \frac{c\gamma}{\omega_p n'}\frac{dn}{dz}\right) + \frac{c}{\omega_p}\frac{n\gamma}{n'}\frac{T_e}{mc^2}\theta^2. \tag{38}$$

4. OTHER COLLECTIVE SLOWING-DOWN MECHANISMS

When an intense beam enters a dense plasma, an inverse current of plasma electrons is produced in the plasma in accordance with the induction law. If the beam injection time is shorter than the skin time, $4\pi\sigma a^2/c^2$, then the density of the inverse current j is practically equal to the beam-current density. Effects of focusing of a beam by its own magnetic field and of plasma heating by dissipation of the inverse current, and the corresponding slowing down of the beam, were considered in [4]. But the Coulomb electric conductivity of the plasma σ increases rapidly with increasing temperature. For a strong heating of the target, it is therefore necessary to have very large beam-current densities.

a) The plasma turbulence due to the unstable beam can greatly decrease σ. The increase of the plasma-electron scattering frequency is connected with the nonlinear effect of scattering of Langmuir waves by electrons

$$\nu_{\text{eff}} \approx (w/nT_e)^2(kr_D)^3\omega_p. \tag{39}$$

Using the estimate (29), we obtain

$$\nu_{\text{eff}} \approx \omega_p(\delta/\omega_p)^2(kr_D)^{-3}; \tag{40}$$

ν_{eff} is larger than ν_{ei} if

$$\frac{n'}{n} > \gamma\theta^2\left[\frac{\nu_{ei}}{\omega_p}(kr_D)^3\right]^{1/2}. \tag{41}$$

The beam mean free path L_σ due to this effect can be estimated from the relation

$$\frac{j^2}{\sigma}\frac{L_\sigma}{c} \equiv \frac{n'^2c^2}{n}m\nu_{\text{eff}}\frac{L_\sigma}{c} = n'mc^2\gamma, \tag{42}$$

whence

$$L_{\sigma\,\text{eff}} \approx \frac{c}{\omega_p}\left(\frac{\gamma n}{n'}\right)^3\theta^4\left(\frac{v_{Te}}{c}\right)^3 \quad (kr_D = v_{Te}/c). \tag{43}$$

b) If the density of the inverse current exceeds the critical value $\alpha en\sqrt{T_e/M}$, where α is the numerical factor, as is well-known, ohmic heating produces a plasma with $T_e > T_i$. In such a plasma, ion-acoustic instability of the current is possible, and can result in turbulent heating of the plasma [6] to a temperature $T_e = (j/en\alpha)^2M$. The condition for the occurrence of turbulent heating can be also written in the form

$$\frac{n'}{n} > \alpha\frac{v_{Te}}{c}\sqrt{\frac{m}{M}}.$$

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239