SOVIET PHYSICS JETP VOLUME 32, NUMBER 6 JUNE, 1971

# CORRECTIONS TO QUASISTATIONARY VALUE OF THE DIELECTRIC TENSOR OF A SMOOTHLY INHOMOGENEOUS ELECTRON PLASMA

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Submitted June 29, 1970

Zh. Eksp. Teor. Fiz. 59, 2160–2164 (December, 1970)

Geometrical optics is used to calculate the correction to the "quasistationary" value of the dielectric tensor $\epsilon^0_{\alpha\beta}$ in a smoothly inhomogeneous electron plasma. The hydrodynamic approximation is used to describe the electron motion. It is shown that besides the antihermitian correction to $\epsilon^0_{\alpha\beta}$, which ensures conservation of the energy flux in each ray tube, there is also a hermitian correction which, however, does not influence the values of the fields in the zeroth approximation of geometrical optics.

As is well known, the dielectric tensor $\epsilon_{\alpha\beta}(\omega, t; k, r)$ of an inhomogeneous and nonstationary dispersive medium can be defined formally as the Fourier transform of the kernel $\epsilon_{\alpha\beta}(t-t', t; r-r', r)$ of the material equation

$$\mathscr{D}_\alpha(\mathbf{r},t)=\int_{-\infty}^{t}dt'\int d^3r'\,\hat{\varepsilon}_{\alpha\beta}(t-t',t;\mathbf{r}-\mathbf{r}',\mathbf{r})\mathscr{E}_\beta(\mathbf{r}',t') \qquad (1)$$

with respect to the difference arguments[1-3], i.e.,

$$\varepsilon_{\alpha\beta}(\omega,t;\mathbf{k},\mathbf{r})=\int_{-\infty}^{t}dt'\int d^3r'\,\hat{\varepsilon}_{\alpha\beta}(t-t',t;\mathbf{r}-\mathbf{r}',\mathbf{r})\,e^{i\omega(t-t')-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}. \qquad (2)$$

The tensor $\epsilon_{\alpha\beta}(\omega, t; \mathbf{k}, \mathbf{r})$ introduced in this manner is practically useless for solving the problem of propagation of electromagnetic waves in an arbitrary inhomogeneous and nonstationary medium, but for smoothly-inhomogeneous and slowly-nonstationary media, when the Maxwell's equations can be solved by geometrical optics, it becomes quite important, since it enters in the main equations of the geometric approximation[1)][4-8,3]. Moreover, within the framework of this approximation it is possible to find also the tensor $\epsilon_{\alpha\beta}$ itself. To this end it is necessary to use in lieu of the phenomenological material equation (1) the microscopic equations of motion of the charges; the latter, like Maxwell's equations, must also be solved by the geometrical optics method.

For a magnetoactive plasma, the tensor $\epsilon_{\alpha\beta}$ was calculated within the framework of the zeroth approximation of geometrical optics in[4-7], using an expansion in terms of the parameter $\mu_1 \sim \rho/L$ ($\rho$ is the Larmor radius and L is the scale of the plasma inhomogeneity), which was assumed small compared with unity but large compared with the "geometrically" small parameter $\mu \sim 1/kL$: $\mu \ll \mu_1 \ll 1$.

In this paper we calculate, by the geometrical optics method, the tensor $\epsilon_{\alpha\beta}$ for a smoothly-inhomogeneous electron plasma in the absence of a magnetic field, when the quantity $\mu \sim 1/kL$ is the only small parameter of the problem.

Following[2], we start with the fact that in a medium whose properties vary smoothly in space and in time the tensor $\epsilon_{\alpha\beta}$ differs little from its "quasistationary" value $\epsilon^0_{\alpha\beta}$, which is calculated for a homogeneous and stationary medium with the same values of the macroscopic parameters (temperature, concentration, etc.) as the considered inhomogeneous and nonstationary medium at the given instant t and at the given point r. In other words, for smoothly-inhomogeneous and slowly-nonstationary media we have

$$\varepsilon_{\alpha\beta}=\varepsilon_{\alpha\beta}{}^0+\delta\varepsilon_{\alpha\beta}+O(\mu^2),\ |\delta\varepsilon_{\alpha\beta}|\sim\mu\ll1. \qquad (3)$$

We substitute (3) in the expression

$$D_\alpha=\varepsilon_{\alpha\beta}E_\beta-i\frac{\partial E_\beta}{\partial x_j}\frac{\partial\varepsilon_{\alpha\beta}}{\partial k_j}-\frac{i}{2}E_\beta\frac{\partial^2\psi}{\partial x_j\partial x_m}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial k_j\partial k_m}+O(\mu^2), \qquad (4)$$

which can be obtained from (1) for stationary media and for monochromatic waves of the type $\mathscr{E}_\beta = E_\beta e^{i\psi - i\omega t}$, $\mathscr{D}_\alpha = D_\alpha e^{i\psi - i\omega t}$ with slowly varying amplitudes $E_\beta$ and $D_\alpha$[2)]. This substitution yields

$$D_\alpha=\varepsilon_{\alpha\beta}{}^0E_\beta-i\frac{\partial E_\beta}{\partial x_j}\frac{\partial\varepsilon_{\alpha\beta}{}^0}{\partial k_j}-\frac{i}{2}E_\beta\frac{\partial^2\psi}{\partial x_j\partial x_m}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial k_j\partial k_m}+\delta\varepsilon_{\alpha\beta}E_\beta+O(\mu^2), \qquad (5)$$

where the second, third, and fourth terms are quantities of order $\mu \ll 1$.

We assume that by solving the microscopic equations of motion for the charges we have succeeded in establishing a connection between $D_\alpha$ and $E_\beta$ in the form (5), in which we separated from $D_\alpha$ the zeroth-order term $\epsilon^0_{\alpha\beta}E_\beta$, and also the first-order terms with $\partial E_\beta/\partial x_j$ and $\partial^2\psi/\partial x_j\partial x_m$. Then the coefficient of $E_\beta$ in the remaining term of the order of $\mu$ can be naturally taken to be the first-order correction $\delta\epsilon_{\alpha\beta}$ to the "quasistationary" value of the tensor $\epsilon^0_{\alpha\beta}$. This is precisely what we shall do below for an inhomogeneous electron plasma whose electron-density gradient is produced by a static electric field $\mathbf{E}_{st}(\mathbf{r})$. In the hydrodynamic approximation in the absence of a static magnetic field, the motion of such a plasma under the influence of a monochromatic field $\vec{\mathscr{E}}$ is described by the following linearized equations:

[1)]In this case $\omega$ and k should be taken to mean the corresponding partial derivatives of the eikonal $\varphi = \varphi(\mathbf{r}, t)$: $\omega = -\partial\varphi/\varphi t$, $k_i = \partial\varphi/\partial x_i$.

[2)]For simplicity we confine ourselves to the stationary case $\omega$ = const, $\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}(\omega, \mathbf{k}, \mathbf{r})$, where $k_j = \partial\psi/\partial x_j$. A generalization of expression (4) to the nonstationary case is contained in [3].

$$-i\omega m n_0 \mathbf{v}_1 = -T\nabla n_1 + n_1 T\frac{\nabla n_0}{n_0} + n_0 e\vec{\mathscr{E}},$$
$$-i\omega n_1 + n_0 \operatorname{div} \mathbf{v}_1 = -\mathbf{v}_1\nabla n_0. \tag{6}$$

Just as in[9,10], we assume that the equilibrium density of the electrons $n_0 = n_0(\mathbf{r})$ is maintained by a static electric field $\mathbf{E}_{st}(\mathbf{r}) = -T\nabla n_0/en_0$.

We seek a solution of the linear equations (6) by the geometrical-optics method, assuming, in analogy with[11,12], that $n_1 = Ne^{i\psi - i\omega t}$, $v_1 = Ve^{i\psi - i\omega t}$. The slow amplitudes N and V satisfy the equations

$$-i\omega m n_0 \mathbf{V} + TkN - n_0 e\mathbf{E} = -T\nabla N + TN\nabla n_0/n_0,$$
$$-i\omega N + n_0 \mathbf{kV} = -n_0 \operatorname{div} \mathbf{V} - \mathbf{V}\nabla n_0.$$

In the right-hand sides of these equations there are separated the terms of first order of smallness with respect to $\mu \sim 1/kL$, so that V and N can be sought in the form of expansions in powers of this small parameter, for example, $N = N_0 + N_1 + \ldots$ . For the amplitude of the zeroth and first approximations we obtain the following systems of equations:

$$-i\omega m n_0 \mathbf{V}_0 + T\mathbf{k}N_0 - n_0 e\mathbf{E} = 0,$$
$$-i\omega N_0 + n_0 \mathbf{kV}_0 = 0, \tag{7}$$

$$-i\omega m n_0 \mathbf{V}_1 + T\mathbf{k}N_1 = -T\nabla N_0 + TN_0\nabla n_0/n_0,$$
$$-i\omega N_1 + n_0 \mathbf{kV}_1 = -n_0 \operatorname{div} \mathbf{V}_0 - \mathbf{V}_0\nabla n_0, \tag{8}$$

After determining $\mathbf{V}_0$ and $\mathbf{V}_1$ from (7) and (8), we find the amplitude of the induced current $\mathbf{I} \cong en_0(\mathbf{V}_0 + \mathbf{V}_1)$, and then also the amplitude of the electric induction $\mathbf{D} = \mathbf{E} + 4\pi i\mathbf{I}/\omega$. Accurate to terms of second order of smallness, $D_\alpha$ can be represented in the form (5) needed by us, with the hermitian and anti-hermitian parts of the sought correction $\delta\epsilon_{\alpha\beta} = \delta\epsilon^h_{\alpha\beta} + \delta\epsilon^a_{\alpha\beta}$ written as follows:

$$\delta\varepsilon_{\alpha\beta}{}^{h} = i(k_\alpha q_\beta - k_\beta q_\alpha), \tag{9}$$

$$\delta\varepsilon_{\alpha\beta}{}^{a} = -\frac{i}{2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial k_j\partial x_j} \cong -i(k_\alpha q_\beta + k_\beta q_\alpha). \tag{10}$$

Here $\epsilon^0_{\alpha\beta}$ is the quasistationary value of the dielectric tensor[3)]:

$$\varepsilon_{\alpha\beta}{}^0 = \varepsilon_\perp\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) + \varepsilon_\parallel\frac{k_\alpha k_\beta}{k^2},$$
$$\varepsilon_\perp = 1 - \frac{4\pi e^2 n_0}{m\omega^2}, \tag{11}$$
$$\varepsilon_\parallel = 1 - \frac{4\pi e^2 n_0}{m\omega^2 - k^2 T} \cong 1 - \frac{4\pi e^2 n_0}{m\omega^2}\left(1 + \frac{k^2 T}{m\omega^2}\right),$$

and q denotes a vector proportional to $\nabla n_0$:

$$\mathbf{q} = \frac{2\pi e^2}{m\omega^2}\frac{T\nabla n_0}{m\omega^2 - k^2 T} \cong \frac{2\pi e^2 T}{m^2\omega^4}\nabla n_0. \tag{12}$$

Both the hermitian and the antihermitian parts of $\delta\epsilon_{\alpha\beta}$ depend on the gradient of the equilibrium electron concentration $\nabla n_0$ and vanish, as expected, at $n_0 = \text{const}$, and also as $T \to 0$, i.e., in a cold plasma.

The obtained value of the anti-hermitian correction (10) agrees with the expression

$$\delta\varepsilon_{\alpha\beta}{}^{a} = \frac{i}{2}\left(\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial\omega\,\partial t} - \frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial x_j\,\partial k_j}\right), \tag{13}$$

which was obtained in[13] from the purely phenomenological condition for the conservation of the adiabatic invariance for non-absorbing media. The first term in (13) was found by Pitaevskiĭ for an isotropic nonstationary medium[2] and was already used, for example, in the study of electroacoustic waves in liquids and in a plasma[14].

In the stationary case $\omega$ = const, $\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}(\omega, \mathbf{k}, \mathbf{r})$, under consideration, the first term in (13) vanishes, and the conservation of the adiabatic invariant simply means conservation of the energy flux in the ray tube (see below). Insofar as the present authors know, Eq. (10) is the first direct (albeit partial, since we are dealing only with the second term of (13)) confirmation of the phenomenological ratio (13). Indirect confirmation of (13) is provided by the results of[11,12], where the conservation of the adiabatic invariant for waves in a magnetoactive nonstationary plasma was proved by geometrical optics, although the correction $\delta\epsilon_{\alpha\beta}$ was not calculated in[11,12].

We note that the correction $\delta\epsilon_{\alpha\beta}$ gives rise to certain singularities of the geometrical-optics equations as compared with the "quasistatic" case $\epsilon_{\alpha\beta} = \epsilon^0_{\alpha\beta}$. When the quantity $\epsilon^0_{\alpha\beta} + \delta\epsilon^h_{\alpha\beta}$ is included as the hermitian part of the tensor $\epsilon_{\alpha\beta}$ in the eikonal equation, the latter contains terms of only second (and not first) order in $\mu$, i.e., the eikonal equation remains the same, accurate to terms $\sim\mu^2$, as when $\epsilon_{\alpha\beta} = \epsilon^0_{\alpha\beta}$. On the other hand, it would be inconsistent to take into account in the eikonal equation the terms quadratic in $\mu$, which has been shown by analysis to lift the polarization degeneracy of the transverse waves, since the second-order correction to $\epsilon^0_{\alpha\beta}$, which we did not take into account, is of the same order. Under these conditions, it is advantageous to include the term with $\delta\epsilon^h_{\alpha\beta}$ not in the zeroth order but in the first order of Maxwell's equations, in analogy with the procedure used in[15].

By including the term with $\delta\epsilon^h_{\alpha\beta}$, and also the term with $\delta\epsilon^a_{\alpha\beta}$ in the first-order equations, it is easy to write down the conditions for the compatibility of these equations. Using (9) and (10), and also the results of[3], we can represent the compatibility conditions for the longitudinal oscillations (for which $\mathbf{E}^0 = \Phi_\parallel\mathbf{k}/k$, $\mathbf{H}^0 = 0$) in the form

$$\operatorname{div}\mathbf{S} = 0,\quad \mathbf{S} = -\frac{\omega}{16\pi}\frac{\partial\varepsilon_\parallel}{\partial k}|\Phi_\parallel|^2\frac{\mathbf{k}}{k}, \tag{14}$$

where S is the energy flux density of the electrostatic oscillations. Equation (14) describes the conservation of the energy flux in the ray tube. As already stated, this is a particular case of the conservation of the adiabatic invariant for monochromatic waves in a stationary medium.

It can be shown that one of the compatibility conditions can be written in the same form also for transverse waves, for which

$$\mathbf{E}_0 = \Phi_1\mathbf{n} + \Phi_2\mathbf{b},$$
$$\mathbf{H}_0 = \sqrt{\varepsilon_\perp}(\Phi_1\mathbf{b} - \Phi_2\mathbf{n}),$$
$$\mathbf{S} = \frac{c\sqrt{\varepsilon_\perp}}{8\pi}\frac{\mathbf{k}}{k}(|\Phi_1|^2 + |\Phi_2|^2)$$

3)The approximate-equality sign in (1)–(12) pertains to the case $k^2T \ll m\omega^2$.

(n and b are respectively the principal normal and binormal to the ray), whereas the second compatibility condition is the law established by Rytov[16,17] for the rotation of the polarization plane

$$d\theta / d\sigma = 1 / T_t$$

where $\theta = \tan^{-1}(\Phi_2/\Phi_1)$ is the length element and $T_t$ is the ray torsion radius.

Thus, for both longitudinal and transverse waves, the anti-hermitian correction (10) ensures conservation of the energy flux in the ray tube, whereas the hermitian part (9) drops out completely from both the eikonal equation and the compatibility conditions of the first-approximation equations. Consequently, in our problem (inhomogeneous plasma in the absence of a magnetic field) it does not influence at all the magnitude of the amplitude of the field $\mathbf{E}_0$ and $\mathbf{H}_0$ of the zeroth approximation. At the same time, as shown by an analysis, the correction $\delta\epsilon^h_{\alpha\beta}$ causes the field vectors to acquire in first order additional components of the order of $\mu$, namely, a longitudinal field component appears for the transverse wave, and a transverse one for the longitudinal wave. These additional components increase strongly in the plasma resonance region $\epsilon_\perp \sim 0$, where a noticeable interaction takes place between the longitudinal and transverse waves[9,10,18].

It seems to us that the consistent geometrical-optics method described in this paper for finding the correction to the "quasistationary" value of the tensor $\epsilon^0_{\alpha\beta}$ can be used also under more general conditions, namely, in the presence of absorption and of an external magnetic field, for a nonstationary plasma, and also in the kinetic description of particle motion in a plasma.

In conclusion, the authors thank S. M. Rytov and N. S. Stepanov for useful discussions.

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Translated by J. G. Adashko
245